REGULARITY FOR VISCOSITY SOLUTIONS OF FULLY NONLINEAR EQUATIONS $F(D^2u) = 0$

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In this talk I will describe some recent work [1], [2] in collaboration with Luis Caffarelli on interior regularity for viscosity solutions of fully nonlinear elliptic equations of the form

$$F(D^2u) = 0. \tag{1}$$

It is well known that viscosity solutions of (1) are $C^{1,\alpha}$ for some $0 < \alpha < 1$, and in the case that the functional F is convex (or concave), they are $C^{2,\alpha}$. We give new and simpler proofs of these results. I will also point out some open problems for equation (1) when the functional F is not convex.

In (1), u is a real function defined in a bounded domain Ω of Euclidean space R^n and D^2u denotes the Hessian of u. F is a real valued function defined on the space of real $n \times n$ symmetric matrices. We assume that F is a uniformly elliptic operator, that is, for any symmetric matrix M and any non-negative definite symmetric matrix N,

$$\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\|,$$

where $\lambda \leq \Lambda$ are two positive constants, which are called ellipticity constants, and $\|N\|$ denotes the maximum eigenvalue of N.

We therefore include important examples of equations, such as Bellman's equations in optimal control theory and Isaacs' equations in differential games.

A continuous function u in Ω is said to be a viscosity subsolution (respectively, viscosity supersolution) of (1) in Ω when the following condition holds: for any $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x_0, we have

$$F(D^2\varphi(x_0)) \geq 0$$

(respectively, if $u - \varphi$ has a local minimum at x_0 then $F(D^2\varphi(x_0)) \leq 0$). We say that u is a viscosity solution of (1) when it is both viscosity subsolution and supersolution.

It is easy to verify that a $C^2(\Omega)$ function u is a viscosity solution of (1) if and only if it is a classical solution. The concept of viscosity solution is very useful when proving existence of solutions by Perron's method.

The following theorem states the $C^{1,\alpha}$ regularity of viscosity solutions of (1). Let u be a viscosity solution of $F(D^2u) = 0$ in the unit ball B_1. Then $u \in C^{1,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + |F(0)|\right),$$

where $0 < \alpha < 1$ and C are constants depending only on n, λ and Λ.

Further regularity can be obtained in the case of concave or convex functionals. We say that F is concave if it is a concave function on the space of symmetric matrices. Note that if F is concave then $G(M) = -F(-M)$ is convex and still uniformly elliptic. Bellman's equations are examples of concave functionals.

Evans and Krylov independently discovered that classical solutions u of concave equations (1) satisfy an interior $C^{2,\alpha}$ *a priori* estimate in terms of $\|u\|_{L^\infty}$. In fact, the following statement is the $C^{2,\alpha}$ regularity of viscosity solutions of concave equations (1). Let F be concave and u be a viscosity solution of $F(D^2u) = 0$ in B_1. Then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + |F(0)|\right), \tag{2}$$

where $0 < \alpha < 1$ and C are constants depending only on n, λ and Λ.

We simplify some steps in the proofs of the previous results. The main tools that we use are the Krylov-Safonov Harnack inequality for solutions of second order linear elliptic equations in nondivergence form with bounded measurable coefficients, and Jensen's approximate solutions of equation (1).

An important open problem is the following. It is not known if estimate (2) is true when F is neither concave nor convex. In this case it is not even known if viscosity solutions of (1) are $C^{1,1}$.

REFERENCES

[1] L.A. Caffarelli and X. Cabré, Fully Nonlinear Elliptic Equations. Colloquium Publications 43, Amer. Math. Soc., July 1995.

[2] X. Cabré and L.A. Caffarelli, Regularity for viscosity solutions of fully nonlinear equations $F(D^2u) = 0$. Submitted to Topol. Methods in Nonl. Analysis.