Algebraic manipulation of variational equations in the computation of local invariant manifolds of hyperbolic periodic orbits

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Abstract. Given a system of ordinary differential equations with a hyperbolic periodic orbit, the derivatives of the Poincaré map associated to its period are needed to obtain the local expansion of the stable and unstable manifolds to the periodic orbit. In order to compute those derivatives the variational equations associated to the system must be found. The main goal of this paper is to explain how to use algebraic manipulation to work out the required variational equations.

1. Introduction

The computation of derivatives of a Poincaré map P associated to a system of ordinary differential equations

$$\dot{z} = F(t, z) \tag{1}$$

can be done by means of the numerical integration of the associated variational equations.

The knowledge of derivatives of the Poincaré map serves to find the local expansion of P that can be used in many related computations. We have studied its application to the computation of local invariant manifolds associated to hyperbolic periodic orbits of a period T in a differential system

$$\left.\begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & f(x) + g(t) \end{array}\right\} \tag{2}$$

associated to a second order differential equation

$$\ddot{x} = f(x) + g(t) \ , \tag{3}$$

where g is a T-periodic function in t.

The computation of these invariant manifolds can be restricted to their intersections at $t = t_0 (\mathrm{mod}\ T)$ by means of a related Poincaré map P. The resulting stable and unstable invariant curves can be computed when the derivatives of P are known at the hyperbolic fixed point $z^* = (x^*, y^*)$ associated to the periodic orbit [BO93]. In this paper some computational techniques to obtain derivatives of a Poincaré map are provided.

Such a kind of systems have been used as test examples to analyze the splitting of separatrices of an unperturbed system $\ddot{x} = f(x)$ by rapidly oscillating perturbations $g(t) = \delta\epsilon^p \sin(t/\epsilon)$. The typical case has been the rapidly forced pendulum with $f(x) = -\sin x$ [HMS88].

Computations of invariant curves for this particular case have been made in [BO91], in order to measure angles between them at its first interseccion. The procedure that has been used to manipulate the derivatives of the Poincaré map algebraically has been specified in [BO93] in order to compute local invariant curves. In the present paper we demostrate the way in which variational equations have been generated to calculate the derivatives of a Poincaré map through algebraic manipulation and numerical integration. The treatment has been extended to a very general case.

Functions $f(x)$ have been restricted to satisfy a linear differential equation with constant coefficients:

$$f^{(D)}(x) = a_{D-1} f^{(D-1)}(x) + \cdots + a_1 f'(x) + a_0 f(x) \ . \tag{4}$$

In the case of the above mentioned rapidly forced pendulum:

$$D = 2 \ , \quad a_1 = 0 \ , \quad a_0 = -1 \ .$$

Other interesting cases are polynomial forces f of degree $D-1$ satisfying

$$f^{(d)} = 0 \quad (d \geq D) \ .$$

In general, each derivative of order d greater than or equal to D can be computed recurrently as a function of the $D-1$ preceding derivatives.

The local expansion of the associated invariant curves to the fixed point z^* of P are there obtained starting from the knowledge of the expansion:

$$P(z^* + se_u + h(s)e_s) = P(z^*) + \sum_{k \geq 1} \frac{1}{k!} D^k P(z^*)[se_u + h(s)e_s]^k$$

that is expressed as a linear combination of vectorial terms

$$P_{i,j} = D^{i+j} P(z^*)[e_u^i, e_s^j] = D^{i+j} P(z^*)[e_u, \overset{(i}{\ldots}, e_u, e_s, \overset{(j}{\ldots}, e_s] \quad (2 \leq i + 2j \leq N) \ .$$

In the above mentionated paper, coefficients c_k ($k = 2 \div N$) of the local invariant curves are computed by means of algebraic manipulation of those terms.

Some techniques to calculate these coeficients are explained below. In section 2, their relation with the variational equations of (1) is shown. Section 3 provides recurrences between variational equations and a procedure to generate the required variational equation in a systematic way. Section 4 refers to an algebraic manipulator implemented to generate the variational equations automatically.

2. Variational equations and derivatives of the Poincaré map

In this section, we have established the relation between variational equations and derivatives of an associated Poincaré map to increase time in period T.

Let us denote the solution of a differential system (1) that passes through z_0 at time t_0 by $z(t; t_0, z_0)$. That is,

$$\dot{z}(t; t_0, z_0) = F(t, z(t; t_0, z_0))$$

with initial condition

$$z(t_0; t_0, z_0) = z_0 \ .$$

For each value of t_0, the image of a point z_0 by the Poincaré map $P \equiv P_{t_0}$ associated to the differential system for period T can be expressed as

$$P(z_0) = z(t_0 + T; t_0, z_0) .$$

Deriving this relation with respect to z_0, an expression for the first derivative of P in z_0 is obtained

$$DP(z_0) = Dz(t_0 + T; t_0, z_0) \tag{5}$$

and it corresponds to the solution of the following *first order variational equation* in $t_0 + T$:

$$(Dz(t; t_0, z_0))^\cdot = D_z F(t, z(t; t_0, z_0)) Dz(t; t_0, z_0) , \tag{6}$$

with initial condition

$$Dz(t_0; t_0, z_0) = I . \tag{7}$$

If e_1 and e_2 are the two columns of identity matrix I, vectors $Dz(t; t_0, z_0)e_1$, $Dz(t; t_0, z_0)e_2$ satisfy the first order variational equation (6) with initial conditions $Dz(t_0; t_0, z_0)e_1 = e_1$, $Dz(t_0; t_0, z_0)e_2 = e_2$ and their values in $t = t_0 + T$ are the two columns of the $DP(z_0)$.

Integrating these first order variational equations from z_0 around the corresponding T-periodic orbit, $DP(z_0)$ is obtained. Its eigenvalues λ_u, λ_s and associated eigenvectors e_u, e_s can then be computed as usual.

We note that $Dz(t; t_0, z_0)e_u$, $Dz(t; t_0, z_0)e_s$ also satisfy the first order variational equation (6) with initial conditions $Dz(t_0; t_0, z_0)e_u = e_u$, $Dz(t_0; t_0, z_0)e_s = e_s$. Thus, their values in $t = t_0 + T$ are $P_{1,0} = DP(z_0)e_u = \lambda_u e_u$, $P_{0,1} = DP(z_0)e_s = \lambda_s e_s$, respectively.

Deriving relation (5) again with respect to z_0, an expression for the second derivative of P is obtained

$$D^2 P(z_0) = D^2 z(t_0 + T; t_0, z_0) , \tag{8}$$

and corresponds to the solution of the following *second order variational equation* in $t_0 + T$

$$\begin{aligned} (D^2 z(t; t_0, z_0))^\cdot &= D_z^2 F(t, z(t; t_0, z_0))[Dz(t; t_0, z_0), Dz(t; t_0, z_0)] \\ &\quad + D_z F(t, z(t; t_0, z_0)) D^2 z(t; t_0, z_0) , \end{aligned} \tag{9}$$

with initial condition

$$D^2 z(t_0; t_0, z_0) = 0 . \tag{10}$$

In the second order expansion of $P(z_0 + s e_u + h(s) e_s)$, the following terms:

$$P_{2,0} = D^2 P(z_0)[e_u, e_u] , \quad P_{1,1} = D^2 P(z_0)[e_u, e_s] , \quad P_{0,2} = D^2 P(z_0)[e_s, e_s] ,$$

are required.

We note that $D^2 z(t; t_0, z_0)[e_u, e_u]$, $D^2 z(t; t_0, z_0)[e_u, e_s]$, $D^2 z(t; t_0, z_0)[e_s, e_s]$ satisfy the second order variational equation (9). They are zero at $t = t_0$ and their values at $t_0 + T$, $P_{2,0}$, $P_{1,1}$, $P_{0,2}$.

Thus, $P_{2,0}$, $P_{1,1}$, $P_{0,2}$ can be computed integrating the second variational equation from t_0 to $t_0 + T$ with null initial conditions.

This procedure can be repeated up to the necessary order of the variational equations. Terms $P_{k,0}$, $P_{k-1,1}$, ..., $P_{0,k}$ can be computed integrating k-th order variational equations from t_0 to $t_0 + T$ with null initial conditions:

$$P_{i,j} = D^{i+j} z(t_0 + T; t_0, z_0)[e_u^i, e_s^j].$$

3. Generation of variational equations

In the following section, some details about the generation of variational equations are given connected with the case under study, in which $z = (x, y)$ satisfies the differential system

$$\dot{x} = y \ , \quad \dot{y} = f(x) + g(t) \ .$$

3.1. Recurrences

Denoting the components of $P_{i,j}$ by v_{ij}, w_{ij}, the variational equations of degree $i+j$ that satisfies $P_{i,j}$ have the form:

$$\dot{v}_{ij} = w_{ij} \ , \quad \dot{w}_{ij} = f_{i,j} \ ,$$

where functions $f_{i,j}$ can be found recurrently as follows.

Using convention $x = v_{00}$, $y = w_{00}$, the starting differential system (with its initial conditions) is rewritten as:

$$\begin{array}{ll} \dot{v}_{00} = w_{00} \ , & \dot{w}_{00} = f(v_{00}) + g(t) \ ; \\ v_{00}(t_0) = x_0 \ , & w_{00}(t_0) = y_0 \ . \end{array}$$

$P_{1,0}$, $P_{0,1}$ satisfy first order variational equations (6) (with their initial conditions) that have the following expression:

$$\begin{array}{ll} \dot{v}_{10} = w_{10} \ , & \dot{w}_{10} = f'(v_{00})v_{10} \ ; \\ v_{10} = 1 \ , & w_{10} = a_u \ . \\ \dot{v}_{01} = w_{01} \ , & \dot{w}_{01} = f'(v_{00})v_{01} \ ; \\ v_{01} = 1 \ , & w_{01} = a_s \ . \end{array}$$

In these expressions, a_u, a_s are the slopes of the eigenvectors e_u, e_s of P, supposed to be unitary in their first component.

$P_{2,0}$, $P_{1,1}$, $P_{0,2}$ satisfy second order variational equations (9) (with null initial conditions) that have the expression:

$$\begin{array}{ll} \dot{v}_{20} = w_{20} \ , & \dot{w}_{20} = f''(v_{00})v_{10}^2 + f'(v_{00})v_{20} \ ; \\ v_{20} = 0 \ , & w_{20} = 0 \ . \\ \dot{v}_{11} = w_{11} \ , & \dot{w}_{11} = f''(v_{00})v_{10}v_{01} + f'(v_{00})v_{11} \ ; \\ v_{11} = 0 \ , & w_{11} = 0 \ . \\ \dot{v}_{02} = w_{02} \ , & \dot{w}_{02} = f''(v_{00})v_{01}^2 + f'(v_{00})v_{02} \ ; \\ v_{02} = 0 \ , & w_{02} = 0 \ . \end{array}$$

First it is noticed that the functions defining the variational equations to compute $P_{i,j}$ can be expressed as follows:

$$f_{i,j} = \sum_{d=1}^{i+j} \alpha_{i,j}^d f^{(d)}(v_{00}) \ . \tag{11}$$

Analyzing the general procedure of generation of variational equations sketched above, we observe that those functions corresponding to variational equations of order $i+j+1$ are deduced from functions corresponding to the variational equations of order $i+j$ according to the following recurrences that recall the derivative of a product:

$$
\begin{aligned}
f_{i+1,j} &= \sum_{d=1}^{i+j}(\partial_1\alpha_{i,j}^d + \alpha_{i,j}^{d-1}v_{1,0})f^{(d)}(v_{00}) + \alpha_{i,j}^{i+j}v_{1,0}f^{(i+j+1)}(v_{00}) \ , \\
f_{i,j+1} &= \sum_{d=1}^{i+j}(\partial_2\alpha_{i,j}^d + \alpha_{i,j}^{d-1}v_{0,1})f^{(d)}(v_{00}) + \alpha_{i,j}^{i+j}v_{0,1}f^{(i+j+1)}(v_{00}) \ . \qquad (12)
\end{aligned}
$$

Consequently, the following recurrences for the coefficients $\alpha_{i,j}^d$ of the d-th derivatives of f are obtained for $i, j \geq 0$:

$$
\begin{aligned}
\alpha_{i+1,j}^d &= \partial_1\alpha_{i,j}^d + \alpha_{i,j}^{d-1}v_{1,0} \quad (d = 1 \div i+j) \\
\alpha_{i+1,j}^{i+j+1} &= \alpha_{i,j}^{i+j}v_{1,0} \qquad (13) \\
\alpha_{i,j+1}^d &= \partial_2\alpha_{i,j}^d + \alpha_{i,j}^{d-1}v_{0,1} \quad (d = 1 \div i+j) \\
\alpha_{i,j+1}^{i+j+1} &= \alpha_{i,j}^{i+j}v_{0,1} \ , \qquad (14)
\end{aligned}
$$

where $\alpha_{0,0}^0 = 1$, $\alpha_{0,0}^d = 0$ for $d \neq 0$ and $\alpha_{i,j}^0 = 0$ for $(i,j) \neq (0,0)$.

It is necessary to explain how $\partial_1\alpha_{i,j}^{(d)}$, $\partial_2\alpha_{i,j}^d$ are defined.

Each coefficient $\alpha_{i,j}^d$ with $d > 0$ is a homogeneous polynomial of degree d in variables $v_{i,j}$ that is written as follows:

$$
\alpha_{i,j}^d = \sum_{|I|=i,|J|=j} a_{I,J}^{i,j} v_{i_1,j_1} \cdots v_{i_d,j_d} \ , \qquad (15)
$$

where $I = (i_1, \ldots, i_d)$, $J = (j_1, \ldots, j_d)$.

Using this notation, ∂_1 means some kind of derivation of $v_{i_1,j_1} \cdots v_{i_d,j_d}$ as a product of d variables, in which derivatives are found but increasing index i by 1. Similarly, ∂_2 means the same but increasing index j by 1. That is:

$$
\begin{aligned}
\partial_1\alpha_{i,j}^d &= \sum_{|I|=i,|J|=j} a_{I,J}^{i,j}\left(v_{i_1+1,j_1} \cdots v_{i_d,j_d} + \cdots + v_{i_1,j_1} \cdots v_{i_d+1,j_d}\right) \ , \\
\partial_2\alpha_{i,j}^d &= \sum_{|I|=i,|J|=j} a_{I,J}^{i,j}\left(v_{i_1,j_1+1} \cdots v_{i_d,j_d} + \cdots + v_{i_1,j_1} \cdots v_{i_d,j_d+1}\right) \ . \qquad (16)
\end{aligned}
$$

3.2. Ordering of the variational equations

The whole terms $P_{i,j}$ can be computed integrating a bulk of variational equations. These equations must be ordered in an adequate way to avoid unnecessary computations of terms $P_{i,j}$ in the computation of the local expression of invariant curves explained in [BO93].

In the computation of coefficient c_k some terms $P_{i,j}$ $(i + 2j < k)$ have already been computed when obtaining the preceding coefficients c_1,..., c_{k-1}. Only the computation of terms $P_{i,j}$ with $i + 2j = k$ is now required. This fact forces an ordering in the computations of terms $P_{i,j}$ in the sense that they are computed when needed.

For successive values of k, the following scheme represents the pairs of indices i,j corresponding to terms $P_{i,j}$ that can be computed by integrating variational equations of order k up to order 9:

k	ij				
0	00				
1	10	01			
2	20	11	02		
3	30	21	12	03	
4	40	31	22	13	04
5	50	41	32	23	14
6	60	51	42	33	24
7	70	61	52	43	34
8	80	71	62	53	44
9	90	81	72	63	54

If only coefficients c_k ($k = 2 \div 9$) are needed, some terms $P_{i,j}$ should not be computed and should be eliminated of the scheme (those that satisfy $i + 2j > 9$). The result is the following scheme of pairs i, j:

k	ij				
0	00				
1	10	01			
2	20	11	02		
3	30	21	12	03	
4	40	31	22	13	04
5	50	41	32	23	14
6	60	51	42	33	
7	70	61	52		
8	80	71			
9	90				

The last scheme represents the pairs of indices ordered according to the value of $i + 2j$ that can be found reading the previous scheme diagonally:

k	ij				
0	00				
1	10				
2	20	01			
3	30	11			
4	40	21	02		
5	50	31	12		
6	60	41	22	03	
7	70	51	32	13	
8	80	61	42	23	04
9	90	71	52	33	14

This scheme corresponds to the order in which the coefficients $P_{i,j}$ are required: it preserves the ordering according to $k = i + 2j$ and, for each value of $i + 2j$, it preserves the ordering given by j. This is the order in which the corresponding variational equations are considered.

The number of order of $P_{i,j}$ is the number of order of $P_{i+2j,0}$ augmented by j. The number of order of the terms of the form $P_{k,0}$ are $q(q+1)+1$, for even $k = 2q$, and $(q+1)^2 + 1$, for odd $q = 2r + 1$.

Due to this fact, term $P_{i,j}$ occupying the place l in this ordering is denoted by P_l. That is:

$$\begin{array}{ll} P_{q(q+1)+j+1} & (i+2j=2q) \ , \\ P_{(q+1)^2+j+1} & (i+2j=2q+1) \ . \end{array}$$

Following this notation, components v_{ij}, w_{ij} of $P_{i,j}$ are denoted by x_{2l}, x_{2l+1}.

The following table indicates the total number of differential equations to integrate together in order to compute the coefficients c_k ($k = 2 \div N$)) of the local invariant curves for different values of N:

N	number of variational equations
2	8
3	12
4	18
5	24
6	32
7	40
8	50
9	60

All these equations in (v_{ij}, w_{ij}) $(0 \leq i+2j \leq N)$ must be integrated together around the periodic orbit, from t_0 to t_0+T, to obtain terms $P_{i,j}$ $(2 \leq i+2j \leq N)$ needed to calculate c_k $(k = 2 \div N)$.

3.3. Ordering of computations

Here we provide a procedure to generate the variational equations in the order specified in the previous section by working out recurrences (13), (14).

First we observe that for $k = 2q$ and for $k = 2q+1$ the same number $q+1$ of terms $P_{i,j}$ ($P_{k,0}$, $P_{k-2,1}$, ..., $P_{k-2q,q}$) have to be computed. We use $r = 0, 1$ to distinguish the even case from the odd one. In this way, $k = 2q+r$ and, for each value of q, there are two values of k: $2q$, $2q+1$.

Given a value of $q > 0$:

- The first q variational equations corresponding to $r = 0$ can be obtained using recurrence (13) over the q variational equations corresponding to $q-1$ and $r = 1$ and the last one, using the recurrence (14) over the last variational equation corresponding to $q-1$ and $r = 0$.

- The variational equations corresponding to $r = 1$ can be obtained using recurrence (13) over the variational equations corresponding to $r = 0$ and the same value of q.

$q-1$	$r=0$	$k=2q-2$	 $(0,q-1)$		
	$r=1$	$k=2q-1$	$(2q-1,0)$ $(2q-3,1)$ ... $(1,q-1)$		
q	$r=0$	$k=2q$	$(2q,0)$ $(2q-2,1)$ ... $(2,q-1)$ $(0,q)$	1 1 1 1 0	$(2q-1,0)$ $(2q-3,1)$ ... $(1,q-1)$ $(0,q-1)$
	$r=1$	$k=2q+1$	$(2q+1,0)$ $(2q-1,1)$ ... $(3,q-1)$ $(1,q)$	0 0 0 0 0	$(2q,0)$ $(2q-2,1)$ ... $(2,q-1)$ $(0,q)$

In the previous table the indices i,j corresponding to variational equations needed to obtain c_k are displayed in the fourth column for general values of q and r ($k = 2q + r$). The sixth column displays the pairs of indices corresponding to the previous variational equation used to derive each variational equation, following the recurrence indicated in the fifth column (1 for recurrence (13), 2 for recurrence (14)).

This has been the selected may of deriving the variational equations in the algebraic manipulator that has been implemented to generate them automatically.

4. Algebraic manipulator for variational equations

This algebraic manipulator can generate the variational equations up to order N involved in the computation of coefficients c_k ($k = 2 \div N$) of the local expansion of the invariant curves.

The corresponding program computes the necessary coefficients $\alpha^d_{i,j}$ by working out recurrences (13), (14). This coefficients have almost all the information about the variational equations involved. This information must be completed with the derivative functions $f^{(d)}(x)$. If functions f are restricted to satisfy the linear differential equation (4), only the first $D-1$ derivatives and the coefficients a_j ($j = 0 \div D-1$) must be provided. The equations obtained by deriving the linear differential equation serve to compute recurrently the derivatives d of order greater or equal than D:

$$f^{(d)}(x) = a_{D-1} f^{(d-1)}(x) + \cdots + a_1 f^{(d-D+1)}(x) + a_0 f^{(d-D)}(x)\ . \qquad (17)$$

When f is a polynomial, all coefficients $a_j (j < D)$ will be null and the derivatives of f greater than or equal to D will be null, which simplifies computations. The program uses a parameter M that is equal to $D-1$ in this polynomial case and equal to zero in other cases.

The only cases that the program does not accept are two obvious ones: $M = 0,\ D = 1,\ M = 1,\ D = 2$.

4.1. Discussion

This algebraic manipulator is linked with other functions in a more general program that computes the local and global invariant manifolds. This manipulator of variational equations and the manipulator of the invariant condition explained in [BO93] are small but very useful pieces in the big puzzle that constitutes the whole program.

The algebraic manipulator of variational equations can offer an elegant output in TEX of the coefficients of the variational equations required. However, the true importance of the constructed manipulator is that the variational equations are generated automatically, stored in memory and can be used by the numerical integrator in the same program without being rewritten and entered to the program.

References

[HMS88] Holmes, P. and Marsden, J. and Scheurle, J. Exponentially small splittings of separatrices withapplications to KAM theory and degenerate bifurcations. *Contemporary Math.*, 81, 213–244 (1988).

[BO91] Benseny, A. and Olivé, C. High precision angles between invariant manifolds for perturbed hamiltonian systems with slow dynamics. *Equadiff 91*, Barcelona (1991).

[BO93] Benseny, A. and Olivé, C. Algebraic manipulation of invariancy condition in the computation of local invariant curves of hyperbolic periodic points. Preprint (1993).