PERIODIC SOLUTIONS AT RESONANCES IN REVERSIBLE AND CONSERVATIVE SYSTEMA

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Abstract

It has often been observed that Hamiltonian and reversible systems show a similar behaviour, in particular when it comes to periodic solutions. In fact, in a number of cases one can extend this similarity in behaviour to the much larger class of *conservative* systems, i.e. systems which have a first integral. In this lecture we discuss a particular example of a problem for which the outcome is the same for reversible and for conservative systems. To fix the ideas: we will say that a system $\dot{x} = f(x)$ is reversible if it lives on an even-dimensional phase space (i.e. $x \in \mathbb{R}^{2n}$), and if there exists a linear operator $R \in \mathcal{L}(\mathbb{R}^{2n})$ such that $R^2 = I$, dim $Fix(R) = n$, and $f(Rx) = -Rf(x)$.

The classical Liapunov Center Theorem says that when the linearization at an equilibrium of a Hamiltonian system has a pair of non-resonant purely imaginary eigenvalues, then the system has a smooth two-dimensional invariant manifold, passing through the equilibrium, tangent to the eigenspace corresponding to the eigenvalues in question, and filled with periodic orbits encircling the equilibrium point. This result easily extends to conservative and to reversible systems; in the last case the equilibrium has to be symmetric (i.e. invariant under R), and the resulting invariant manifold and periodic orbits are then symmetric as well. It should be observed that because of the reversibility or the existence of a first integral one cannot move simple eigenvalues off the imaginary axis by perturbing the system (within the corresponding class, of course); the only way such eigenvalues can splitt off the imaginary axis is by first colliding with other such eigenvalues. This is precisely the situation we want to study: suppose that by changing parameters we move $k \geq 2$ pairs of simple purely imaginary eigenvalues together, and let them collide and then split off the imaginary axis. The question is then what happens to the k one-parameter families of periodic orbits which exist just before the collision, when the k pairs are still distinct but close to each other. Generically the scenario requires at least $k-1$ independent parameters, and the linearization at the collision will be non-semisimple — more precisely: at the critical parameter value the linearization will have a pair of purely imaginary eigenvalues with geometric multiplicity 1 and algebraic multiplicity k.

To study the existence of periodic orbits near such k-fold resonance we use a general reduction result which we proved in some earlier work (see [1]). This result tells us that the

periodic solutions we are looking for are in one-to-one relation with the periodic soltions of a *reduced system* which lives on the generalized eigenspace U corresponding to the pair of critical eigenvalues; this reduced system is again reversible or conservative (depending on the case), but also has an additional S^1-equivariance (the linearization induces a natural S^1-action on U). Moreover, the periodic solutions of the reduced system have a simple form which immediately leads to the algebraic determining equations for such periodic solutions. Finally, the reduced system can be determined up to any desired order by bringing the original vectorfield in an appropriate normal form. A similar reduction result can be obtained for the Hamiltonian case (see [2]). For our particular problem it is sufficient to bring the linear part of the equation in normal form; to perform this reduction we use ideas which go back to Arnold. For both the reversible and the conservative case the determining equations finally reduce to a polynomial scalar equation, whose solutions can be explicitly discussed.

The bifurcation sets in parameter space are just diffeomorphic to the well known cuspoïds from singularity theory: a fold for $k = 2$, a cusp for $k = 3$, a swallowtail for $k = 4$, etc. The resulting bifurcation pictures reveal some simple patterns for the appearance and disappearance of branches of periodic orbits; also, the results generalize the bifurcation scenarios found at a Krein collision in Hamiltonian systems.

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References

[1] J. Knobloch and A. Vanderbauwhede. A general reduction method for periodic solutions in conservative and reversible systems. *TU Ilmenau Preprint No M 11/94*, 1994.

[2] A. Vanderbauwhede and J.-C. van der Meer. A general reduction method for periodic solutions near equilibria in Hamiltonian systems. To appear in: *Normal Forms and Homoclinic Chaos*, Fields Institute Communications, A.M.S.