Uniqueness of the optimal for a nonlinear elliptic control problem arising from population dynamics

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Abstract. An optimal control problem for a nonlinear elliptic equation of Volterra-Lotka type is considered. Under certain assumptions, the existence of at least an optimal is shown and an optimality system is derived. Then, this system is used for proving the uniqueness.

1. Introduction.

In this work we study an optimal control problem for a nonlinear elliptic equation of Volterra-Lotka type. More concretely, we consider the B.V.P.

$$\begin{aligned} -\Delta u(x) = u(x)[a(x) - f(x) - b(x)u(x)],\ x \in \Omega, \\ u(x) = 0,\ x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded and regular domain in $\mathbb{R}^N$.
Previous equation arises from Population Dynamics, in the study of the evolution of some biological species and it models the steady-state solutions of the corresponding nonlinear evolution problem (see [10]). The coefficients which appear in (1.1) can be interpreted as follows: The function u means the species concentration, a represents its intrinsic growth rate, b denotes the crowding effect and f plays the role of control.

We take the space of admissible controls as

$$L^\infty_+(\Omega) = \{g \in L^\infty(\Omega) : g(x) \geq 0 \text{ a.e. in } \Omega\}.$$

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Under certain assumptions (see hypothesis [H] below), equation (1.1) will have, for each admissible control f, an unique maximal nonnegative solution denoted by $u_{b,f}$ (really, such solution depends on Ω, a, b and f, but in our approach the domain Ω and the function a will be considered fixed).

The benefit functional is defined by

$$J(f) = \int_\Omega (\lambda u_{b,f}(x) f(x) - f^2(x))dx \tag{1.2}$$

which represents the difference between economic revenue and cost. The strictly positive constant λ may be interpreted as the ratio between the price of the species and the cost of the control.

We are interested in the existence, uniqueness and approximation of $f \in L^\infty_+(\Omega)$ such that

$$J(f) = \sup_{g \in L^\infty_+(\Omega)} J(g) \tag{1.3}$$

and we will denote this control problem by $P_{\Omega,a,b,\lambda}$.

The existence of controls $f \in L^\infty_+(\Omega)$ maximizing (1.2) (this controls will be called optimal controls) have been shown by the authors in [3], although the benefit may be zero. Moreover, in previous paper we give conditions to get optimal benefit positive.

Similar problems have been studied by Leung and Stojanovic [7, 8, 9]. In [8] these authors have studied (1.1) but with Neumann boundary conditions, and took the space of the controls

$$C_\delta = \{g \in L^\infty(\Omega) : 0 \leq g(x) \leq \delta \text{ a.e. in } \Omega\},$$

where $0 < \delta < \inf_{x \in \Omega} a(x)$. In this particular situation they prove the existence of optimal control $f \in C_\delta$. Under suitable conditions, they describe the optimal control f in terms of the solution of an appropriated elliptic system (the optimality system).

Section 2 recalls some results and fix notation. In section 3, we will give necessary conditions for an element to be an optimal control. We will derive the optimality system and under certain conditions involving the function b, the uniqueness of solutions of a certain class is proved. Hence, we obtain the uniqueness of the optimal control.

2. Preliminary results.

For $e \in L^\infty(\Omega)$, we denote $\underline{e} = \operatorname{ess}\inf_{x\in\Omega} e(x)$, $\overline{e} = \operatorname{ess}\sup_{x\in\Omega} e(x)$.
Let us consider (1.1). From now on, we assume the following hypothesis:

[H] $\quad a, b \in L^\infty(\Omega)$, $\underline{b} > 0$,
$f \in L^\infty_+(\Omega)$.

It is known (see [1, 5]) that, for each $f \in L^\infty_+(\Omega)$, there exists a maximal nonnegative weak solution, $u_{b,f}$ of (1.1) such that

$$\begin{gathered} u_{b,f}(x) > 0 \ , \ \forall x \in \Omega \Leftrightarrow \lambda_1(-a+f) < 0 \\ \text{or} \\ u_{b,f} \equiv 0 \Leftrightarrow \lambda_1(-a+f) \geq 0 \end{gathered} \tag{2.1}$$

where, for $q \in L^\infty(\Omega)$, $\lambda_1(q)$ denote the principal eigenvalue of the eigenvalue problem

$$\begin{gathered} -\Delta u(x) + q(x)u(x) = \lambda u(x), \ \ x \in \Omega \\ u(x) = 0, \ \ x \in \partial\Omega. \end{gathered} \tag{2.2}$$

Some known results about $P_{\Omega,a,b,\lambda}$ are:

Theorem 2.1 (Existence of optimal control).
Assume [H]. Then problem $P_{\Omega,a,b,\lambda}$ has solution, i.e., $\exists f \in L^\infty_+(\Omega)$ such that
$$J(f) = \sup_{g\in L^\infty_+(\Omega)} J(g).$$

The basic idea for the proof of this theorem, is that the possible optimal controls must be bounded; in fact, if $f \in L^\infty_+(\Omega)$ is an optimal control then

$$f \leq \frac{\overline{a}\lambda}{\underline{b}} \ a.e. \ in\ \Omega. \tag{2.3}$$

The next theorem characterize when the optimal benefit is positive.

Theorem 2.2 (Positivity of benefit).

$$\sup_{g\in L^\infty_+(\Omega)} J(g) > 0 \ \Leftrightarrow \ \lambda_1(-a) < 0. \tag{2.4}$$

This justifies the hypothesis $\lambda_1(-a) < 0$, which will be assumed in many of the next results.

Previous theorems have been proved in [3].

3. Optimality system and uniqueness of the optimal.

In this section we give some necessary conditions for an element $f \in L^\infty_+(\Omega)$ to be an optimal control (the optimality system). Amongst other things, it will allow to prove the uniqueness of the optimal control, when the data of the problem satisfy some restrictions.

To derive the optimality system we need some properties of $u_{b,f}$ with respect to f (continuity, derivability in some weak sense, etc.). These properties may be found in [2, 4, 6]. By using them we can prove (see [4])

Theorem 3.1 ([4]).
Assume [H] and $\lambda_1(-a) < 0$. Then, there exists $\Lambda_1 \equiv \Lambda_1(\Omega, a, b)$, a positive constant, such that if $0 < \lambda \leq \Lambda_1$, any optimal control $f \in L^\infty_+(\Omega)$, can be expressed in the form

$$f = \frac{\lambda}{2} u_{b,f}(1 - P_{b,f}) \tag{3.1}$$

where the pair $(u_{b,f}, P_{b,f}) \equiv (u, p)$, satisfies

$$0 \leq p \leq 1, \; u > 0, \quad a.e. \; in \; \Omega \tag{3.2}$$

and the optimality system

$$\begin{aligned} -\Delta u &= u\left(a - [b + \frac{\lambda}{2}(1-p)]u\right), \; in \; \Omega, \\ -\Delta p + p(-a + 2bu) &= \frac{\lambda}{2} u(1-p)^2, \; in \; \Omega, \\ u = p &= 0, \; on \; \partial\Omega. \end{aligned} \tag{3.3}$$

Theorem 3.2 (See [4] again).
If λ is small enough, then there exists an unique solution (u, p) of system (3.3) satisfying conditions (3.2).

And so we obtain

COROLLARY 3.3 (Uniqueness of the optimal).
Under the conditions of Theorem 3.1, there exists a positive constant $\Lambda_2(\leq \Lambda_1)$ such that, if $\lambda < \Lambda_2$ then the problem $P_{\Omega,a,b,\lambda}$ has an unique optimal control which is described by the formula (3.1) "via" the unique solution of system (3.2), (3.3).

Until now we have studied the control problem $P_{\Omega,a,b,\lambda}$ when Ω and the functions a, b are fixed. Below we study problem $P_{\Omega,a,b,\lambda}$ when λ is not necessary small. In particular, for Ω, a and λ fixed, we give sufficient conditions on function b to obtain uniqueness of the optimal.

The most simple case is that where the function b is a constant. With this assumption we obtain

COROLLARY 3.4. *Let us consider the problem $P_{\Omega,a,b,\lambda}$ with b a positive constant function. Assume that the constant λ, the domain Ω and the function a are fixed satisfying $[H]$ and $\lambda_1(-a) < 0$. Choose $\Lambda_2(\Omega, a, 1)$ as in previous corollary.*
Then, if

$$b \geq \frac{\lambda}{\Lambda_2(\Omega, a, 1)} \tag{3.4}$$

the problem $P_{\Omega,a,b,\lambda}$ has an unique optimal.

Sketch of the proof. It is sufficient to observe that

$$b u_{\Omega,a,b,f} = u_{\Omega,a,1,f} \tag{3.5}$$

where $u_{\Omega,a,b,f}$ means the maximal nonnegative solution of (1.1). As a consequence, we have that for each $f \in L^\infty_+(\Omega)$,

$$\begin{aligned} J_{\Omega,a,b,\lambda}(f) &\equiv \int_\Omega (\lambda u_{\Omega,a,b,f}(x) f(x) - f^2(x))dx = \\ &= \int_\Omega (\frac{\lambda}{b} b u_{\Omega,a,b,f}(x) f(x) - f^2(x))dx = J_{\Omega,a,1,\frac{\lambda}{b}}(f) \end{aligned} \tag{3.6}$$

Therefore, Corollary 3.3 may be applied to obtain the desired conclusion.

Besides the previous case, it is possible to study others, such as that where Ω, the function a and the parameter λ are fixed and b is not necessarily a constant function. It does not seem possible to do this last case, by proving a similar relation to (3.5). However a similar treatment to that considered in the previous results can be done, obtaining a result about the uniqueness of the optimal, which involves, in a first place a restriction on the quantity $\overline{b}/\underline{b}$ (basically, this quantity must belong to the interval $[1, 2)$) and secondly, $\underline{b}$ must be sufficiently large.

THEOREM 3.5 (Uniqueness of the optimal).
Let us suppose [H] and $\lambda_1(-a) < 0$. Let $M \in [1, 2)$ satisfying $\overline{b} \leq M\underline{b}$. Then, there exists a positive constant $\gamma \equiv \gamma(\Omega, a, \lambda, M)$ such that, if $\underline{b} \geq \gamma$, $P_{\Omega,a,b,\lambda}$ has an unique solution.

The main idea to prove the above theorem is that, under this conditions, any optimal control, f, admits an expression as (3.1), where (3.3) and (3.2) remain for $u_{b,f}$ and $P_{b,f}$ defined like Theorem 3.1.

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