Existence of solutions to some semilinear elliptic equations from Dynamic Programming

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1 Introduction.

Involved to each $x \in \mathbb{R}^{\mathrm{N}}$, $\mathrm{N} \geq 1$, we consider the stochastic process

$$\mathcal{X}_t^x = x + \sqrt{2}\mathcal{B}_t$$

where $\{\mathcal{B}_t\}_{t\geq 0}$ is a standard Brownian motion. In order to simplify we omit the technical details. So that, what we are considered is the dynamical evolution from a *deterministic state* to a *stochastic state* given by a Gaussian variable verifying

$$\mathbb{E}[\mathcal{X}_t^x] = x \quad \text{and} \quad \mathcal{V}ar[\mathcal{X}_t^x] = 2t$$

for every $t \geq 0$. (As usually, by $\mathbb{E}$ we denote the expectation of a random variable). In this paper, states are restricted to remain in an open set $\Omega \subset \mathbb{R}^{\mathrm{N}}$, then one defines the *random variable*

$$\tau_x = \sup\{t \geq 0 : \ \mathcal{X}_t^x \in \Omega\}. \qquad (\textit{the first exit time from } \Omega).$$

Remark 1

Ito's Rule enables us to obtain the estimates

$$\mathbb{E}[\tau_x] \leq \frac{\mathrm{R}^2 - |x|^2}{2\mathrm{N}} \quad \text{and} \quad \mathbb{E}[\tau_x^2] \leq \frac{\mathrm{N}}{\mathrm{N}+2}\left[\frac{\mathrm{R}^2 - |x|^2}{2\mathrm{N}}\right]^2 + \frac{2\mathrm{R}^2}{\mathrm{N}(\mathrm{N}+2)}\mathbb{E}[\tau_x],$$

provided $\Omega \subset \mathbf{B}_{\mathrm{R}}(0)$. Equalities hold if and only if $\Omega = \mathbf{B}_{\mathrm{R}}(0)$, for which

$$\mathbb{E}[\tau_x^2] = \frac{\mathrm{N}}{\mathrm{N}+2}\left(\mathbb{E}[\tau_x]\right)^2 + \frac{2\mathrm{R}^2}{\mathrm{N}(\mathrm{N}+2)}\mathbb{E}[\tau_x]. \quad \square$$

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We are interested in to study *value functions* involved to *continuous controls*. More precisely, for each $v \in \mathcal{C}(\overline{\Omega})$ we define the *cost* function relative to any *initial state* $x \in \overline{\Omega}$

$$\left.\begin{array}{rcl} \mathcal{J}_v(x) & = & \overbrace{\mathbb{E}\left[\int_0^{\tau_x} f(\mathcal{X}_s^x)\mathrm{exp}\left(-\lambda\int_0^s |v(\mathcal{X}_r^x)|^{\mathrm{m}-1}\mathrm{d}r\right)\mathrm{d}s.\right.}^{\text{interior payment}} \\ & + & \underbrace{\left.\varphi(\mathcal{X}_{\tau_x}^x)\mathrm{exp}\left(-\lambda\int_0^{\tau_x} |v(\mathcal{X}_s^x)|^{\mathrm{m}-1}\mathrm{d}s\right)\right]}_{\text{boundary payment}} \end{array}\right\} \tag{1}$$

here λ and m are two positive constants and $f : \overline{\Omega} \to \mathbb{R}$, $\varphi : \partial\Omega \to \mathbb{R}$ are continuous functions on their arguments, then in [Di-So] one proves that $\mathcal{J}_v \in \mathcal{C}(\overline{\Omega})$. Thus, we are interested in *to control the discount term*

$$\mathrm{exp}\left(-\lambda\int_0^s |v(\mathcal{X}_r^x)|^{\mathrm{m}-1}\mathrm{d}r\right)\mathrm{d}s$$

over the (stochastic) *interior payment per unit time* $f(\mathcal{X}_s^x)$. So that, Ito's Rule proves that

$$\mathcal{S}_t^v\phi(x) = \mathbb{E}\left[\phi(\mathcal{X}_t^x)\mathrm{exp}\left(-\lambda\int_0^t |v(\mathcal{X}_s^x)|^{\mathrm{m}-1}\mathrm{d}s\right)\right]$$

defines a linear semigroup whose *infinitesimal generator* is

$$\mathcal{L}^v\phi(x) = \Delta\phi(x) - \lambda|v(x)|^{\mathrm{m}-1}\phi(x).$$

Then, (1) solves, in viscosity sense (see [Cr-Is-Li]), the boundary value problem

$$(\mathcal{P}_v) \qquad \left\{\begin{array}{rcll} -\Delta w(x) + \lambda|v(x)|^{\mathrm{m}-1}w(x) & = & f(x), & x \in \Omega, \\ w(x) & = & \varphi(x), & x \in \partial\Omega. \end{array}\right.$$

Indeed, first of all Ito'Rule leads to

$$\mathrm{d}\left[\mathcal{J}_v(\mathcal{X}_t^x)\mathrm{exp}\left(-\int_0^t |v(\mathcal{X}_s^x)|^{\mathrm{m}-1}\mathrm{d}s\right)\right]$$

$$= \mathrm{exp}\left(-\int_0^t |v(\mathcal{X}_s^x)|^{\mathrm{m}-1}\mathrm{d}s\right)\left[-\lambda|v(\mathcal{X}_t^x)|^{\mathrm{m}-1}\mathcal{J}_v(\mathcal{X}_t^x)\mathrm{d}t + \nabla_x\mathcal{J}_v(\mathcal{X}_t^x)\cdot\mathrm{d}\mathcal{B}_t + \Delta_x\mathcal{J}_v(\mathcal{X}_t^x)\mathrm{d}t\right].$$

Since $\mathbb{E}[\mathrm{d}\mathcal{B}_t] = 0$, taking expectation, integrating from 0 to t, multiplying by t^{-1} and letting $t \to 0$, definition of infinitesimal generator proves

$$\mathcal{L}^v\mathcal{J}_v(x) = \Delta\mathcal{J}_v(x) - \lambda|v(x)|^{\mathrm{m}-1}\mathcal{J}_v(x).$$

On the other hand, by specific arguments of the Dynamic Programming

$$\mathcal{J}_v(x) = \mathbb{E}\left[\int_0^t f(\mathcal{X}_s^x)\mathrm{exp}\left(-\lambda\int_0^s |v(\mathcal{X}_r^x)|^{\mathrm{m}-1}\mathrm{d}r\right)\mathrm{d}s + \mathcal{J}_v(\mathcal{X}_t^x)\mathrm{exp}\left(-\lambda\int_0^t |v(\mathcal{X}_s^x)|^{\mathrm{m}-1}\mathrm{d}s\right)\right]$$

for $t \leq \tau_x$, then

$$-\frac{1}{t}\mathbb{E}\left[\int_0^t f(\mathcal{X}_s^x)\exp\left(-\lambda\int_0^s |v(\mathcal{X}_r^x)|^{\mathrm{m}-1}\mathrm{d}r\right)\mathrm{d}s\right] \to -f(x) \quad \text{as } t \to 0$$

concludes

$$-\Delta\mathcal{J}_v(x) + \lambda|v(x)|^{\mathrm{m}-1}\mathcal{J}_v(x) = f(x), \quad x \in \Omega,$$

in the viscosity sense, whenever $\partial\Omega$ only admits *regular points*.

The main goal of this paper is to extend the above lines in order to study the *semilinear boundary value problem*

$$(\mathcal{P}) \qquad \begin{cases} -\Delta u(x) + \lambda|u(x)|^{\mathrm{m}-1}u(x) &= f(x), \qquad x \in \Omega, \\ u(x) &= \varphi(x), \qquad x \in \partial\Omega. \end{cases}$$

Denoting

$$\mathcal{T}v = \mathcal{J}_v$$

it is clear that any eventual fixed point $\mathcal{T}u = u$ solves the semilinear boundary value problem. From uniqueness results we will obtain then that the unique solution of $(\mathcal{P})$ admits the *implicit representation*

$$\left.\begin{aligned} u(x) \;&=\; \mathbb{E}\left[\int_0^{\tau_x} f(\mathcal{X}_s^x)\exp\left(-\lambda\int_0^s |u(\mathcal{X}_r^x)|^{\mathrm{m}-1}\mathrm{d}r\right)\mathrm{d}s\right. \\ &\quad+\; \left.\varphi(\mathcal{X}_{\tau_x}^x)\exp\left(-\lambda\int_0^{\tau_x}|u(\mathcal{X}_s^x)|^{\mathrm{m}-1}\mathrm{d}s\right)\right]. \end{aligned}\right\} \tag{2}$$

2 A Fixed Point Problem.

Since $\mathcal{J}_v$ verifies $\mathcal{J}_v \in \mathcal{C}(\overline{\Omega})$, then

$$v \mapsto \mathcal{T}v = \mathcal{J}_v$$

defines a map

$$\mathcal{T} : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega}).$$

In order to prove that $\mathcal{T}$ admits a fixed point, whenever $\mathcal{C}(\overline{\Omega})$ is equipped with the supremum norm, we consider some steps:

1. **Uniform boundednes of $\mathcal{J}_v$.**

From

$$\begin{aligned} \mathcal{J}_v(x) \;&=\; \mathbb{E}\left[\int_0^{\tau_x} f(\mathcal{X}_s^x)\exp\left(-\lambda\int_0^s |v(\mathcal{X}_r^x)|^{\mathrm{m}-1}\mathrm{d}r\right)\mathrm{d}s\right. \\ &\quad+\; \left.\varphi(\mathcal{X}_{\tau_x}^x)\exp\left(-\lambda\int_0^{\tau_x}|v(\mathcal{X}_s^x)|^{\mathrm{m}-1}\mathrm{d}s\right)\right]. \end{aligned}$$

and Remark 1, it follows

$$\begin{aligned} \|\mathcal{J}_v\|_{\mathrm{L}^\infty(\Omega)} \;&\leq\; \|f\|_{\mathrm{L}^\infty(\Omega)}\mathbb{E}[\tau_x] + \|\varphi\|_{\mathrm{L}^\infty(\Omega)} \\ &\leq\; \|f\|_{\mathrm{L}^\infty(\Omega)}\frac{\mathrm{R}^2}{2\mathrm{N}} + \|\varphi\|_{\mathrm{L}^\infty(\partial\Omega)} \doteq \mathcal{R}. \end{aligned}$$

So that, we have proved that assumptions imply in the space $\mathcal{C}(\overline{\Omega})$

$$\mathcal{T}(\mathcal{C}(\overline{\Omega})) \subseteq \overline{\mathbf{B}}_{\mathcal{R}}(0).$$

2. **Uniform continuity of $\mathcal{T}$.**

Straightforward computations show, for any $v,\ \hat{v} \in \mathcal{C}(\overline{\Omega})$ and $x \in \overline{\Omega}$,

$$\begin{aligned}
\mathcal{J}_v(x) - \mathcal{J}_{\hat{v}}(x) &\leq \lambda \|f\|_{\mathrm{L}^\infty(\Omega)} \mathbb{E}\left[\int_0^{\tau_x}\int_0^t \left||v(\mathcal{X}_s^x)|^{\mathrm{m}-1} - |\hat{v}(\mathcal{X}_s^x)|^{\mathrm{m}-1}\right| \mathrm{d}s\mathrm{d}t\right] \\
&\quad + \lambda \|\varphi\|_{\mathrm{L}^\infty(\partial\Omega)} \mathbb{E}\left[\int_0^{\tau_x} \left||v(\mathcal{X}_s^x)|^{\mathrm{m}-1} - |\hat{v}(\mathcal{X}_s^x)|^{\mathrm{m}-1}\right| \mathrm{d}s\right] \\
&\leq \lambda \left(\|f\|_{\mathrm{L}^\infty(\Omega)} \frac{\mathbb{E}\left[\tau_x^2\right]}{2} + \|\varphi\|_{\mathrm{L}^\infty(\partial\Omega)} \mathbb{E}\left[\tau_x\right]\right) \left\||v|^{\mathrm{m}-1} - |\hat{v}|^{\mathrm{m}-1}\right\|_{\mathrm{L}^\infty(\Omega)} \\
&\underbrace{\leq}_{\mathrm{m}\geq 2} \lambda \left(\|f\|_{\mathrm{L}^\infty(\Omega)} \frac{\mathbb{E}\left[\tau_x^2\right]}{2} + \|\varphi\|_{\mathrm{L}^\infty(\partial\Omega)} \mathbb{E}\left[\tau_x\right]\right) (\mathrm{m}-1)\, \mathcal{R}^{\mathrm{m}-2} \|v - \hat{v}\|_{\mathrm{L}^\infty(\Omega)} \\
&\leq \lambda \left(\|f\|_{\mathrm{L}^\infty(\Omega)} \frac{\mathrm{R}^4}{4\mathrm{N}^2} + \|\varphi\|_{\mathrm{L}^\infty(\partial\Omega)} \frac{\mathrm{R}^2}{2\mathrm{N}}\right) (\mathrm{m}-1) \mathcal{R}^{\mathrm{m}-2} \|v - \hat{v}\|_{\mathrm{L}^\infty(\Omega)}.
\end{aligned}$$

So that, $\mathcal{T}$ is a Lipschitz continuous map.

3. **Existence and uniqueness to** $(\mathcal{P})$**.**

Since

$$\mathcal{T}\left(\overline{\mathbf{B}}_{\mathcal{R}}(0)\right) \subseteq \overline{\mathbf{B}}_{\mathcal{R}}(0),$$

the continuity of $\mathcal{T}$ implies, by Fixed Point Brower Theorem, the existence of a $u \in \overline{\mathbf{B}}_{\mathcal{R}}(0)$ such that

$$\mathcal{T}u = u$$

which consecuently solves $(\mathcal{P})$. Finally, the monotonicity of the term

$$r \mapsto \lambda |r|^{\mathrm{m}-1} r$$

enables to proves that $(\mathcal{P})$ only admits a unique solutions that, *a fortiori*, is given by (2). So that, our main contribution is

Theorem 1

Under assumptions $f \in \mathcal{C}(\overline{\Omega})$, $\varphi \in \mathcal{C}(\partial\Omega)$ *and* Ω *bounded open set of* $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N} \geq 1$, *whose boundary points are exclusively regular points, problem* $(\mathcal{P})$ *has a unique viscosity solution* $u \in \mathcal{C}(\overline{\Omega})$*,verifying* (2)*, provided* $\mathrm{m} \geq 2$.

Moreover, if $f \in \mathcal{C}^{0,\alpha}(\Omega)$, $\varphi \in \mathcal{C}(\partial\Omega)$ *and* $\partial\Omega$ *satisfying the exterior sphere condition, one has* $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^{2,\alpha}(\Omega)$*. Consecuently,* u *is the unique classical solution of problem* $(\mathcal{P})$. □

Remark 2

¿From the regularity

$$\mathcal{J}_v \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^{2,\alpha}(\Omega),$$

provided $f \in \mathcal{C}^{0,\alpha}(\Omega)$, $\varphi \in \mathcal{C}(\partial\Omega)$ and $\partial\Omega$ satisfying the exterior sphere condition, it follows that

$$u = \mathcal{J}_u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^{2,\alpha}(\Omega).$$

Then u solves in the classical sense $(\mathcal{P})$. $\square$

3 Some extensions.

Above lines can be extended to a more general situations. So, process $\mathcal{X}_t^x$ can be replaced by the solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}\mathcal{X}_t^x = b(\mathcal{X}_t^x, \alpha_t)\mathrm{d}t + \sigma(\mathcal{X}_t^x, \alpha_t)\mathrm{d}\mathcal{B}_t & t \geq 0, \\ \mathcal{X}_0^x = x \end{cases}$$

under suitable properties on b and σ. Here $\alpha_{(\cdot)}$ are measurable controls functions taking values on a convex set $\mathcal{A}$. For each $v \in \mathcal{C}(\overline{\Omega})$ we define

$$\begin{aligned} \mathcal{J}_{v,\alpha_{(\cdot)}}(x) &= \mathbb{E}\left[\int_0^{\tau_x} f(\mathcal{X}_s^x, \alpha_t)\exp\left(-\lambda \int_0^s |v(\mathcal{X}_r^x)|^{\mathrm{m}-1}\mathrm{d}r\right)\mathrm{d}s\right. \\ &+ \left.\varphi(\mathcal{X}_{\tau_x}^x)\exp\left(-\lambda \int_0^{\tau_x} |v(\mathcal{X}_s^x)|^{\mathrm{m}-1}\mathrm{d}s\right)\right]. \end{aligned}$$

Now we are interested in to consider the *optimum value* function

$$\mathcal{J}_v(x) = \inf_{\alpha_{(\cdot)}} \mathcal{J}_{v,\alpha_{(\cdot)}}(x), \quad x \in \overline{\Omega}.$$

The arguments of the Dynamic Programming show

$$\begin{aligned} \mathcal{J}_v(x) &= \inf_{\alpha_{(\cdot)}} \left\{\mathbb{E}\left[\int_0^t f(\mathcal{X}_s^x, \alpha_t)\exp\left(-\lambda \int_0^s \lambda |v(\mathcal{X}_r^x)|^{\mathrm{m}-1}\mathrm{d}r\right)\mathrm{d}s\right.\right. \\ &+ \left.\left.\mathcal{J}_v(\mathcal{X}_t^x)\exp\left(-\lambda \int_0^t |v(\mathcal{X}_s^x)|^{\mathrm{m}-1}\mathrm{d}s\right)\right]\right\}. \end{aligned}$$

for $t \leq \tau_x$. It is a basic tool in order to prove that $\mathcal{J}_v$ solves, in the viscosity sense, the Hamilton-Jacobi-Bellman equation

$$\sup_{\alpha\in\mathcal{A}} \left\{ \sum_{i,j=1}^{\mathrm{N}} a_{ij}(x,\alpha)\frac{\partial^2 w(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{\mathrm{N}} b_i(x,\alpha)\frac{\partial w(x)}{\partial x_i} - f(x,\alpha)\right\} + \lambda|v(x)|^{\mathrm{m}-1}w(x) = 0,$$

for $x \in \Omega$, where

$$a_{ij}(x,\alpha) = \sum_{l=1}^{\mathrm{N}} \sigma_{il}(x,\alpha)\sigma_{lj}(x,\alpha).$$

Clearly, one satisfies

$$\mathcal{J}_v(x) = \varphi(x), \quad x \in \partial\Omega.$$

Then the fixed point argument enables us to prove that u is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation

$$\sup_{\alpha\in\mathcal{A}}\left\{\sum_{i,j=1}^{\mathrm{N}} a_{ij}(x,\alpha)\frac{\partial^2 u(x)}{\partial x_i\partial x_j} + \sum_{i=1}^{\mathrm{N}} b_i(x,\alpha)\frac{\partial u(x)}{\partial x_i} - f(x,\alpha)\right\} + \lambda|u(x)|^{\mathrm{m}-1}u(x) = 0,$$

with

$$u(x) = \varphi(x), \quad x \in \partial\Omega,$$

is given implicitly by

$$\begin{aligned} u(x) &= \inf_{\alpha(\cdot)} \mathbb{E}\left[\int_0^{\tau_x} f(\mathcal{X}_s^x,\alpha_t)\exp\left(-\lambda\int_0^s |u(\mathcal{X}_r^x)|^{\mathrm{m}-1}\mathrm{d}r\right)\mathrm{d}s\right. \\ &+ \left.\varphi(\mathcal{X}_{\tau_x}^x)\exp\left(-\lambda\int_0^{\tau_x}|u(\mathcal{X}_s^x)|^{\mathrm{m}-1}\mathrm{d}s\right)\right]. \end{aligned}$$

Other important extensions occur whenever *explosion on the boundary* are admited, *i.e*

$$\varphi = +\infty \quad \text{on a subset } \Gamma \subseteq \partial\Omega.$$

In [Di-So] one uses adequate truncation schemes in order to obtain *exploxive solutions* verifying adequate implicit formulae. In some sense, it extends results of [Di-Le], for $\Gamma = \partial\Omega$, by analytic arguments and [LeG], for $\Gamma \subset \partial\Omega$, by stochastic methods.

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