Math 228
4-30-1994

Review Session for the Final - 1.

Example. Solve the system

$$\vec{x}\,' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

Classify the critical point $(0,0)$, determine whether it is stable, asymptotically stable, or unstable, and describe how the solutions behave as $t \to \infty$.

The characteristic polynomial is $\lambda^2 + 5\lambda + 4 = (\lambda+1)(\lambda+4)$. The eigenvalues are $\lambda = -1$ and $\lambda = -4$.

Since the eigenvalues are real and distinct, it is probably easiest to find the solution by computing eigenvectors.

For $\lambda = -1$,

$$A + I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}.$$

The two rows must be multiples, so I can ignore the second row. From the first row, I see that I want $\langle a, b\rangle$ such that

$$(-2)\cdot a + 1 \cdot b = 0.$$

An obvious solution is $\langle a, b\rangle = \langle 1, 2\rangle$.

Next, for $\lambda = -4$,

$$A + I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

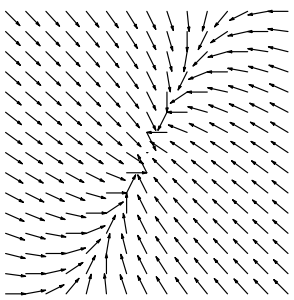
Ignore the second row. From the first row, I see that I want $\langle a, b\rangle$ such that

$$1 \cdot a + 1 \cdot b = 0.$$

An obvious solution is $\langle a, b\rangle = \langle 1, -1\rangle$.

The solution is

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$



$\det = 4$ and $\mathrm{tr} = -5$, so $\det < \frac{1}{4}\,\mathrm{tr}^2$. The critical point is an asymptotically stable improper node. Therefore, all the solutions approach $(0,0)$ as $t \to \infty$ □

Example. Determine whether the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

is diagonalizable.

A is diagonalizable if and only if it has 3 independent eigenvectors.

The matrix is triangular, so the eigenvalues lie on the main diagonal. They are $\lambda = 2$ (double) and $\lambda = 1$.

For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $b = 0$ and $c = 0$. Therefore, the eigenvectors are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Setting $a = 1$ yields $\langle 1, 0, 0 \rangle$.

For $\lambda = 1$,

$$A - I = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $a = 0$ and $b - c = 0$, or $b = c$. Therefore, the eigenvectors are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ c \end{bmatrix} = c \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Setting $c = 1$ yields $\langle 0, 1, 1 \rangle$.

There are only two independent eigenvectors. The matrix is *not* diagonalizable. □

Example. Solve

$$y'' + 2y' + y = 3te^{-t}, \qquad y(0) = 4, \quad y'(0) = 2,$$

using Laplace transforms.

Transform both sides:

$$\left(-2 - 4s + s^2 Y(s)\right) + 2\left(-4 + sY(s)\right) + Y(s) = \frac{3}{(s+1)^2},$$

$$-10 - 4s + (s+1)^2 Y(s) = \frac{3}{(s+1)^2}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{3}{(s+1)^4} + \frac{4s + 10}{(s+1)^2}.$$

The first term may be inverted immediately:

$$\mathcal{L}^{-1}\left(\frac{3}{(s+1)^4}\right) = \frac{1}{2}t^3 e^{-t}.$$

To invert the second term, try to write it in terms of $s+1$'s:

$$\frac{4s+10}{(s+1)^2} = \frac{4s+4}{(s+1)^2} + \frac{6}{(s+1)^2} = \frac{4}{s+1} + \frac{6}{(s+1)^2}.$$

Now

$$\mathcal{L}^{-1}\left(\frac{4}{s+1} + \frac{6}{(s+1)^2}\right) = 4e^{-t} + 6te^{-t}.$$

The solution is

$$y(t) = \frac{1}{2}t^3e^{-t} + 4e^{-t} + 6te^{-t} \quad \square.$$

Example. Solve

$$y'' + 2y' + y = 3te^{-t}, \qquad y(0) = 4, \quad y'(0) = 2,$$

using undetermined coefficients.

The operator form is $(D+1)^2y = 3te^{-t}$.
The homogeneous solution is

$$y_h = c_1e^{-t} + c_2te^{-t}.$$

The right side $3te^{-t}$ is killed by $(D+1)^2$. Apply this to both sides of the original equation to determine the form of a particular solution:

$$(D+1)^4y_p = (D+1)^2(3te^{-t}) = 0.$$

Therefore,

$$y_p = Ae^{-t} + Bte^{-t} + Ct^2e^{-t} + Dt^3e^{-t}.$$

Ae^{-t} and Bte^{-t} are homogeneous solutions; throw them out, leaving

$$y_p = Ct^2e^{-t} + Dt^3e^{-t}.$$

Differentiate:

$$y_p' = 2Cte^{-t} - Ct^2e^{-t} + 3Dt^2e^{-t} - Dt^3e^{-t} = \left(2Ct + (-C+3D)t^2 - Dt^3\right)e^{-t},$$

$$y_p'' = 2Ce^{-t} - 4Cte^{-t} + Ct^2e^{-t} + 6Dte^{-t} - 6Dt^2e^{-t} + Dt^3e^{-t} = \left(2C + (-4C+6D)t + (C-6D)t^2 + Dt^3\right)e^{-t}.$$

Hence,

$$\begin{array}{rcccccccccc} y_p'' + 2y_p' + y_p & = & 2Ce^{-t} & + & (-4C+6D)te^{-t} & + & (C-6D)t^2e^{-t} & + & Dt^3e^{-t} & & \\ & & & + & 4Cte^{-t} & + & (-2C+6D)t^2e^{-t} & - & 2Dt^3e^{-t} & & \\ & & & & & + & Ct^2e^{-t} & + & Dt^3e^{-t} & & \\ & = & 6Ce^{-t} & + & 6Dte^{-t} & & & & & = & 3te^{-t}. \end{array}$$

Comparing coefficients, $C = 0$ and $D = \dfrac{1}{2}$, so

$$y_p = \frac{1}{2}t^3e^{-t}.$$

The general solution is

$$y = c_1e^{-t} + c_2te^{-t} + \frac{1}{2}t^3e^{-t}.$$

Insert the initial conditions and solve for the constants: $4 = y(0) = c_1$, so

$$y = 4e^{-t} + c_2 te^{-t} + \frac{1}{2}t^3 e^{-t}.$$

Differentiate:

$$y' = -4e^{-t} + c_2 e^{-t} - c_2 te^{-t} + \frac{3}{2}t^2 e^{-t} - \frac{1}{2}t^3 e^{-t}.$$

Next, $2 = y'(0) = -4 + c_2$ and $c_2 = 6$. Hence,

$$y = 4e^{-t} + 6te^{-t} + \frac{1}{2}t^3 e^{-t}. \square$$

Example. Solve

$$y'' + 2y' + y = 3te^{-t}, \qquad y(0) = 4, \quad y'(0) = 2,$$

using variation of parameters.

$y_1 = e^{-t}$ and $y_2 = te^{-t}$ are independent homogeneous solutions. The general solution will have the form $y = u_1 y_1 + u_2 y_2$.

Set up the variation of parameters equation:

$$\begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 3te^{-t} \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = e^{2t} \begin{bmatrix} e^{-t} - te^{-t} & -te^{-t} \\ e^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} 0 \\ 3te^{-t} \end{bmatrix} = \begin{bmatrix} -3t^2 \\ 3t \end{bmatrix}.$$

Solve for u_1 and u_2:

$$u_1' = -3t^2, \quad u_1 = -t^3 + c_1.$$

$$u_2' = 3t, \quad u_2 = \frac{3}{2}t^2 + c_2.$$

Therefore,

$$y = -t^3 e^{-t} + c_1 e^{-t} + \frac{3}{2}t^3 e^{-t} + c_2 te^{-t} = c_1 e^{-t} + c_2 te^{-t} + \frac{1}{2}t^3 e^{-t}.$$

You can find the constants exactly as we did in the previous example. $\square$

Example. Solve for y in terms of x:

$$(e^{3x} - 2y)\,dx = x\,dy.$$

The equation is not separable, or homogeneous. If I rewrite it as

$$(e^{3x} - 2y)\,dx - x\,dy = 0$$

I find that $M_y = -2$ while $N_x = -1$, so it is not exact. It would become exact upon multiplication by an integrating factor, but there is an easier way.

Solve for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{e^{3x}}{x} - 2\frac{y}{x}.$$

I now see that the equation is first-order linear:

$$\frac{dy}{dx} + 2\frac{y}{x} = \frac{e^{3x}}{x}.$$

Compute the integrating factor:

$$I = \exp \int 2\frac{1}{x}\,dx = \exp(2\ln x) = \exp\ln(x^2) = x^2.$$

Then

$$yx^2 = \int xe^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c.$$

(A common mistake is to forget to put in the arbitrary constant at this step. You have to put it in *before* dividing by x^2!)

Here's the work for the integral:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int dx \\ + & x & & e^{3x} \\ & & \searrow & \\ - & 1 & & \dfrac{1}{3}e^{3x} \\ & & \searrow & \\ + & 0 & \rightarrow & \dfrac{1}{9}e^{3x} \end{array}$$

$$\int xe^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c.$$

Hence,

$$y = \frac{1}{3x}e^{3x} - \frac{1}{9x^2}e^{3x} + \frac{c}{x^2}. \square$$

©1994 by Bruce Ikenaga