Smooth linearization for strongly dissipative two-dimensional maps

M. Bosch and J. C. Tatjer

Dptm. de Matemàtica Aplicada i Anàlisi
Univ. de Barcelona
08071 Barcelona

Abstract

This work deals with smooth one-parameter families of maps $F_b : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $|b| < b_0$ such that there exists a strongly dissipative saddle fixed point for all b, and when $b = 0$ the second component is identically zero. For this kind of families we prove the existence of a smooth linearization depending smoothly on the parameter. This result is a generalization of Sternberg's theorems to the special singular case in which one of the eigenvalues can be equal to zero. It can be applied to the well known Hénon family of maps.

1 Introduction and statement of results

A basic result in dynamical systems theory is

Theorem 1 *(Hartman-Grobman) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a diffeomorphism and $p = f(p)$ an hyperbolic fixed point. Then f is locally topologically conjugate to $Df(p)$, that is, there exists an homeomorphism h from a neighbourhood of 0 to a neighbourhood of p, such that $h^{-1} \circ f \circ h = Df(p)$.*

We are concerned with two variations of this result. The first is that adding suitable non-resonant conditions on the eigenvalues of $Df(p)$ it is possible to obtain some smoothness on h ([1, 2, 3, 4]).

On the other hand, if we consider the singular case $f = (f_1, 0) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, when the saddle p has eigenvalues λ with $|\lambda| > 1$ and $\mu = 0$, the conjugation $h = (u, v)$ must satisfy for (x, y) near p

$$f_1(u(x, y), v(x, y)) = u(\lambda x, 0)$$

$$0 = v(\lambda x, 0)\,.$$

From this we deduce $v(x,0) = 0$ and then $u(x,0)$ can be obtained from $f_1(u(x,0),0) = u(\lambda x, 0)$. Finally $u(x,y) = g(x, v(x,y))$ using the implicit function theorem.

Our goal in this communication is to link the two last remarks in the following way. We will consider a one-parameter family of smooth maps with a saddle fixed point, which are diffeomorphisms except for a special value of the parameter when the map will be singular (in the last sense), and we will prove that there exists a smooth linearization, also smooth with respect to the parameter, and including the singular case. More exactly we will give some hints of the proof of the following result.

Theorem 2 *Let* $F_b : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $|b| \le b_0$, $F_b(x,y) = (F_b^1(x,y), F_b^2(x,y))$ *be a family of* C^∞*-maps in* (x,y,b)*, such that*

- $F_0 = (F_0^1, 0)$*;*
- $F_b(0,0) = (0,0)\ \forall b$ *(it is not restrictive to move the fixed point to the origin);*
- $DF_b(0,0)$ *has eigenvalues* $\lambda = \lambda(b)$ *and* $\mu = \mu(b)$ *verifying* $|\lambda(0)| > 1$, $\mu(0) = 0$ *and* $\dot\mu(0) \neq 0$*.*

Then $\forall n \in \mathbb{N}\ \exists b_1 = b_1(n) > 0$ *such that* $\forall |b| < b_1\ \exists h_b, C^n$ *in* (x,y,b) *wich conjugates locally* F_b *to* $L_b(x,y) = (\lambda(b)x, \mu(b)y)$*, that is* $F_b \circ h_b = h_b \circ L_b$ *in a neighbourhood of* $(0,0)$ *independent on* b*.*

Some remarks are useful before to proceed with the proof.

1. For a fixed $b \neq 0$, this result is similar to Sternberg's theorems. Also in the case $b = 0$ the conjugation is immediate. In both cases the conjugation h_b is not unique. The new contribution of this work is that we will choose the family h_b such that it is smooth with respect to b in a neighbourhood of $b = 0$.

2. In the case $b \neq 0$, the change of variables h_b, first defined in a neighbourhood of $(0,0)$, can be extended, using forward and backward iterations of F_b, to a neighbourhood of the coordinate axes. This is not possible to do when $b = 0$ because F_0 is not invertible. In general, it is not possible to obtain such a change in a neighbourhood of the coordinate axes, as we will see in the case of the Hénon map.

2 Supression of non-resonant terms

In this section we use the normal form theory in order to obtain a formal linearization such that it admits the limit case $b = 0$.

Conditions on the eigenvalues λ and μ prevent the existence of resonances of order n fixed arbitrarily large if $|b|$ is small enough and different from zero. Effectively, reducing $|b|$ as much as we need, we can suppose $|\lambda(b)| > 1$ and $|\mu|$ very small. This implies that the relations $\lambda = \lambda^i \mu^j$ or $\mu = \lambda^i \mu^j$ are not possible for $2 \leq i + j \leq n$ with n large. We remark that in order to increase the bound n of non-existence of resonances it will be necessary to decrease the range of values $|b|$ allowed.

Then, we can apply well known results on normal forms. For each $b \neq 0$ small enough, we eliminate increasing order terms of F_b using polynomial change of variables (in the form Identity+ Homogeneous) in a neighbourhood of $(0,0)$. Exactly, we define $F_{b,2} = F_b$ and for $k \geq 2$

$$F_{b,k+1} = H_k \circ F_{b,k} \circ H_k^{-1}$$

where

$$F_{b,k}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x + \sum_{i=0}^{k} a_{i,k-i} x^i y^{k-i} + O_{k+1}(x,y) \\ \mu x + \sum_{i=0}^{k} b_{i,k-i} x^i y^{k-i} + O_{k+1}(x,y) \end{pmatrix}$$

and

$$H_k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \sum_{i=0}^{k} \alpha_{i,k-i} x^i y^{k-i} \\ y + \sum_{i=0}^{k} \beta_{i,k-i} x^i y^{k-i} \end{pmatrix} .$$

The coefficients $\alpha_{i,k-i}$, $\beta_{i,k-i}$ in each change H_k can be easily expressed in terms of the coefficients $a_{i,k-i}$, $b_{i,k-i}$ and λ, μ. Using the hypotesis $\mu(0) = 0$, $\dot{\mu}(0) \neq 0$ and $F_0 = (F_0^1, 0)$ it is possible to write

$$\mu(b) = b\tilde{\mu}(b) \ , \ \ b_{i,k-i} = b\tilde{b}_{i,k-i} \ \forall i,k \ .$$

A straighforward calculation give the explicit expressions for the coefficients of the change H_k and their limits when $b \to 0$:

$$\alpha_{i,k-i} = \frac{a_{i,k-i}}{\lambda - \lambda^i b^{k-i} \tilde{\mu}^{k-i}} \longrightarrow \begin{cases} \frac{a_{k,0}}{\lambda - \lambda^k} & i = k \\ \frac{a_{i,k-i}}{\lambda} & i < k \end{cases} ,$$

$$\beta_{i,k-i} = \frac{b\tilde{b}_{i,k-i}}{b\tilde{\mu} - \lambda^i b^{k-i} \tilde{\mu}^{k-i}} \longrightarrow \begin{cases} 0 & i = k \\ \frac{\tilde{b}_{k-1,1}}{\tilde{\mu} - \lambda^{k-1}\tilde{\mu}} & i = k-1 \\ \frac{\tilde{b}_{i,k-i}}{\tilde{\mu}} & i < k-1 \end{cases} .$$

We remark that all the limits have sense because the denominators do not vanish.

Thus we have seen that for each fixed n, if $|b|$ is small enough (including the case $b = 0$) we can linearizate F_b "up to order n" and the linearization is as smooth as we want in (x, y, b).

3 Proof of the existence of the smooth linearization map

For the last section we can consider our original one-parameter family in the form

$$F_b \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x + f(b,x,y) \\ \mu y + bg(b,x,y) \end{pmatrix} ,$$

whith f and g of order 3 in (x, y). We divide the proof in three steps.

First step: Linearization on the unstable manifold.

Doing the change $x = u_1(\overline{x})$, $y = \overline{y} + v_1(\overline{x})$, where $(u_1(\overline{x}), v_1(\overline{x}))$ is a parametrization of $W^u_F((0,0))$ such that

$$F_b(u_1(\overline{x}), v_1(\overline{x})) = (u_1(\lambda\overline{x}), v_1(\lambda\overline{x}))$$

the map is transformed into

$$F_b \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x + yf(b,x,y) \\ \mu y + byg(b,x,y) \end{pmatrix}$$

(here f and g are different from the original ones).

Now $W^u_F((0,0))$ is locally the axis $y = 0$, and F_b restricted to $W^u_F((0,0))$ is $x \to \lambda x$. We remark that u_1 and u_2 depend also on b.

Second step.

By means of a change like $x = \overline{x}$, $y = \overline{y} + \overline{y}v_2(\overline{x})$ we try to move F_b into a new expression

$$F_b \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x + yf(b,x,y) \\ \mu y + by^2 g(b,x,y) \end{pmatrix} .$$

It results that v_2 must satisfy

$$\mu v_2(x) + b(1 + v_2(x))g(x, 0, b) = \mu v_2(\lambda x) .$$

Obviously, if $b = 0$ the last functional equations is, in fact, $0 = 0$ because $\mu(0) = 0$. Then any v_2 is a solution. Thus, we must first simplify the common factor b in the equation and then solve the new equation. As before, v_2 depends also on b.

Third step.

We look for a change

$$\begin{array}{rcl} x & = & \overline{x} + \overline{y} u_2(\overline{x}, \overline{y}, b) \\ y & = & \overline{y} + \overline{y}^2 v_2(\overline{x}, \overline{y}, b) \end{array}$$

which linearizes the last F_b.

After simplification of common factors y and by^2, u_2 and v_2 must satisfy

$$\begin{array}{rcl} u_2 & = & \lambda^{-1}\mu u_2(\lambda x, \mu y, b) - \lambda^{-1}(1 + y v_2) f(x + y u_2, y + y^2 v_2, b) \\ v_2 & = & \mu v_2(\lambda x, \mu y, b) - \tilde{\mu}(1 + y v_2)^2 g(x + y u_2, y + y^2 v_2, b) \end{array}$$

where u_2 and v_2 are evaluated in (x, y, b) when it is not especificate.

The solution of these local functional equations can be reduced to the solution of the following global ones:

$$\begin{array}{rcl} u_2 & = & \lambda^{-1}\mu u_2(L(x), M(y), b) + \overline{F}_1(x, y, u_2, v_2, b) \\ v_2 & = & \mu v_2(L(x), M(y), b) + \overline{F}_2(x, y, u_2, v_2, b) \end{array}$$

where L, M are bounded, $|\dot{L}|_\infty \leq |\lambda|$, $|\dot{M}|_\infty \leq |\mu|$, $|D_{(u_2,v_2)}(\overline{F}_1, \overline{F}_2)|_\infty \leq \epsilon$ (arbitrarily small) and $(\overline{F}_1, \overline{F}_2)$ coincides with (f, g) in a neighbourhood of $(x, y, u_2, v_2, b) = (0, 0, 0, 0, 0)$.

Using the contraction theorem, it is easy to see that these equations admit a continous solution. The idea to get differentiability is:

1. Differentiate the functional equations.

2. See that the new equations are contractions in the derivatives.

3. Apply the fiber contraction theorem.

For instance, if we differentiate with respect to x:

$$\begin{array}{rcl} D_1 u_2 & = & \lambda^{-1}\mu \dot{L} D_1 u_2 + D_1\overline{F}_1 + D_3\overline{F}_1 D_1 u_2 + D_4\overline{F}_1 D_1 v_2 \\ D_1 v_2 & = & \mu \dot{L} D_1 v_2 + D_1\overline{F}_2 + D_3\overline{F}_2 D_1 u_2 + D_4\overline{F}_2 D_1 v_2\ . \end{array}$$

If $|b|$ is small enough then $|\lambda^{-1}\mu\dot{L}|_\infty < 1$, $|\mu\dot{L}|_\infty < 1$ and $|D_3\overline{F}|_\infty$, $|D_4\overline{F}|_\infty < \epsilon$. Then we have a contraction with respect to $(D_1 u_2, D_1 v_2)$.

We can do the same in the other derivatives.

4 Application: the Hénon map

The standard form of the Hénon map is

$$f_{a,b}(x,y) = (1 + y - ax^2, bx) ,$$

where $(x,y) \in \mathbb{R}^2$ and a, b are real parameters. For $b \neq 0$, $f_{a,b}$ is a global diffeomorphism on $\mathbb{R}^2$, but when $b = 0$ it collapses all the plane into the x axis.

We restrict ourselves to the case $a > 0$ when there are two fixed points $P^{\pm} = (x_0^{\pm}, y_0^{\pm})$ with $x_0^+ > 0$ and $x_0^- < 0$. Moreover P^- is always a saddle, while P^+ is also a saddle when $a > 3(1-b)^2/4$.

We study the local linearization of the Hénon map in a neighbourhood of a saddle when b is near the singular case $b = 0$. In this special case, P^- has eigenvalues $\mu_- = 0$, $\lambda_- = -2ax_0^- > 0$ and P^+ has eigenvalues $\mu_+ = 0$, $\lambda_+ = -2ax_0^+ < 0$.

Let (x_0, y_0) be one of the saddle fixed points, and μ, λ its eigenvalues. Performing explicit calculations and taking $b = 0$ one gets the linearizing change

$$\begin{aligned} w_1(x,y) &= u(x) + \tfrac{4x_0^3}{u(x)\dot{u}(x)} y \\ w_2(x,y) &= \tfrac{8ax_0^3}{\dot{u}(x)} y + \tfrac{16ax_0^6}{u(x)^2 \dot{u}(x)^2} y^2 \end{aligned}$$

where $u(x)$ satisfies

$$1 - au(x)^2 = u(\lambda x) , \quad u(0) = x_0 .$$

For any fixed $x = \overline{x}$, $\{(w_1(\overline{x},y), w_2(\overline{x},y)),\ y \in \mathbb{R}\}$ is a parabola ($w_2 = a(w_1 - u(\overline{x})^2)$). Now we study the domain of w_1 and w_2.

- Case $x_0 = x_0^-$, $a > 0$:

  It is easy to see that there exists a critical value $x_c > 0$ such that $u(x_c) = 0$ and $u(x)\dot{u}(x) \neq 0 \ \ \forall x \in (-\infty, x_c)$.

  Then w_1, w_2 are defined for $(x,y) \in ((-\infty, x_c) \times \mathbb{R}) \bigcup (\mathbb{R} \times \{0\})$.

- Case $x_0 = x_0^+$, $a > 1$:

  In this case there exist two critical values $x_{c,1} < 0 < x_{c,2}$ such that $u(x_{c,1}) = 0$, $\dot{u}(x_{c,2}) = 0$ and $u(x)\dot{u}(x) \neq 0 \ \ \forall x \in (x_{c,1}, x_{c,2})$.

  Now w_1, w_2 are defined for (x,y) in

  $$((x_{c,1}, x_{c,2}) \times \mathbb{R}) \cup ((-\infty, x_{c,1}) \times \{0\}) \cup ((x_{c,2}, +\infty) \times \{0\}) .$$

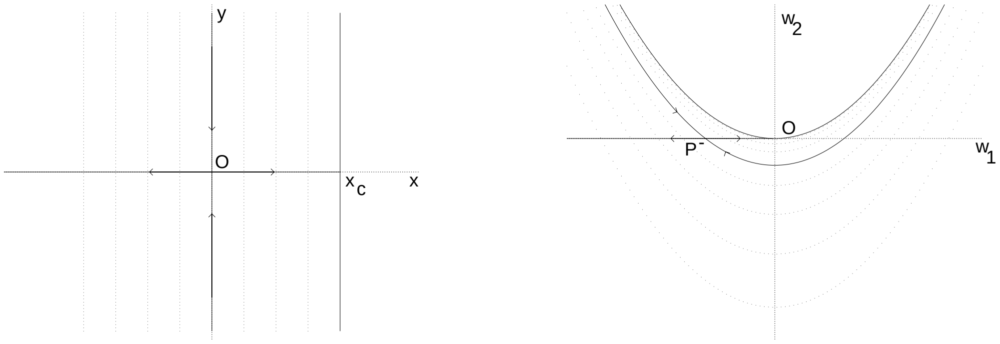

Figure 1: Domain (left) and image (right) of the change of variables in the case $x_0 = x_0^-$, $a > 0$. The line $x = x_c$ is mapped to the upper parabola.

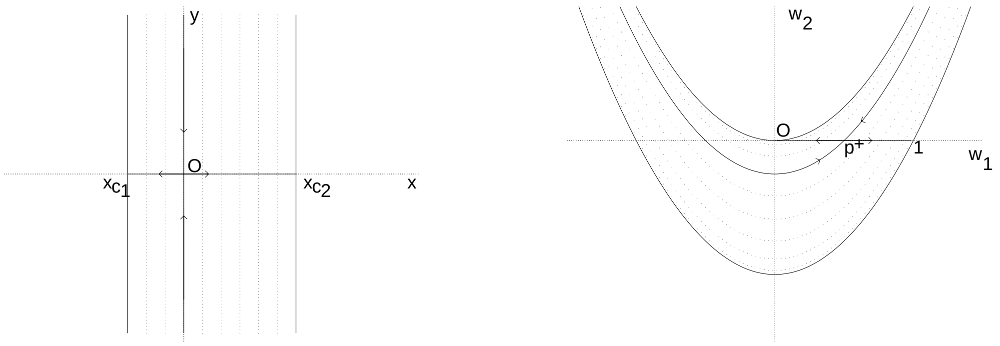

Figure 2: Domain (left) and image (right) of the change of variables in the case $x_0 = x_0^+$, $a > 1$. The line $x = x_{c_1}$ is mapped to the upper parabola, the line $x = x_{c_2}$ is mapped to the lower parabola.

References

[1] S. Sternberg: On the structure of local homeomorphisms of euclidean n-space II. Amer. J. Math. 80 (1958) 623-632.

[2] P. Hartman: Ordinary Differential Equations, 2on. ed. Birkhäuser (1982).

[3] G. R. Sell: Smooth linearization near a fixed point. Amer. J. Math. 107 (1985) 1035-1091.

[4] D. Stowe: Linearization in Two Dimensions. J. Diff. Eq. 63 (1986) 183-226.