DIRICHLET PROBLEMS WITH ASYMMETRIC NONLINEARITIES

Salvador VILLEGAS and David ARCOYA

Departamento de Análisis Matemático, Universidad de Granada. [1]
18071-Granada, Spain.

1 Introduction

In this note we deal with the existence of solutions of Dirichlet problems like

$$\left.\begin{array}{rcl} -\Delta u &=& \lambda u^- + g(x,u), \;\; x \in \Omega \\ u &=& 0, \;\; x \in \partial\Omega \end{array}\right\} \qquad (1.1)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\lambda \in \mathbb{R}$, $u^- = \mathrm{Min}\ \{0, u\}$ and $g : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function.

Observe that the nonlinearity $f(x,u) = \lambda u^- + g(x,u)$ given by the right hand in the equation in (1.1) is asymmetric in the sense that its asymptotic behavior at $+\infty$ is distinct to this one at $-\infty$. We refer, for instance, to [2], [4], [5] and [6] for this kind of asymmetric problems.

If $\lambda_1 > 0$ denotes the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions, we have to remark that the existence of solutions of (1.1) can be obtained in an easy way provided that $\lambda < \lambda_1$. Indeed, in this case, it is sufficient to minimize the Euler functional associated to (1.1) (which is coercive). We are interested in proving existence of solutions of (1.1) for $\lambda > \lambda_1$. To this end we use bifurcation theory.

Notation. We let $K : C(\bar{\Omega}) \to X \equiv C^1(\bar{\Omega})$ denote the Green operator of $-\Delta$ with zero Dirichlet boundary conditions. With this notation, problem (1.1) is equivalent to

$$F(\lambda, u) \equiv u - K[\lambda u^- + g(u)] = 0\,, \;\; u \in X.$$

[1] This work was partially supported by DGICYT, Ministry of Education and Science (Spain), under grant number PB92-0941.

Hereafter we use the same symbol to denote both the function and the associated Nemytskii operator. Remind that K is (continuous and) compact, and hence it makes sense to consider the topological degree of $F(\lambda, u)$.

We say that λ_∞ is a bifurcation point from infinity for (1.1) if there exist $\{\lambda_n\} \to \lambda_\infty$ and $\{u_n\} \subset X$ such that $F(\lambda_n, u_n) = 0$ and $\|u_n\|_X \to +\infty$.

2 Existence of solutions

Our next result proves that the unique possible bifurcation point from infinity for (1.1) is λ_1.

LEMMA 1 *Assume that g is a continuous and bounded function in $\bar{\Omega} \times \mathbb{R}$ and let $\lambda \neq \lambda_1$. Then λ is not a bifurcation point from infinity for* (1.1).

Proof. Suppose that $\{\lambda_n\} \to \lambda_\infty$ and $\{u_n\} \subset X$ satisfy $F(\lambda_n, u_n) = 0$ and $\|u_n\|_X \to +\infty$. Then for $w_n = u_n / \|u_n\|_X$ we have

$$w_n = K\left(\lambda_n w_n^- + \frac{g(x, u_n)}{\|u_n\|_X}\right)$$

By the compactness of K we deduce that (up to a subsequence) $w_n \to w_0$ in X with $w_0 = K(\lambda^* w_0^-)$. Since $w_0 \neq 0$, the maximun principle gives us $\lambda_\infty = \lambda_1$ and the lemma is proved. □

REMARK 2 *With the notation of the previous proof we can deduce that $w_n \to w_0$ in X with $w_0 < 0$ in Ω and $\frac{\partial w_0}{\partial n} > 0$ on $\partial\Omega$. This implies that $u_n < 0$ in Ω for large n. Consequently, if there is a bifurcation from infinity for* (1.1) *then for λ near the bifurcation value, solutions of large norm are indeed negative.*

In order to prove that λ_1 is a bifurcation point from infinity, denote $\Sigma = \{(\lambda, u) \in \mathbb{R} \times X \,/\, F(\lambda, u) = 0\}$.

THEOREM 3 ([3]) *Assume that g is a continuous and bounded function in $\bar{\Omega} \times \mathbb{R}$ such that $g(x, 0) > 0$ for every $x \in \bar{\Omega}$. Then λ_1 is a bifurcation point from infinity for* (1.1). *More specifically, there exists a set $\Sigma_\infty \subset \Sigma$ that bifurcates from infinity and satisfies*

a) $\{(\lambda, z) \in \mathbb{R} \times X \,/\, z \neq 0$ *and* $(\lambda, z\|z\|^{-2}) \in \Sigma_\infty\} \cup \{\lambda_1, 0)\}$ *is connected.*

b) $Proj_\lambda \Sigma_\infty = \{\lambda \in \mathbb{R} \,/\, \exists u \in X \text{ with } (\lambda, u) \in \Sigma_\infty\}$ *is bounded below and unbounded above.*

Sketch of the proof. To show the existence of Σ_∞ bifurcating from infinity we follow the ideas in [1]. First, we observe that the problem of finding a bifurcation from infinity may be reduced to a problem of bifurcation from zero by using the Kelvin transformation $z = u\|u\|^{-2}$ $(u \neq 0)$. Indeed, if we define $\widetilde{F}(\lambda, z) = \|u\|^{-2} F(\lambda, u)$ for $u \neq 0$ and $\widetilde{F}(\lambda, 0) = 0$, then λ_1 is a bifurcation point from infinity for (1.1) iff it is a bifurcation from the trivial solution $z = 0$ for the equation $\widetilde{F}(\lambda, z) = 0$.

Now, if we show that the index (or local degree) of $\widetilde{F}(\lambda, \cdot)$ changes when λ crosses λ_1, we deduce by the global bifurcation theorem of Rabinowitz [7] that λ_1 is actually a bifurcation point from zero for $\widetilde{F}(\lambda, z) = 0$. In order to prove this change in the index, consider $\lambda < \lambda_1 < \bar{\lambda}$. By Lemma 1 there exists $r_0 > 0$ such that

$$\widetilde{F}(\lambda t, 0) \neq 0 \quad \forall t \in [0,1] \quad \forall 0 < \|z\|_X \leq r_0$$

and

$$(1-s)z + s\widetilde{F}(0, z) \neq 0 \quad \forall s \in [0,1] \quad \forall 0 < \|z\|_X \leq r_0$$

Hence, by the homotopy invariance property of the degree it follows that the index

$$\begin{aligned} i(\widetilde{F}(\lambda, \cdot), 0, 0) &= deg(\widetilde{F}(\lambda, \cdot), B_{r_0}(0), 0) = deg(\widetilde{F}(0, \cdot), B_{r_0}(0), 0) \\ &= deg(Id, B_{r_0}(0), 0) = 1 \end{aligned}$$

On the other hand, let Φ_1 be the eigenfunction associated to λ_1 such that $\Phi_1 > 0$ in Ω and $\|\Phi_1\|_X = 1$ and choose $R_0 > 0$ such that for every $t \geq 0$ the problem

$$\left.\begin{aligned} -\Delta u &= \bar{\lambda} u^- + g(x, u) - t\Phi_1, \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega \end{aligned}\right\} \tag{2.2}$$

has no solution $u \in X$ with $\|u\|_X \geq R_0^{-1}$. Then, we deduce again from homotopy that

$$i(\widetilde{F}(\bar{\lambda},\cdot),0,0) = deg(\widetilde{F}(\bar{\lambda},\cdot),B_{R_0}(0),0) = deg(\widetilde{F}(\bar{\lambda},\cdot)-\Phi_1,B_{R_0}(0),0) = 0$$

Therefore, by Rabinowitz theorem, there exists an unbounded and connected subset $\widetilde{\Sigma_0}$ in the closure of $\tilde{\Sigma} = \{(\lambda,z) \in \mathbb{R}\times X \,/\, z \neq 0,\ \widetilde{F}(\lambda,z) = 0\}$ such that $(\lambda_1,0) \in \widetilde{\Sigma_0}$. Since $g(.,0) \not\equiv 0$, $u = 0$ is not a solution of $F(\lambda,u) = 0$, or equivalently, $\widetilde{F}(\lambda,z) = 0$ does not possess bifurcation points from infinity. Thus $Proj_\lambda \widetilde{\Sigma_0}$ is unbounded. The set Σ_∞ we are looking for is $\Sigma_\infty = \{(\lambda,u) \in \mathbb{R}\times X \,/\, u \neq 0,\ (\lambda,z) = (\lambda, u\|u\|^{-2}) \in \widetilde{\Sigma_0}\}$. The proof is completed if we show that $Proj_\lambda \Sigma_\infty = Proj_\lambda\left(\widetilde{\Sigma_0} - \{(\lambda_1,0)\}\right)$ is bounded below. We argue by contradiction assuming that $Proj_\lambda\Sigma_\infty$ is unbounded below. Then there exists $(\lambda_0,u_0) \in \Sigma_\infty$ such that $g(x,s) + \lambda_0 s > 0\ \forall s \leq 0$, and by the maximun principle $u_0 > 0$. This implies that Σ_∞ "connects" the positive solution u_0 with negative solutions (see Remark 2) which clearly is a contradiction. □

Remarks:

i) More precise description of Σ_∞ may be given. For instance, if we define $\underline{a}(x) = \liminf_{u\to-\infty} g(x,u)$ and $\bar{a}(x) = \limsup_{u\to-\infty} g(x,u)$ for every $x \in \Omega$, then:

   a) If $\int_\Omega \underline{a}\Phi_1\,dx > 0$ then Σ_∞ bifurcates from the right of $\lambda = \lambda_1$.

   b) If $\int_\Omega \bar{a}\Phi_1\,dx < 0$ then Σ_∞ bifurcates from the left of $\lambda = \lambda_1$.

ii) As it has been observed in the previous proof, we can replace the hypothesis $g(x,0) > 0\ \forall x \in \bar{\Omega}$ by the more general assumption $g(.,0) \not\equiv 0$ in order to state the existence of Σ_∞ bifurcating from infinity with $Proj_\lambda\Sigma_\infty$ unbounded. But, note that the unboundness of $Proj_\lambda\Sigma_\infty$ is not sufficient to obtain the existence of "nontrivial solution" of (1.1), i.e. solutions of (1.1) for $\lambda > \lambda_1$.

COROLLARY 4 ([3]) *Under the hypotheses of Theorem* 3, *the problem* (1.1) *has at least one solution for every* $\lambda > \lambda_1$. □

References

[1] Ambrosetti, A. and Hess, P., *Positive Solutions of Asymptotically Linear Elliptic Eigenvalue Problems.* J. Math. Anal. Appl. **73** (1.980), 411-422.

[2] Arcoya, D. and Villegas, S., *Nontrivial solutions for a Neumann problem with a nonlinear term asymptotically linear at $-\infty$ and superlinear at* $+\infty$. Math. Z. **219** (1.995), 499-513.

[3] Arcoya, D. and Villegas, S., *Article in preparation* .

[4] De Figueiredo, D.G., *On superlinear elliptic problems with nonlinearities interacting only with higher eiganvalues.* Rocky Mountain J. Math. **18** (1.988), 287-303.

[5] De Figueiredo, D.G. and Ruf, B., *On a superlinear Sturm-Liouville equation and a related bouncing problem.* J. Reine Angew. Math. **421** (1.991), 1-22.

[6] Lazer, A.C. and McKenna P.J., *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis.* SIAM Review **32** (1990), 537-578.

[7] Rabinowitz, P.H., *On bifurcation from infinity.* J. Diff. Equ. **14** (1973), 462-475.