On the Construction of Minimal Sets in Dynamical Systems

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For the discrete dynamical system (X, f) where X is a compact metric space and $f \in C(X, X)$ a continuous map, an extension of the notion of periodic orbit is given by the notion of minimal set. We recall that given a point $x \in X$, the orbit of x by f , $Orb_f(x)$, is given by the set $(f^n(x))_{n=1}^{\infty}$ where $f^n = f \circ f^{n-1}$ and $f^0 = Identity$.

We say that $\oslash \neq E \subseteq X$ is minimal if

(a) E is closed
(b) E is strongly invariant, that is, f(E)=E.
(c) E has no subset holding the conditions (a) and (b).

The notion of minimal set was introduced for homeomorphisms between compact metric spaces in the setting of the ergodic theory. Given a homeomorphism $T : X \to X$, $\oslash \neq E \subseteq X$ is minimal if $T\mid_E$ is minimal, that is, if for every $x \in X$, the orbit of x is dense in E. When a transformation T is minimal over some closed set it results "irreductible" in the sense that it plays the same rôle that an ergodic transformation in the setting of measure-preserving transformations.

When f is an endomorphism in X, then f restricted to any of these minimal sets behaves in a simple way according to the point of view of the dynamics, that is , f restricted to such sets has zero topological entropy.

The aim of this paper is to review some known results from the literature and to give some new others concerning the construction and characterization of minimal sets in different situations for X, In most cases we will offer some indications of the proofs.

I.- $\mathbf{X}=[0,1] = I$

In this case, any finite set can be a minimal set since there exists a continuous map for which the points of the finite set form a periodic orbit. We have not the same situation for closed infinite sets. Not all of these sets can be minimal according to the following result

Proposition 1

Let E$\subsetneq$ I be an infinite minimal set for a continuous map f. Then E is a Cantor set (a non empty, closed, perfect and totally disconnected set)

Proof

It can be seen in $[Bl, Co]$ page 92.

The converse problem is also solved. In fact, it is not difficult to see that given any Cantor set $E \subset I$ we can get a contonuous map f such that E is minimal with respect to f. First in $[Bl, Co]$ page 147 is constructed a map g such that the classical middle-third Cantor set E_0 is minimal with respect to g. Secondly as it is proved in $[H, Y]$, for any Cantor set $E \subseteq I$ there exists a homeomorphism $h : E \to E_0$. The complement of E in I is a union of countably many open intervals. E results a minimal set for the continuous map $f : I \to I$ given by $f(x) = h^{-1} \circ g \circ h(x)$ if $x \in E$ and defined linearly on any component of the complement of E.

There is a closed connection between the minimal sets and the almost periodic points as was put in evidence by Birkhoff.

Definition 2

Let X be a compact metric space and f$\in$ C(X,X). $x \in X$ is said to be almost periodic for f if for evey $\epsilon > 0$ there is a positive integer N such that for every positive integer i is

$$d'(\left\{f^{i+1}(x), f^{i+2}(x), ..., f^{i+N}(x)\right\}, x) < \epsilon$$

It is evident that all periodic points are almost periodic. More difficult is to obtain explicit expressions of almost periodic points which are nor periodic.Using the tent map $f(x) = 1- \mid 2x - 1 \mid$ we can obtain some of these examples.

Let $x \in I$ be the point written in binary expansion as $x = .j_1 j_2 j_3 ....$where $j_1 = 0, j_2 = 1$,$j_3 = 1$, $j_4 = 0$ and for each positive integer $i \geq 2$, $j_{2^i+1} = j_1, j_{2^i+2} = j_2$, ..., $j_{2^i+2^i-2} = j_{2^i-2}$, $j_{2^i+2^i-1} = 1 - j_{2^i-1}$ and $j_{2^i+2^i} = j_{2^i}$.

It is easy to see that if $n \geq 2$ and $k = 0, 1, 2, ...$then $f^{k2^n}(x) = 2^{k2^n}x - \left[2^{k2^n}x\right]$, where $[y]$ denotes the integer part of y. $f^{k2^n}(x)$ can be obtained by moving the binary point to the right $k2^n$ places and then taking the fractional part of the number. This implies that the orbit of x by f is infinite and hence x is not periodic.

Given $\epsilon > 0$ we can find a positive integer m such that is $\frac{1}{2^{2^m-1}} < \epsilon$, and set $N = 2^m$. Then, for every positive integer p we can compute f^{p+1},f^{p+2}, ..., f^{p+N} and find that it is possible to obtain a positive integer q such that $p + q = k2^n$ with $1 \leq q \leq N$.In this case we have

$$d'(\left\{f^{p+1}(x), f^{p+2}(x), ..., f^{p+N}(x)\right\}, x) \leq d(x, f^{p+q}(x)) = d(x, f^{k2^n}(x)) < \frac{1}{2^{2^n-1}} < \epsilon$$

The conclusion is that x is almost periodic.

The following result explains the connection between the minimal sets and the almost periodic points

Theorem 3

Let X be a continuum (a compact and connected metric space), $f\in C(X;X)$ and $x\in X$. Then $\overline{Orb_f(x)}$ is minimal if and only if x is almost periodic.

Proof

See $[Mu]$.

The application of the above theorem allows us to construct minimal sets in other situations.

Theorem 4

Let $f\in$ C(I,I) such that f^2 has a dense orbit. Then, there is a point $x\in X$ not periodic

but almost periodic.

Sketch of the proof

It is not difficult to see that there is a point $y \in I$ of period 3 for f^2 and as consequence there is $z \in I$ almost periodic but not periodic for f^2.

Finally it can be proved that $\overline{Orb_f(z)} = A$ does not contain periodic orbits. Therefore A is minimal.

Following Block and Coven $[Bl, C]$ it can be seen that if for a continuous map $f \in C(I, I)$ we have a point $x \in \omega(f) \setminus \overline{Per(f)}$ ($\omega(f) = \cup_{x\in I}\omega_f(x)$ and $Per(f)$ is the set of all periodic points of f) then $\omega_f(x)$(the set of limit points of $Orb_f(x)$) is an infinite minimal set. Using this result we can construct besides an example of a continuous map f for which $\omega(f) \neq \overline{Per(f)}$.

From the measure point of view, the periodic orbits are uniquely ergodic sets since they support a unique invariant measure. Are all the infinite minimal sets uniquely ergodic?. If the answer would be affirmative, this type of Cantor sets would be the natural extension of the periodic orbits. But this is not the case since it is possible to construct Cantor sets which with respect to a map can support only one invariant measure, exactly two or even a finite number of them (see $[B]$).

When f has some differentiability properties ($f \in C^1(I, I)$) the situation is different. Not all the Cantor sets can be minimal. We obtain the following result where with $\lambda(A)$ we denote the Lebesgue measure of A.

Proposition 5

Let $E\subset I$ be a minimal set for a map $f\in C^1(I,I)$, then $\lambda((a, b) \cap E) = 0$ for any $(a, b) \subset I$. Let $E\subset I$ be a Cantor set such that $\lambda((a, b) \cap E) = 0$ for any $(a, b) \subset I$, then can be obtained a map $g\in C^1(I,I)$ such that E is minimal for g.

Proof

See $[B, L]$.

II.- $\mathbf{X=S^1}$

This case is more complicated than the interval I since besides the periodic orbit and Cantor sets, the S^1 can also be a minimal set and other possibilities could occur. An easy example shows that S^1 can be obtained as a minimal set.

Example 6

$X = \{e^{i\alpha} : 0 \leq \alpha \leq 2\pi\}$, $\dot{f} : X \to X$ given by $f(e^{i\alpha}) = e^{i(\alpha+\frac{\pi}{\sqrt{2}})}$.

It is easy to show that for any $\alpha \in [0, 2\pi]$, $X = \overline{Orb_f \alpha}$ which implies that X is a minimal set for f.

Using the notion of *mat sets (minimal and almost twist)*, Misiurewicz has given some results which contain a partial answer to the problem of characterizing the minimal sets.

Let $f : S^1 \to S^1$ be a continuous map of degree one of the circle onto itself and let $F : R \to R$ be its lifting. We denote by $e : R \to S^1$ the natural projection.

Definition 7

$A\subset S^1$ is a twist set (respectively an almost twist set) if F restricted to $e^{-1}(A)$ is increasing (respectively non-decreasing)

Definition 8

$A \subset S^1$ *is a mat set if* A *is* $non-empty, closed, invariant$ $(that\ is\ f(A) \subset A)$, $\min imal\ and\ an\ almost\ twist\ set.$

Definition 9

A point $x \in S^1$ is a twist point (respectively, an almost twist point) if its orbit is a twist set (respectively, an almost twist set)

If $A \subset S^1$ is an almost twist set, then for every $x \in e^{-1}(A)$ the limit

$$\rho(x) = \lim_n \tfrac{1}{n}(F^n(x) - x)$$

exists and is independent of the choice of x. We call it *a rotation number of A.* . If the above limit does not exist then we have *the rotation interval.* The construction of minimal set is closely related to the rotations intervals.

Theorem 10

Let $f : S^1 \to S^1$ be a continuous map of degree one. Then for every α from the rotation interval of f there exists a mat set A with $\rho(A) = \alpha$.

Theorem 11

Let f:$S^1 \to S^1$ be a continuous map of $\deg ree$ *one. Then for every rational α from the rotation interval of f there exists a periodic twist point $x\in S^1$ with $\rho(x) = \alpha$*

Theorem 12

Let A be a mat set with ρ (A) irrational. Then $A = S^1$ or A is homeomorphic to the Cantor set.

The proof of the las three theorems can be found in $[M]$.

To our knowledge the converse problem is not known and also a complete characterization of the minimal sets even in the case of maps of degree different of one.

III.- X=$[0,1]\times[0,1]=I$

In this setting we will consider triangular maps on I^2 which are given by $F(x,y)=(f(x),g(x,y))$. It is easy to prove that if $E\subset I^2$ is a minimal set for F then the projection of E with respect to the first coordinate, E_1, is a minimal set for f.

If E is not a periodic orbit then E_1 must be a Cantor set. What about the other projection E_2?. The answer is still open but some partial result can be given.

(a)E can contain a vertical segment.

This can be seen in the construction given in $[F,P,S]$.

(b) $E=E_1\times[0,1]$ where E_1 is a Cantor set for f.

We conjecture that it would be possible but until now we have not being able to construct an example. The complete characterization of minimal sets, even the topological structure of the ω-limit sets for triangular maps is open.

IV.- X=(I,f), the inverse limit space

Given $f\in C(I,I)$, let $(I,f)=\{(x_0,x_1,x_2,...):f(x_{i+1})=x_i\}$. (I,f) is a compact and connected metric space under the metric

$$d(\underset{\longleftrightarrow}{x},\underset{\longleftrightarrow}{y})=\sum_{i=0}^{\infty}\frac{|x_i-y_i|}{2^i}$$

For each $n=0,1,2...$ let $\prod_n:(I,f)\to I$ be the $n-th$ projection map, $\prod_n(\underset{\longleftrightarrow}{x})=x_n$. For each n, $\prod_n$ is continuous and given $\underset{\longleftrightarrow}{x}\in(I,f)$, $\underset{\longleftrightarrow}{x}=(x_0,x_1,x_2,...)$, let induce the homeomorphism $\widehat{f}:(I,f)\to(I,f)$ defined by

$$\widehat{f}(\underset{\longleftrightarrow}{x})=(f(x_0),x_0,x_1,x_2,...).$$

There exists a close relation between the minimal sets on (I,f) and those in I produced by f according to the following results.

Proposition 13

Let $f\in C(I,I)$ and $E\subset I$ be a minimal set for f.

Let $\widehat{E}=\left\{\underset{\longleftrightarrow}{x}\in(I,f):\text{for each } n\geq 0, \prod_n(\underset{\longleftrightarrow}{x})\in E\right\}$

Then $\widehat{E}$ is a minimal set in (I,f) with respect to the induced homeomorphism $\widehat{f}$.

Proof

See $[Mu]$

Proposition 14

Let $f\in C(I,I)$ and $\widehat{E}\subset(I,f)$ be a minimal set for $\widehat{f}$. Let

$$E=\left\{ x \in I : \textit{there is an } \underset{\longleftrightarrow}{x} \in \widehat{E} \textit{ and } n > 0 \textit{ such that } x = \prod_n(\underset{\longleftrightarrow}{x}) \right\}$$

Then E is a minimal set for f.

Proof

See also '$[Mu]$.

The minimal sets in (I, f) can be partially characterized by the following result which says that in some cases the unique minimal sets in (I, f) are precesily periodic orbits and Cantor sets.

Proposition 15

Let f∈ $C(I, I)$ such that f^2 has a dense orbit. Then the minimal sets in (I,f) for $\widehat{f}$ are either orbits or Cantor sets.

Proof

Can be partially seen in $[Mu]$.

What other topological structures can appear as minimal sets in (I, f) is an open problem and it is also open the converse problem.

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