SOLVING PARTIAL DIFFERENTIAL EQUATIONS BY LIE ALGEBRAIC METHODS

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1 Introduction

In this paper we use Lie agebraic methods to obtain explicit expressions that approximate the solution (provided it exists) to the Cauchy problem defined by

$$\frac{\partial}{\partial t} f(t;\mathbf{x}) = A(t;\mathbf{x}) f(t;\mathbf{x}), \qquad f(0;\mathbf{x}) = g(\mathbf{x}), \tag{1}$$

where g is an arbitrary analytic function, $\mathbf{x} \equiv (x_1, x_2, \ldots, x_m) \in \mathcal{R}^m$ and

$$\begin{aligned} A(t;\mathbf{x}) \;\; = \;\; & \sum_{i,j=1}^{m} a_{ij}(t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{m} b_{ij}(t) x_i \frac{\partial}{\partial x_j} + \sum_{i,j=1}^{m} c_{ij}(t) x_i x_j \\ & + \sum_{j=1}^{m} d_j(t) \frac{\partial}{\partial x_j} + \sum_{j=1}^{m} e_j(t) x_j + h(t). \end{aligned} \tag{2}$$

The coefficients $a_{ij}(t)$, etc, of the differential operator $A(t;\mathbf{x})$ are defined in an open interval and are complex-valued analytic functions.

Problems of this type appear frequently in the literature. They include the time-dependent linear Fokker-Planck equation [8], the Schrödinger equation and the Helmholtz equation in the paraxial approximation for waves propagation.

One should note that $A(t;\mathbf{x})$ is an element of a Lie algebra Q of finite dimension n under the bracket operation $[B_1, B_2] = B_1 \circ B_2 - B_2 \circ B_1$, with $B_1, B_2 \in Q$ and $\circ$ denotes the operator composition.

If A does not depend explicitly of time t, then we can write the solution of Eq.(1) as

$$f(t;\mathbf{x}) \equiv U(t) f(0;\mathbf{x}) = e^{tA} g(\mathbf{x}), \tag{3}$$

where $\exp(tA)$ should be interpreted as an element in the Lie group associated with Q. Thus one can use the properties of the Lie algebra Q to study the operator $\exp(tA)$. This has been done by Steinberg [9] for $m = 1$. More specifically, a suitable basis for Q, with constant generators A_i, $i = 1, \ldots, n$, is chosen and then the elements $\exp(tA_i)$ are computed. Next, ordering formulas of Baker, Campbell, Hausdorff and Zassenhaus type are used to write the evolution operator $U(t)$ in the form

$$U(t) = \exp(f_1(t)A_1) \exp(f_2(t)A_2) \cdots \exp(f_n(t)A_n), \tag{4}$$

where the $f_i(t)$ are t-dependent functions linked to the constant coefficients of the operator $A(\mathbf{x})$. One natural method of calculating the $\{f_i\}$ is to differentiate (4) with respect to t

and then solve the result for $\frac{df_i}{dt}$, which yields a system of nonlinear ordinary differential equations with constant coefficients that can be easily integrated. This technique can be generalized to the case of explicitly time-dependent operators $A(t;\mathbf{x})$, although now the ordinary differential equations that determine the functions $f_i(t)$ in Eq. (4) cannot be solved, in general, by quadratures for arbitrary coefficients of the operator A.

In this paper we present a modified version of the above algorithm, based entirely on Lie algebraic methods, for solving approximately the Cauchy problem (1) when the coefficients of A are arbitrary functions of time. As a result, no ordinary differential equations for the functions $f_i(t)$ must be solved. The method consists of finding a low-dimensional faithful matrix representation $\hat{Q}$ of the Lie algebra Q and then applying Lie algebraic techniques to obtain the solution of the corresponding image of our partial differential equation in $\hat{Q}$. If the associated Lie groups are also isomorphic, one can get in a straightway explicit expressions for the functions $f_i(t)$ appearing in Eq. (4) [4], and thus a solution for the Cauchy problem (1) in closed-form.

Conventionally, the solution of Eq. (1) is written as a time-ordered exponential operator

$$U(t) = P\left[\exp\left(\int_0^t A(s)ds\right)\right] \equiv I + \sum_{n=1}^{\infty}\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n A(t_1)\ldots A(t_n), \qquad (5)$$

but this aproach presents two main drawbacks in relation to the previous scheme. First of all, the treatment depends on whether the coefficients in (2) are constant or not; in the first case the time-ordered exponential reduces to an ordinary one, (Eq. 3), whereas in the latter one has to evaluate the formal series (5) explicitly. Secondly, it is not easy to evaluate the action of the operator $U(t)$ on $f(0;\mathbf{x})$ and to study the influence of the single factors A_i on the time evolution of $f(t;\mathbf{x})$.

Here, we are forced to assume that the solution to the Cauchy problem defined by Eq. (1) is uniquely determined provided the initial data $g(\mathbf{x})$ is chosen in some appropriate space of functions, at least for t sufficiently small. The resulting flow $(U(t)g)(\mathbf{x})$ will then be on the given function space. The verification of these hypothesis leads to very difficult problems on existence and uniqueness of solutions [7] that we shall not consider in this work. Here we will obtain results which are of a formal nature, but nevertheless will have direct practical applications.

2 The algebraic method

Suppose the linear operator $A(t;\mathbf{x})$ can be expressed in the form

$$A(t;\mathbf{x}) = \sum_{i=1}^{n} a_i(t)A_i(\mathbf{x}), \qquad n \text{ finite}, \qquad (6)$$

where the $a_i(t)$ $(i = 1,\ldots,n)$ are scalar functions of time, and $A_1, A_2, \ldots, A_n$ are time-independent operators that form a basis of the Lie algebra Q under the commutator product.

Let us suppose we have found a low-dimensional faithful matrix representation $\hat{Q}$ of

Q. Using this isomorphism we can consider the associated equation

$$\frac{d\vec{f}}{dt} = \hat{A}(t)\vec{f}, \tag{7}$$

which will be referred as the image equation of (1) in the matrix representation. Here $\hat{A}(t)$ is the $s \times s$ matrix in $\hat{Q}$, image of $A(t;\mathbf{x}) \in Q$, and $\vec{f}(t) \in \mathcal{C}^s$. Equivalently, we can consider the linear equation

$$\frac{d\hat{U}(t)}{dt} = \hat{A}(t)\hat{U}(t), \quad \hat{U}(0) = \hat{I}, \tag{8}$$

where $\vec{f}(t) = \hat{U}(t)\vec{f}(0)$ and the matrix $\hat{A}(t)$ can be written as

$$\hat{A}(t) = \sum_{i=1}^{n} a_i(t)\hat{A}_i, \tag{9}$$

with $\hat{A}_i$ the element of the basis of $\hat{Q}$ associated with the operator $A_i(\mathbf{x})$.

Wei and Norman [11] have shown that if $\hat{U}(t)$ is a solution of Eq. (8), then there exists a neighborhood of $t = 0$ in which it can be represented in the form

$$\hat{U}(t) = \exp(f_1(t)\hat{A}_1)\exp(f_2(t)\hat{A}_2)\cdots\exp(f_n(t)\hat{A}_n), \tag{10}$$

where the $f_i(t)$ are scalar functions of time. Moreover, the $f_i(t)$ satisfy a set of differential equations which depend only on the Lie algebra $\hat{Q}$ and the $a_i(t)$'s. This representation is global for all solvable Lie algebras.

Now if the Lie groups associate with the Lie algebras $\hat{Q}$ and Q are also isomorphic, it is possible to express the solution of Eq. (1) as $f(t;\mathbf{x}) \equiv U(t)f(0;\mathbf{x})$, with

$$U(t) = \exp(f_1(t)A_1)\exp(f_2(t)A_2)\cdots\exp(f_n(t)A_n). \tag{11}$$

Finally, by computing explicitly the flows

$$\left(\exp(f_i A_i)g\right)(\mathbf{x}), \quad i = 1, \ldots, n \tag{12}$$

we obtain a formal expression for the solution of the Eq. (1) in a neighborhood of $t = 0$.

In the general case of a time-dependent operator $A(t;\mathbf{x}) \in Q$, the set of differential equations that determine the scalar functions $f_i(t)$, or equivalently the system (8), cannot be solved by quadratures. Instead, approximate methods of resolution are required. Among them, we will use the so-called Fer factorization [5], which allows to construct explicit convergent approximations to the solution in a neighborhood of $t = 0$. The main features of this method have been analyzed in the reference [3], where its properties as a symplectic integrator have also been established for linear Hamiltonian systems.

For the linear equation (8) we can apply the results of Fer, which are summarised in the following

Theorem 2.1 *Let $\hat{A}(t)$ and $\hat{U}(t)$ be two bounded linear operators acting on a Euclidean space, with $\| \hat{A}(t) \|$ a piecewise continuous function. Then* *(a) The solution of the initial value problem*

$$\frac{d\hat{U}(t)}{dt} = \hat{A}(t)\hat{U}(t), \quad \hat{U}(0) = \hat{I} \tag{13}$$

may be expressed in the form

$$\hat{U}(t) = e^{F_1} \cdots e^{F_n} \hat{U}_n, \tag{14}$$

with

$$\frac{d\hat{U}_n}{dt} = H_n(t)\hat{U}_n, \qquad \hat{U}_n(0) = \hat{I},$$

$$F_{n+1} = \int_0^t H_n(t')dt', \qquad H_0 \equiv \hat{A}(t), \tag{15}$$

$$H_{n+1} = e^{-F_{n+1}} H_n e^{F_{n+1}} - \int_0^1 dx e^{-xF_{n+1}} H_n e^{xF_{n+1}} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} j}{(j+1)!} [F_{n+1}^j, H_n],$$

$$[F_{n+1}^j, H_n] = \underbrace{[F_{n+1}, [F_{n+1}, \ldots [F_{n+1}}_{j \text{ times}}, H_n] \ldots]]$$

$(n > 0)$ and therefore it can be written as an infinite product

$$\hat{U}(t) = e^{F_1} e^{F_2} \cdots e^{F_n} \cdots \tag{16}$$

(b) This infinite product is convergent for times t such that

$$\int_0^t \parallel \hat{A}(t') \parallel dt' < \xi, \tag{17}$$

where ξ is the non-zero solution of the equation

$$\xi = \int_0^{\xi} \frac{1 - e^{2x}(1 - 2x)}{2x} dx \qquad (\xi \simeq 0.861) \tag{18}$$

When the functions involved in Eq. (14) belong to a solvable Lie algebra, then a finite product of exponentials is attained for the evolution operator $\hat{U}(t)$. Otherwise, we must truncate the infinite product for $\hat{U}(t)$ in the n-th term, $n = 1, 2, \ldots$ by doing $\hat{U}_n = \hat{I}$. Thus we obtain an approximate expression for the evolution operator in the form

$$\hat{U}(t) \simeq \hat{V}_n(t) = e^{F_1} e^{F_2} \cdots e^{F_n}. \tag{19}$$

In that case we have the following result concerning the error bound of the approximation [2]

Theorem 2.2 *Let $E_n(t)$ be the difference between the exact and the approximate solution of Eq. (13),*

$$E_n(t) = \hat{U}(t) - \hat{V}_n(t). \tag{20}$$

Then

$$\parallel E_n(t) \parallel \leq K_n(t) \exp(\sum_{i=0}^{n} K_i(t)), \qquad n \geq 1, \tag{21}$$

with

$$K_0 \equiv \int_0^t \parallel \hat{A}(t') \parallel dt'$$

and

$$K_{n+1} = \int_0^{K_n} \frac{1 - e^{2x}(1-2x)}{2x} dx.$$

If we denote $K_0 = \alpha\xi$, with $0 < \alpha < 1$, then it can be shown that

$$\parallel E_n \parallel \leq \alpha^{2^n} g_n(\alpha)\xi, \tag{22}$$

where g_n is a function that tends to a constant as n increases. In this way, Fer's factorization provides a reliable method to obtain convergent approximate solutions to the linear equation (8) and thus convergent expressions for the characteristic ordering functions $f_i(t)$ of Eq. (10) in a neighborhood of $t = 0$ in terms of quadratures. This method allows to compute explicitly the region of convergence and the error bounds of the approximation.

3 Examples

In this section we apply the above procedure to solve some physically important linear partial differential equations. In particular, we consider the one-dimensional time-dependent Fokker-Planck equation and the Lambropoulos equation.

Example 1. As a first application we take

$$A(t;x) = a(t)\partial^2 + b(t)x\partial + d(t)\partial + h(t), \tag{23}$$

where the notation $\partial \equiv \frac{\partial}{\partial x}$ has been used. This corresponds to a one-dimensional Fokker-Planck equation whose diffusion and drift coefficients are both arbitrary functions of the time. It is a linear second-order partial differential equation of parabolic type [8].

If we identify the operators $A_1 = I$, $A_2 = x\partial$, $A_3 = \partial$, $A_4 = \partial^2$ as the basis of the Lie algebra Q in this case, then the basic commutators are given by

$$[A_2, A_3] = -A_3, \quad [A_2, A_4] = -2A_4, \quad [A_3, A_4] = 0 \tag{24}$$

and therefore the sub-algebra $L \equiv \langle A_2, A_3, A_4 \rangle$ is solvable. It is easy to realize that a matrix representation for these operators is provided by

$$\hat{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \hat{A}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{A}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{25}$$

thus specifying the matrix image of Eq. (1) with the operator A given by (23) in the form of Eq. (7), or equivalently,

$$\frac{d\hat{U}}{dt} = \hat{A}(t)\hat{U}(t), \tag{26}$$

with $\hat{A}(t) = \sum_{i=1}^{4} a_i(t)\hat{A}_i$ and $a_1(t) = h(t)$, $a_2(t) = b(t)$, $a_3(t) = d(t)$, $a_4(t) = a(t)$.

Now the Wei-Norman factorization (10), when applied to the matrix equation (26), leads to the expression

$$\hat{U}(t) = e^{f_1} \begin{pmatrix} 1 & f_3 & f_4 + \frac{1}{2} f_3^2 \\ 0 & e^{f_2} & e^{f_2} f_3 \\ 0 & 0 & e^{2f_2} \end{pmatrix}, \tag{27}$$

with the functions $f_1(t), \ldots, f_4(t)$ to be determined.

In this case, by applying Fer's factorization (Theorem 2.1) to Eq. (26) we obtain the exact solution as

$$\hat{U}(t) = e^{F_1} e^{F_2}, \tag{28}$$

where

$$F_1 = \sum_{i=1}^{4} \alpha_i(t) \hat{A}_i, \qquad \alpha_i(t) = \int_0^t a_i(s) ds \tag{29}$$

and

$$F_2 = \alpha_3^{(2)}(t) \hat{A}_3 + \alpha_4^{(2)}(t) \hat{A}_4, \qquad \alpha_i^{(2)} = \int_0^t h_i^{(1)}(s) ds$$

$$\begin{aligned} h_3^{(1)}(t) &= \frac{1}{\alpha_2^2}(-\alpha_2 e^{\alpha_2} + e^{\alpha_2} - 1)(\alpha_3 a_2 - \alpha_2 a_3) \\ h_4^{(1)}(t) &= \frac{1}{2\alpha_2^2}(-2\alpha_2 e^{2\alpha_2} + e^{2\alpha_2} - 1)(\alpha_4 a_2 - \alpha_2 a_4). \end{aligned} \tag{30}$$

If we evaluate explicitly the exponentials of Eq. (28) and compare the matrix thus obtained with the expression (27), after some algebra we obtain the exact expressions for the ordering functions $f_i(t)$ in terms of quadratures

$$\begin{aligned} f_1(t) &= \int_0^t h(s) ds \\ f_2(t) &= \int_0^t b(s) ds \\ f_3(t) &= \int_0^t d(s) e^{f_2(s)} ds \\ f_4(t) &= \int_0^t a(s) e^{f_2(s)} ds. \end{aligned} \tag{31}$$

The same expressions can be obtained, of course, by writting down and solving the differential equations satisfied by the functions $f_i(t)$ [13]. This is possible here because the Lie algebra involved is solvable.

Finally, by using the easily derivable expressions [10, 13]

$$\exp\left[a(t) x \frac{\partial}{\partial x}\right] g(x) = g\left(e^{a(t)} x\right) \tag{32}$$

$$\exp\left[a(t) \frac{\partial}{\partial x}\right] g(x) = g\left(x + a(t)\right) \tag{33}$$

$$\exp\left[a(t) \frac{\partial^2}{\partial x^2}\right] g(x) = \frac{1}{\sqrt{4\pi a(t)}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(y-x)^2}{4a(t)}\right] g(y) dy \tag{34}$$

we find for $f(t;x)$,

$$f(t;x) = \frac{e^{f_1(t)}}{\sqrt{4\pi f_4(t)}} \int_{-\infty}^{+\infty} dy f(0;y) \exp\left[\frac{-\left[y-(xe^{f_2(t)}+f_3(t))\right]^2}{4f_4(t)}\right], \tag{35}$$

a result previously obtained in the references [10, 13] with different Lie algebraic techniques.

Example 2. Next we consider the operator $A(t;x)$ given by

$$A(t;x) = a(t)\partial^2 + b(t)x\partial + c(t)x^2, \tag{36}$$

where $a(t)$, $b(t)$, $c(t)$ are complex valued analytic functions. This constitutes a generalization of a linear Fokker-Planck equation. If we denote

$$A_1 = I, \quad A_2 = \frac{1}{4}(1+2x\partial), \quad A_3 = \frac{1}{2}x^2, \quad A_4 = \frac{1}{4}\partial^2, \tag{37}$$

then these operators form a basis of the Lie algebra Q, the basic commutation relations are

$$[A_2, A_3] = A_3, \quad [A_2, A_4] = -A_4, \quad [A_3, A_4] = -A_2 \tag{38}$$

and the sub-algebra $\langle A_2, A_3, A_4\rangle$ can be identified with $SU(1,1)$, which is not solvable. A matrix representation of the $SU(1,1)$ generators is provided by

$$\hat{A}_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{A}_3 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \hat{A}_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{39}$$

and the image of the operator $A(t;x)$ under this representation can be written as Eq. (9) with $a_1(t) = -\frac{1}{2}b(t)$, $a_2(t) = 2b(t)$, $a_3(t) = 2c(t)$, $a_4(t) = 4a(t)$.

If we apply the Wei-Norman factorization to the image equation (8), the corresponding solution can be now represented as

$$\hat{U}(t) = e^{f_1}\begin{pmatrix} e^{f_2/2}(1-\frac{1}{2}f_3f_4) & -f_3e^{f_2/2} \\ \frac{1}{2}f_4e^{-f_2/2} & e^{-f_2/2} \end{pmatrix} \tag{40}$$

in a neighborhood of $t=0$. In this case the system of differential equations that determine the functions f_i cannot be solved by quadratures for arbitrary coefficients $a_i(t)$. On the other hand, Fer's expansion provides an iterative procedure for obtaining convergent approximations to the matrix $\hat{U}(t)$. More specifically, by applying Theorem 2.1 we get up to order n

$$\hat{U}(t) \simeq \hat{V}_n(t) = e^{F_1}e^{F_2}\cdots e^{F_n}, \tag{41}$$

with

$$\begin{aligned} F_1 &= \sum_{j=1}^{4}\alpha_j^{(1)}(t)\hat{A}_j & \alpha_j^{(1)}(t) &= \textstyle\int_0^t a_j(s)ds \\ F_{i+1} &= \textstyle\sum_{j=2}^4 \alpha_j^{(i+1)}(t)\hat{A}_j & \alpha_j^{(i+1)}(t) &= \textstyle\int_0^t h_j^{(i)}(s)ds, \quad i=1,\ldots,n-1, \end{aligned} \tag{42}$$

where $h_j^{(i)}(t)$, $j=2,3,4$, are the coordinates of the matrix H_{i+1} with respect to the basis $\{A_i\}$ (Eq. (15)), which depend both on the coefficients $h_j^{(i-1)}$ and $\alpha_j^{(i)}$ [3]. A simple calculation shows that

$$e^{F_i} = (\cosh\omega_i)I_2 + \frac{\sinh\omega_i}{\omega_i}B_{(i)}, \tag{43}$$

where

$$\omega_i \equiv \frac{1}{2}\sqrt{\alpha_2^{(i)^2} - 2\alpha_3^{(i)}\alpha_4^{(i)}}, \qquad B_{(i)} = \frac{1}{2}\begin{pmatrix} \alpha_2^{(i)} & -2\alpha_3^{(i)} \\ \alpha_4^{(i)} & -\alpha_2^{(i)} \end{pmatrix} \tag{44}$$

and I_2 denotes the 2×2 identity matrix.

In this way we can write an approximation to the matrix $\hat{U}(t)$ in the form

$$\hat{V}_n(t) \equiv e^{\alpha_1^{(1)}} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \tag{45}$$

whence, by comparing with Eq. (40), we get

$$\begin{aligned} f_1(t) &= \alpha_1^{(1)}(t) = -\frac{1}{2}\int_0^t b(s)ds & f_2(t) &= -2\log u_{22} \\ f_3(t) &= -u_{12}u_{22} & f_4(t) &= 2u_{21}{u_{22}}^{-1}, \end{aligned} \tag{46}$$

i.e., approximate explicit expressions for the ordering functions $f_i(t)$ in terms of quadratures. This procedure converges to the true solution $\hat{U}(t)$ as $n \to \infty$ and therefore to the functions f_i in time intervals $[0, t[$ such that

$$\int_0^t \parallel \hat{H}(s) \parallel ds < \xi, \tag{47}$$

with

$$\hat{H}(t) = \begin{pmatrix} b(t) & -2c(t) \\ 2a(t) & -b(t) \end{pmatrix}. \tag{48}$$

Finally, the solution of Eq. (1), with $A(t;x)$ given by (36), can be found in closed-form as in the preceding example, thus obtaining

$$f(t;x) = \frac{1}{\sqrt{\pi f_4(t)}} \exp\left[f_1(t) + \frac{1}{4}f_2(t) + \frac{1}{2}f_3(t)x^2 e^{f_2}\right] \int_{-\infty}^{+\infty} dy\, f(0;y) \exp\left[\frac{-(y - xe^{f_2(t)/2})^2}{f_4(t)}\right]. \tag{49}$$

Example 3. As a third and last example we consider the equation [12, 14]

$$\left(\frac{\partial^2}{\partial x \partial y} + b(t)x\frac{\partial}{\partial x} + b(t)y\frac{\partial}{\partial y} + c(t)xy + \frac{\partial}{\partial t}\right) f(t;x,y) = 0 \tag{50}$$

subject to the initial condition $f(0;x,y) = \phi(x,y)$. This partial differential equation was introduced and solved by Lambropoulos [6] when the coefficients b and c are constants. Later Wilcox [12] obtained a closed-form solution by normal-ordering exponential operators techniques. In the following we apply the method outlined in the previous section to solve the general case.

If we introduce the operators

$$A_1 = I, \quad A_2 = \frac{1}{2}(1 + x\partial_x + y\partial_y), \quad A_3 = \frac{1}{2}xy, \quad A_4 = \partial^2_{xy}, \tag{51}$$

then Eq. (50) can be written as Eq. (1) with $A(t;x,y) = \sum_{i=1}^4 a_i(t)A_i$ and $a_1(t) = b(t)$, $a_2(t) = -2b(t)$, $a_3(t) = -2c(t)$, $a_4 = -1$. Moreover, we have the basic commutation relations (38) of the algebra $SU(1,1)$, so the same steps of the Example 2, when applied to this case, lead to the ordering functions

$$\begin{array}{ll} f_1(t) = \int_0^t b(s)ds & \varphi(t) \equiv e^{-f_2(t)/2} = u_{22}(t) \\ f_3(t) = -u_{12}u_{22} & f_4(t) = 2u_{21}u_{22}^{-1}, \end{array} \tag{52}$$

where the coefficients $u_{ij}(t)$ are evaluated with the Fer factorization. If we denote

$$\begin{aligned} \delta(t) &= -f_1(t) - \frac{1}{2}f_2(t) = \log\varphi(t) - \int_0^t b(s)ds \\ \Omega(t) &= \frac{1}{\varphi(t)}u_{12}(t) \\ \beta(t) &= -2u_{21}(t)\frac{1}{\varphi(t)}, \end{aligned} \tag{53}$$

the solution of Eq. (50) is then given by

$$f(t;x,y) = \exp\left[-\delta(t) - \Omega(t)xy\right] R\left(t; \frac{x}{\varphi(t)}, \frac{y}{\varphi(t)}\right), \tag{54}$$

under the assumption that

$$R(t;x,y) \equiv \exp\left[-\beta(t)\partial^2_{xy}\right]\phi(x,y) \tag{55}$$

exists. In the particular case of constant coefficients, Fer's expansion leads to the exact solution $\hat{U}(t) = e^{F_1}$, or, equivalently,

$$\begin{aligned} \varphi(t) &= \cosh\omega t + \frac{b}{\omega}\sinh\omega t \\ \delta(t) &= \log\varphi(t) - bt \\ \beta(t) &= \frac{\gamma}{c\varphi} \\ \Omega(t) &= \frac{\gamma}{\varphi}, \end{aligned} \tag{56}$$

with

$$\omega \equiv \sqrt{b^2 - c}, \qquad \gamma \equiv \frac{c}{\omega}\sinh\ \omega t. \tag{57}$$

This is the solution obtained by Wilcox [12] for Eq. (50) in the time-independent case.

4 Summary and conclusions

In this paper we have applied Lie algebraic methods for obtaining closed-form solutions of a wide class of partial differential equations. This technique can be viewed as a useful alternative to more conventional methods.

Some advantages of the procedure are at hand. First, there is no formal distinction in treating equations with constant or time dependent coefficients, since they appear in a linear combination of operators of a finite-dimensional Lie algebra. Second, the solution is represented in factorized form, which allows the treatment of each individual factor and avoids the use of time-ordered exponentials. Third, this method can be, at least formally, extended to algebras with higher dimensionality without any conceptual problem, although technical difficulties may appear.

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