Propagation of Trigger Waves in a Disordered Medium

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ABSTRACT

The behaviour of trigger waves propagating in disordered media is analized by means of a Ballast resistor model. We find several differents behaviours depending on the relative size of the wave to the correlation lenght of the field. Analysis and simulations are compared by using Generalized Poisson Random Fields. We also investigate waves of transition from an unstable state, concluding that the *natural* velocity is also selected in the case of random fields.

1. Introduction

Disorder is an essential component of many physical systems. Their effects are well known in linear systems [1,2] and are under investigation in nonlinear systems [3]. Examples of physical systems in which disorder is relevant are porous media, amorphous semiconductors, glasses, granular media [1,2] etc.. Transport phenomena associated to these systems are influenced by disorder. If disorder is strong enough the behaviour of such systems becomes anomalous. Tipical effects produced by disorder are subdiffusion of excitations and particles, localization of waves and particles, backscattering of light, weak electrical conduction [1,2], etc. Some of these effects can be satisfactory explained by means of linear models whose evolution equations have inhomogeneous random coefficients (Random fields). The main effect of disorder in such linear systems consists of a change in the diffusivity of the corresponding excitation. Matematically it can be made evident by averaging the evolution equations obtaining, in more or less rigorous way, equations with renormalized diffusion coefficients [4].

When the disorder is very strong the diffusion laws change dramatically giving rise to new phases in which the diffusion lenght varies in time with a characteristic exponent determined by the intensity of disorder. In some cases it is possible to perform an overall analysis of these phases and to relate the exponent of the diffusion lenght with the degree of disorder [4].

In a nonlinear system the effect of disorder is not well known. The collective behavior, tipical of a nonlinear system, leads to the formation of structures that could interact with disorder in a more global manner .

The aim of this contribution is to illustrate these different behaviors by means of reaction-diffusion models with analytical solutions. We first analyze disorder effects in the propagation of a trigger wave in the Ballast resistor (Sects. 2, 3 and 4). In this model there are several lenghts and characteristic times. Under influence of disorder the trigger wave has different behaviors. It can be either like a particle in a positive velocity field or like a structure driven by a time depending noise.

To conclude, we briefly discuss some properties of waves that perform transitions from unstable stationary states (Sect. 5).

2. The Model of Disordered Ballast Resistor

The Ballast resistor is a conducting wire inmersed in a bath with constant temperature T_B [5]. If $T(x,t)$ is the temperature of the wire at site x and time t, the evolution equation of this function will be given by:

$$c\partial_t T(x,t) = \partial_x D(x)\partial_x T(x,t) - q(T(x,t) - T_B) + I^2 R(T) \tag{1}$$

where the first term of the right hand side corresponds to heat diffusion, $D(x)$ being the heat conductivity, the second one represents the heat transfer to the bath (q is the heat transfer coefficient) and the last is the electrical power density. I is the electrical intensity, c is the specific heat and $R(T)$ is the resistence as function of the temperature.

We focus on the case of a superconducting wire. Then $R(T) = R_0\theta(T - T_c)$, being $\theta(x)$ the step function, T_c the critical temperature and R_0 a constant resistence. In an ordered case, that is, with $D(x) = D$ (constant), we can write Eq. (1) in the form:

$$\partial_t T(x,t) = D\partial^2_{xx} T(x,t) + f(T) \tag{2}$$

with

$$f(T) = -T + (T_3 - T_1)\theta(T - T_2) + T_1 \quad , \tag{3}$$

being T_1, T_2 and T_3 functions of c, q, T_B, T_c, R_0 and I.

This is a special case of a bistable medium, that is, a system with two steady states which are stable under sufficiently small perturbations. Stronger perturbations can cause transitions between these states. The fundamental form of a pattern in bistable media is a trigger wave, which represents a propagation front of transition from one stationary state into the other [6].

The piecewise linear form of $f(T)$, Eq. (3), allows an exact time assymptotic solution. The final result is a trigger wave of width $L_F = (\frac{D}{q})^{\frac{1}{2}}$ propagating with a velocity given by $v = \sqrt{\frac{qD}{c^2}}$ [7]. We take as widht of the wave the lenght in which the slope changes appreciably.

The effects of disorder are modeled taking $D(x)$ as a Generalized Poissonian Random Field generated with exponential pulses $e^{-x/l}$ (l is the correlation lenght of the noise) with random heights exponentially distributed with mean value w_0 [8]. The position of the pulses has a Poissonian distribution with parameter λ. Moments, cumulants and probabilities can be analytically calculated.

3. Adiabatic propagation.

We define an adiabatic propagation as a case in which the wave formation is faster than the wave propagation. This condition can be analytically expressed by $\frac{L_F^2}{D} \ll \frac{L_F}{v}$, where the first term is a estimation of the formation time of the wave as the time in which the diffusion is able to connect points over a lenght L_F. The term on the right is obviously the propagation time of the wave through the same distance.

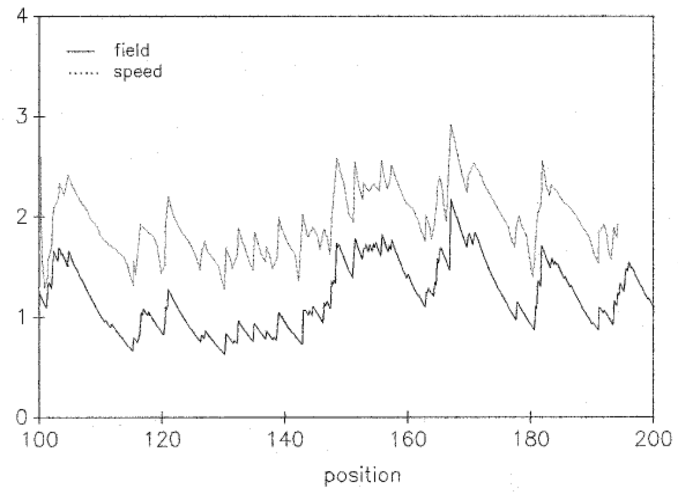


Fig.1

If moreover the correlation lenght of the random field is longer than L_F, the wave should propagate adiabatically following the variation of the field. We have corroborated this fact performing simulations with a Generalized Poisson Random Field. In Fig. 1 we show the random field (solid line) and the velocity of the wave in one of these cases of adiabatic propagation. It can be seen that the wave follows the variations of the field.

If we consider the wave in a moving frame with velocity v, and define $T_s(x,t) = T(x+vt)$, $T_s(x,t)$ is then governed by Eq.:

$$\partial_t T_s(x,t) = \partial^2_{xx} T_s + \partial_x \xi_{x+vt} \partial_x T_s + v \partial_x T_s - T_s + T_h \theta(T_s - T_c) \tag{4}$$

with $\partial_t T_s(x,t) = 0$, and the velocity and width can be exactly calculated, obtaining:

$$v(x) = (z - \frac{1}{z})\sqrt{(1+\xi_x)} \tag{5}$$

and

$$L_F(x) = z\sqrt{(1+\xi_x)}$$

with $z = (\frac{T_h}{T_c} - 1)^{\frac{1}{2}}$, $T_h = \frac{I^2 R_0}{q} - T_B$ and $\xi_x = \frac{D(x) - <D(x)>}{<D(x)>}$ (transformed random field with zero mean value).

The center of the wave moves like a directed random walk with velocity $v(x)$. Finally, the conditions for this case can now be more precisely formulated as $\frac{T_h}{T_c} \ll \frac{3}{2}$ and $l_c \gg z\langle\sqrt{1+\xi}\rangle$. This is an interesting case because it is possible to see analytically the effect of strong localization on the propagation of the wave. The wave behaves like a particle in a positive random velocity field.

In Fig. 2 we show the mean velocities versus λl in an adiabatic case, obtained solving numerically the Ballast model with a Generalized Poissonian Random Field. The analytical prediction (solid line) in this case of large correlation lenght and $\lambda\omega_0 = 1$, is given by:

$$< v > = \left(z - \frac{1}{z}\right)\left\langle(1+\xi_x)^{\frac{1}{2}}\right\rangle = \left(z - \frac{1}{z}\right)\left(\frac{\omega_0}{l}\right)^{1/2}\frac{\Gamma(\lambda l + 1/2)}{\Gamma(\lambda l)}$$

In the same figure we plot the dispersion of the velocity of the wave (down), in comparison with the analytical solution $(< v^2 > = (z - \frac{1}{z})^2)$.

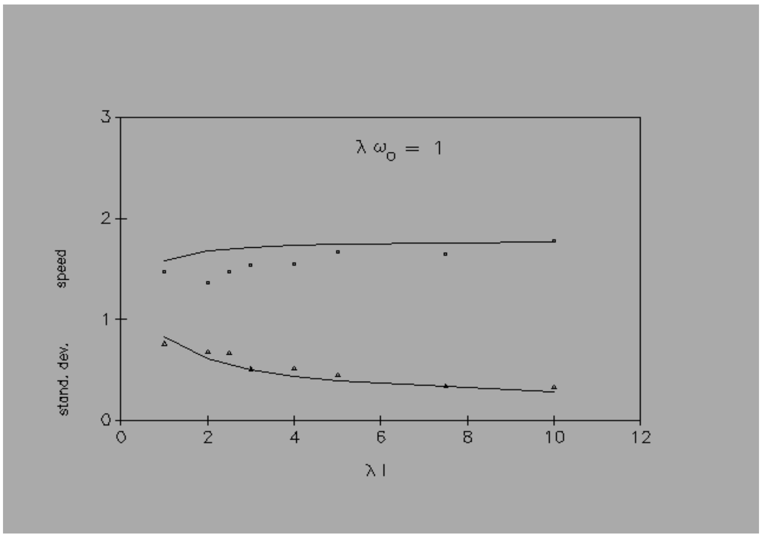
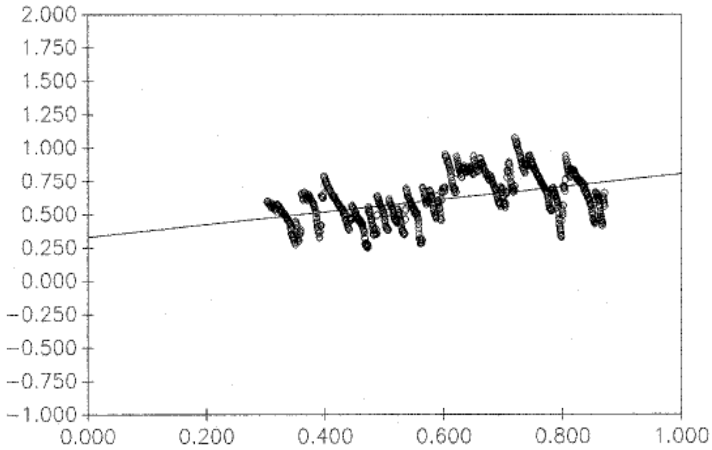


Fig.2

Finally, by plotting the mean velocity versus the random diffusion coeficient $D(x)$, we check than $v(x) \approx D(x)^{\frac{1}{2}}$, as appear in Fig. 3, where we have obtained $lnv(x) = 0.472 lnD(x) + 0.33$.



2.000
1.750
1.500
1.250
1.000
0.750
0.500
0.250
0.000
−0.250
−0.500
−0.750
−1.000
0.000
0.200
0.400
0.600
0.800
1.000

Fig.3

4. Fast propagation.

In a non adiabatic case the process of formation and propagation of the pulse are simultaneous. Then all terms in the equation (4) are important and the random field is now time-depending. Here the correlation lenght also plays an important role because if it is larger than L_F the noise can be considered as global (only-time depending). In a general case, it is very difficult to obtain analytical predictions. Here we restrict ourselves to the case of weak intensity of the noise. We average Eq. (4) with $\langle\theta(T_s - T_c)\rangle \sim \theta(\overline{T_s} - T_c)$ in order to get a stationary equation for the conditioned wave, obtaining :

$$\partial_{xx}\overline{T_s} + \partial_x F(x) + v\partial_x\overline{T_s} - \overline{T_s} + T_h\theta(\overline{T_s} - T_c) = 0 \tag{6}$$

where $F(x)$ can be calculated at the first order in the noise intensity, $\lambda\omega_0^2$, by means of the Novikov theorem as a function of T_s and the random field. The resulting equation suggests that for short correlation lenghts (white noise limit) the relevant parameter is the intensity of the noise.

This fact can be corroborated with numerical simulation, as can be seen in Figs. 4 a), b) and c) where we see that the mean velocity does not depend on λl.

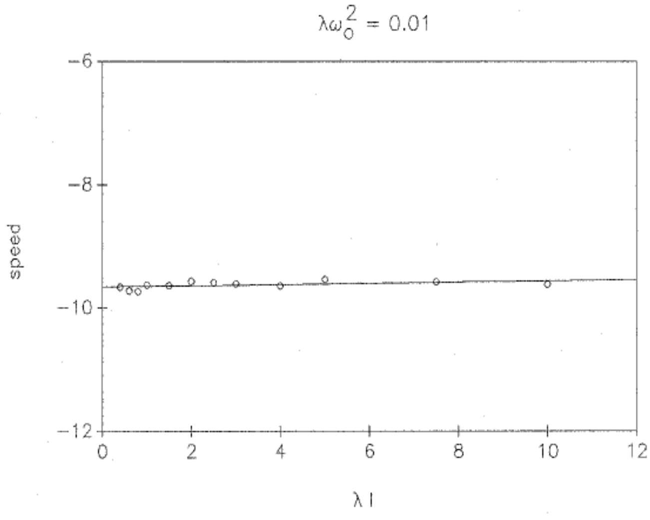


Fig.4 a)

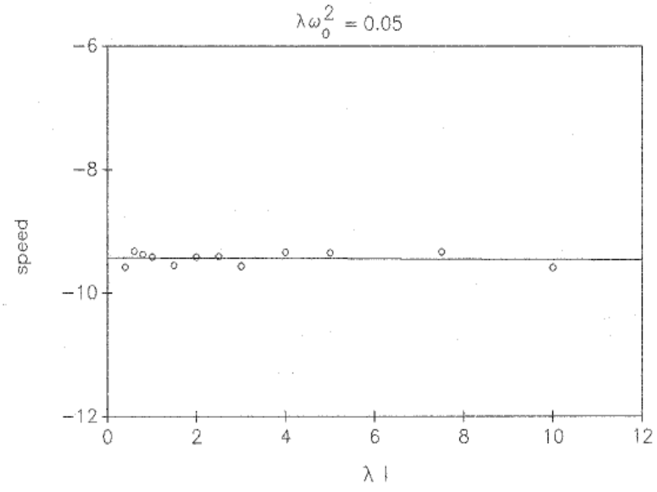


Fig.4 b)

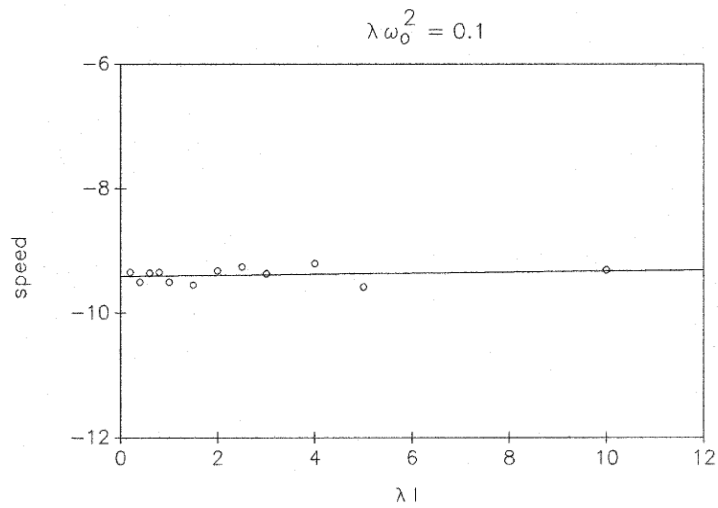


Fig.4 c)

Finally, in Fig. 5 it is seen the linear dependence of velocity with the intensity of noise, $\lambda\omega_0^2$

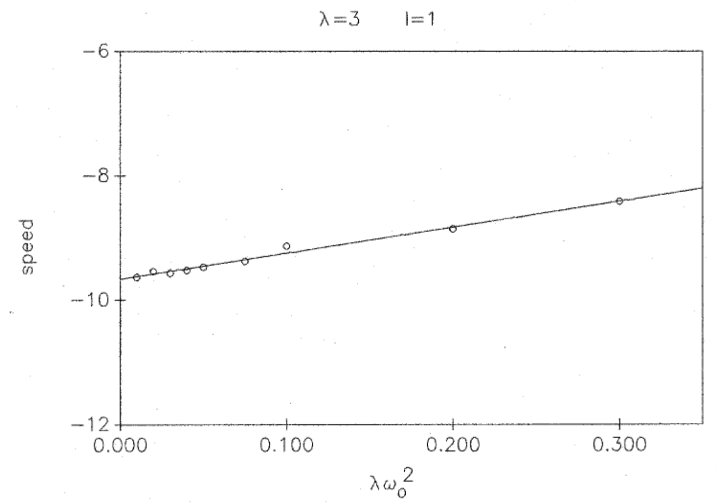


Fig.5

5. Waves of Transition from an Unstable State.

It is also interesting to study properties of waves that perform transitions from unstable stationary states. This case can be modeled by Eq. (2), where the nonlinear function $f(T)$ has two roots, T_1 and T_2, corresponding to an unstable and a stable stationary states, respectively. The mathematical theory of waves that perform a transition from an unstable stationary state was constructed in 1937 by Kolmogorov, Petrovskii and Piskunov [9].

In contrast to the case of trigger waves in bistable media, boundary conditions do not specify a unique solution. There is a family of solutions that correspond to transition waves moving at different velocities. All possible propagation velocities are bounded from below. The slowest transition wave propagating with a minimal velocity corresponds to *natural* propagation [6].

We focus in the special case of models with piecewise-linear approximation of $f(T)$, that is:

$$f(T) = \begin{cases} T - T_1, & \text{if } T < T_2 \\ T_3 - T, & \text{if } T > T_2 (> T_1) \end{cases}$$

In this case, we have obtained an equation for the velocity depending on T_1, T_2, T_3 and an aditional parameter c (non-uniqueness):

$$\lambda_1(T_2 - T_3) = cv^2\sqrt{1 - 4/v^2} + \lambda_2(T_2 - T_1)$$

where:

$$\lambda_{1,2} = \frac{v^2}{2}\left\{1 - \sqrt{1 \pm 4/v^2}\right\}$$

For certain values of T_1, T_2, T_3 it is possible to calculate the family of values for the velocity, and its minimun value. Numerical simulations of Eq. (2) corroborate that this slowest velocity corresponds to *natural* propagation.

As in the previous case, we have consider disorder in the diffusion coefficient by using Generalized Poissonian Random Fields. Here it is interesting to investigate the adiabatic case since the velocity is not well determined. We have performed simulations (Fig. 6) concluding that the natural velocity is always selected. Comparing Figs. 6 and 1 one can see how in this last case the field is not instantaneously followed by the velocity. This can be an effect of the selection of velocities. More detailed investigations are in course.

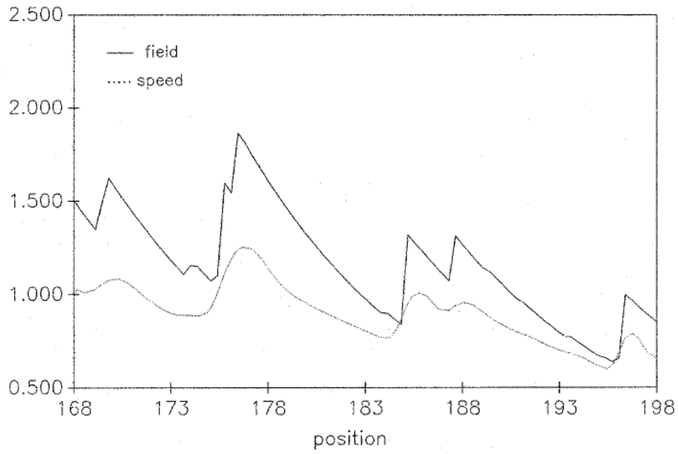


Fig.6

Acknowledgments

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References.

1. P.A. Lee and T.V. Ramakrishan, Rev. Mod. Phys. 57, 287 (1985). S. Alexander, J. Bernasconi, W.R. Schneider, R. Orbach, Rev. mod.Phys. 53, 175 (1981).

2. Chance and Matter, ed. Souletie, J. Vanimeus and R. Stora (Elsevier, Amsterdan,1987).

3. W. Zimmermann, M. Seesselberg, F. Petruccione P.R.E. 48, 2699 (1993).

4. J.W. Haus and K.W. Kher, Phys. Rep. 150,263 (1987). J.P. Bouchaud, A. Georges, Phys. Reports 4-5,127,293 (1990)

5. D. Bedeaux and P. Mazur. Physica A180, 295 (1992).

6. Foundations of Synergetics I. Distributed Active Systems, A.S. Mikhailov, Ed. Springer-Verlag (1990).

7. C. Schat and H.S. Wio, Physica A180, 295 (1992).

8. E. Hernández-García, L. Pesquera, M.A. Rodriguez and M. San Miguel, Phys. Rev. A 36, 5774 (1987).

9. A.N. Kolmogorov, I.G. Petrovskii, N.S. Piskunov: Bull. Mosk. Gos. Univ., Sekt. Matematika i Mekhanika 1, 1-26 (1937).