2-22-1994

Review Problems on Second Order Equations.

Example. In some cases, you may need to proceed in two steps. First, you use **reduction of order** to obtain a complete set of homogeneous solutions. Next, you use those solutions and **variation of parameters** to solve a nonhomogeneous equation.

Consider the equation

$$x^2y'' + xy' - y = x^2(x+1).$$

Suppose I'm given that $y_1 = x$ is a homogeneous solution. Let me remind you of what that means. If I compute the derivatives, I get

$$y_1 = x, \quad y_1' = 1, \quad y_1'' = 0.$$

Therefore,

$$x^2y_1'' + xy_1' - y_1 = 0 + x - x = 0.$$

So $y_1 = x$ is indeed a homogeneous solution.

I will use reduction of order to find a second homogeneous solution. Thus, I'm temporarily looking at

$$x^2y'' + xy' - y = 0.$$

I try to find a solution of the form $y = vy_1 = vx$. The derivatives are

$$y_1' = xv' + v, \quad y'' = xv'' + 2v'.$$

Plug these into the equation:

$$\begin{array}{rcccccccc} x^2y'' + xy' - y & = & x^3v'' & + & 2x^2v' & & & & \\ & & & + & x^2v' & + & xv & & \\ & & & & & - & xv & & \\ & = & x^3v'' & + & 3x^2v' & & & = & 0. \end{array}$$

Usually, you reduce order at this point, setting $u = v'$, so $u' = v''$. But notice that x^3-$3x^2$ combination? In fact, by the product rule, $x^3v'' + 3x^2v'$ is just the derivative of x^3v'! So the equation is

$$(x^3v')' = 0, \quad \text{and hence} \quad x^3v' = c_1.$$

Now

$$v' = \frac{c_1}{x^3}, \quad \text{so} \quad v = -\frac{c_1}{2x^2} + c_2.$$

Finally,

$$y = vx = -\frac{c_1}{2x} + c_2x.$$

Ignore the constants — the given solution was x (the second term), o the new solution is $\dfrac{1}{x}$ (the first term). That is, $y_2 = \dfrac{1}{x}$ is a second homogeneous solution.

Now return to the original nonhomogeneous equation, which was

$$x^2y'' + xy' - y = x^2(x+1).$$

(Aren't problems like these great? You're getting your money's worth, for sure! Heh.)

First, I divide through by x^2 to put the equation in standard form:

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = x + 1.$$

Variation of parameters tells me to expect a solution of the form

$$y = u_1 y_1 + u_2 y_2 .$$

I know $y_1 = x$ and $y_2 = \dfrac{1}{x}$, so the problem is to determine u_1 and u_2.
I set up the variation of parameters equation:

$$\begin{bmatrix} x & \dfrac{1}{x} \\ 1 & -\dfrac{1}{x^2} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x+1 \end{bmatrix}.$$

Invert the coefficient matrix and put the inverse on the right:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = -\frac{x}{2} \begin{bmatrix} -\dfrac{1}{x^2} & -\dfrac{1}{x} \\ -1 & x \end{bmatrix} \begin{bmatrix} 0 \\ x+1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2}(x+1) \\ -\dfrac{1}{2}x^2(x+1) \end{bmatrix}.$$

Solve for u_1 and u_2:

$$u_1' = \frac{1}{2}(x+1), \quad u_1 = \frac{1}{4}x^2 + \frac{1}{2}x + c_1,$$

$$u_2' = -\frac{1}{2}x^2(x+1), \quad u_2 = -\frac{1}{8}x^4 - \frac{1}{6}x^3 + c_2.$$

The solution is

$$y = x\left(\frac{1}{4}x^2 + \frac{1}{2}x + c_1\right) + \frac{1}{x}\left(-\frac{1}{8}x^4 - \frac{1}{6}x^3 + c_2\right) = \frac{1}{8}x^3 + \frac{1}{3}x^2 + c_1 x + \frac{c_2}{x}. \ \square$$

Example. Solve $(D^2+1)y = 2x\sin x$.

The homogeneous solution is

$$y_h = c_1\cos x + c_2 \sin x.$$

The particular solution y_p satisfies

$$(D^2+1)y_p = 2x\sin x.$$

If I do the equation as is, I would notice that $x\sin x$ is killed by $(D^2+1)^2$. Apply $(D^2+1)^2$ to both sides:

$$(D^2+1)^3 y_p = (D^2+1)^2(2x\sin x) = 0.$$

Hence,

$$y_p = A\cos x + B\sin x + Cx\cos x + Dx\sin x + Ex^2\cos x + Fx^2\sin x.$$

$\cos x$ and $\sin x$ are homogeneous solutions, so I throw them out. This leaves

$$y_p = Cx\cos x + Dx\sin x + Ex^2\cos x + Fx^2\sin x.$$

Note that I need the second derivative y_p'' to plug into the original equation, and each differentiation will generate two terms for each term, by the product rule. Wow! — that's pretty disgusting!

Instead, I will use complex exponentials. Go back and complexify the original y_p equation. To do this, just recall that $\sin x$ is the imaginary part of e^{ix}. So replace $\sin x$ with e^{ix}, yielding

$$(D^2+1)y_p = 2xe^{ix}.$$

To kill e^{ix}, I should use $D-i$. But the x in front tells me the root is doubled, so I need to apply $(D-i)^2$:

$$(D-i)^2(D^2+1)y_p = (D-i)^2(2xe^{ix}) = 0.$$

Now here's a slightly tricky point — what's y_p? You might be tempted to write down

$$y_p = Ae^{ix} + Bxe^{ix} + C\cos x + D\sin x. \quad \text{(BOGUS)}$$

Then you throw out the homogeneous solutions Ae^{ix}, $C\cos x$, $D\sin x$, leaving $y_p = Bxe^{ix}$. Try it — it won't work.

The problem is that

$$D^2 + 1 = (D-i)(D+i).$$

So the equation *really* is

$$(D-i)^3(D+i)y_p = 0.$$

This means that

$$y_p = Ae^{ix} + Bxe^{ix} + Cx^2e^{ix} + De^{-ix}.$$

Now Ae^{ix} and De^{-ix} are homogeneous solutions, so throwing them out leaves

$$y_p = Bxe^{ix} + Cx^2e^{ix} = (Bx + Cx^2)e^{ix}.$$

Differentiate:

$$y_p' = (B + 2Cx)e^{ix} + (Bix + Cix^2)e^{ix} = (B + 2Cx + Bix + Cix^2)e^{ix},$$

$$y_p'' = (2C + Bi + 2Cix)e^{ix} + (Bi + 2Cix - Bx - Cx^2)e^{ix} = (2C + 2Bi + 4Cix - Bx - Cx^2)e^{ix}.$$

(To be sure, the computations are a bit tedious, but I think they're better than they would be in the uncomplexified case. It hurts less this way, really!)

Plug in:

$$y_p'' + y_p = (2C + 2Bi + 4Cix)e^{ix} = 2xe^{ix}.$$

Cancelling the e^{ix}'s and equating corresponding coefficients, I get $C = -\frac{i}{2}$ and $B = \frac{1}{2}$. So

$$y_p = \frac{1}{2}xe^{ix} - \frac{i}{2}x^2e^{ix}.$$

Since the original equation had $\sin x$, and since $\sin x$ is the imaginary part of e^{ix}, I need the imaginary part of this complex y_p. Expand it:

$$y_p = \frac{1}{2}x(\cos x + i\sin x) - \frac{i}{2}x^2(\cos x + i\sin x) = \left(\frac{1}{2}x\cos x + \frac{1}{2}x^2\sin x\right) + i\left(\frac{1}{2}x\sin x - \frac{1}{2}x^2\cos x\right).$$

The imaginary part is

$$\frac{1}{2}x\sin x - \frac{1}{2}x^2\cos x,$$

and this is a particular solution to the original real equation.

The solution to the original equation is

$$y = c_1\cos x + c_2\sin x + \frac{1}{2}x\sin x - \frac{1}{2}x^2\cos x. \ \square$$

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