Nuclear mapping and a Riemann approach to vector valued integration

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Abstract

The purpose of this paper is to show a natural setting under which we consider vector valued functions integrable in the generalized Riemann sense, using nuclear mappings. The results of this paper contain all of the results published already in [17-22].

K. Kunugi introduced the method of ranked space as a method for the mathematical analysis in 1954 (cf. [6]). Following this, we introduced the notion of ranked union space, and showed that the spaces $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{D}$, $\mathcal{D}'$, etc., occurring in the distribution theory of L. Schwartz [24], can be treated as ranked union spaces called "Ranked (UCs-N) Spaces" in short, without changing the notion of convergence for the countable sequence of points ([9], [11]). As other typical ranked (UCs-N) spaces we know the spaces $C([0,1],\mathcal{S})$, $C([0,\infty),\mathcal{S})$, $C([0,1],\mathcal{D})$, $C([0,\infty),\mathcal{D})$, $C([0,1],\mathcal{S}')$, $C([0,\infty),\mathcal{S}')$, $C([0,1],\mathcal{D}')$, $C([0,\infty),\mathcal{D}')$, etc. ([13]). This paper is concerned with the McShane and Henstock-Kurzweil integrals. We tried a Riemann approach to integration for the functions taking values in a ranked Hilbertian (UCs-N) space endowed with the nuclearity, and knew that the ranked Hilbertian (UCs-N) space endowed with the nuclearity is a natural setting under which we consider the vector valued integration in the generalized Riemann sense. In this paper, we will explain the results obtained without use of the notion of ranked space. Because, we do not necessarily need the theory of ranked space on the study of the Riemann type integration. We refer to the book [1] for the terminology concerning Hilbert space etc.

I. (UCs-N) spaces

1. (UCs-N) spaces. Let X be a vector space over the real numbers, and let $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$ be a family of subspaces X_α of X such that a sequence of semi-norms $\{p^\alpha_m\}_{m=0}^\infty$ is defined on X_α for each α. Suppose that they satisfy the following conditions (I)-(V).

(I) $\cup_{\alpha\in\Sigma}X_\alpha = X$.

(II) Σ is an upward directed set with the ordering $\leq$ (see [27, p. 103] for the definition).

(III) $\alpha \leq \beta$ if and only if $X_\alpha \subset X_\beta$.

(IV) For each $\alpha \in \Sigma$, $p^\alpha_0(x) \leq p^\alpha_1(x) \leq \ldots$ for $x \in X$.

(V) If $\alpha \leq \beta$, then $p^\alpha_m(x) \geq p^\beta_m(x)$, for $x \in X_\alpha$ and $m = 0, 1, 2, \ldots$.

Convergence, Cauchy sequence, Separation axiom

In the space X mentioned in the above, the notion concerned with "convergence" is defined only for the countable sequence of points as follows.

(C_1) A sequence x_i $(i = 1, 2, \ldots)$ is said to be *convergent to* x in X if and only if there exists an α such that x_i $(i = 1, 2, \ldots)$ and x are contained in X_α and the sequence is convergent to x in the space X_α topologized by $\{p^\alpha_m\}_{m=0}^\infty$ (cf. [2, p. 15]).

(C_2) A sequence x_i $(i = 1, 2, \ldots)$ is said to be a *Cauchy sequence* in X if and only if there exists an α such that x_i $(i = 1, 2, \ldots)$ are contained in X_α and the sequence is a Cauchy sequence in the space X_α topologized by $\{p^\alpha_m\}_{m=0}^\infty$.

(C_3) The space X is said to be *separated* if $x = y$ whenever $\lim x_i = x$ and $\lim x_i = y$.

By (C_1) and (C_2), we see that the space X is separated if and only if for every $\alpha \in \Sigma$, the space X_α topologized by $\{p^\alpha_m\}_{m=0}^\infty$ is separated.

Definition 1. If X is a vector space over the real numbers endowed with $(X_\alpha, \{p^\alpha_m\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$ satisfying (I)-(V) and if, on X, convergence, Cauchy sequence and separation axiom are defined by (C_1), (C_2) and (C_3), respectively, then the space X is called a (UCs-N) *space* defined by $(X_\alpha, \{p^\alpha_m\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$ ((UCs-N) standing for "union of countably semi-normed spaces"), and each space $(X_\alpha, \{p^\alpha_m\}_{m=0}^\infty)$ is called a *component space* of the (UCs-N) space X. We sometimes use the same notation $(X_\alpha, \{p^\alpha_m\}_{m=0}^\infty)$ to denote the space X_α topologized by $\{p^\alpha_m\}_{m=0}^\infty$.

In particular:

[1] When Σ is a set consisting of a single element, say α, and $p^\alpha_0(x) = p^\alpha_1(x) = \cdots$, the space X is called a (s-N) *space*. In particular, if the common semi-norm is a norm, the space X is called a (N) *space*. We usually denote

such a (s-N) space X by (X, p) as a vector space endowed with a semi-norm p.

[2] When Σ is a set consisting of a single element, say α, and $p_0^\alpha(x) \leq p_1^\alpha(x) \leq \cdots$, the space X is called a (Cs-N) *space*. In particular, if p_m^α is a norm for each m, then X is called a (CN) *space*. We usually denote such a (Cs-N) space X by $(X, \{p_m\}_{m=0}^\infty)$.

[3] When $p_0^\alpha(x) = p_1^\alpha(x) = \cdots$ for each $\alpha \in \Sigma$, the space X is called a (Us-N) *space*. In particular, if the common semi-norm is a norm, the space X is called a (UN) *space*.

[4] When the space X is a (Us-N) space (resp. (UN) space) such that Σ is a countable set, the space X is called a (CUs-N) *space* (resp. (CUN) *space*).

[5] When, for each $\alpha \in \Sigma$, p_m^α is a norm for $m = 0, 1, 2, \ldots$, the (UCs-N) space X is called a (UCN) *space*.

[6] When the space X is a (UCs-N) space (resp. (UCN) space) such that Σ is a countable set, the space X is called a (CUCs-N) *space* (resp. (CUCN) *space*).

Example 1

The spaces $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{D}$ and $\mathcal{D}'$ can be treated as follows without changing the meaning of convergence for a countable sequence of points.

(1) $\mathcal{S}$: a complete separated (CN) space ([10, (2), p. 198], [11, Proposition 1 p. 355])

(2) $\mathcal{S}'$: a separated (CUN) space with complete component spaces ([10, II, p. 806], [11, Theorem 1, p. 359])

(3) $\mathcal{D}$: a separated (CUCN) space with complete component spaces ([10, p. 195]. [11, Propositions 2 and 4, pp. 356, 357])

(4) $\mathcal{D}'$: a separated (UCs-N) space with complete component spaces ([11, Theorem 2, p. 364], [11, Propositions 2 and 4, pp. 356, 357])

2. (UCs-N) space valued continuous functions. Let X be a (UCs-N) space with component spaces $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$. An X-valued function f defined on $[a, b]$ is said to be *continuous at* $t \in [a, b]$ if there are an $\alpha \in \Sigma$ and a neighborhood $V(t)$ of t in $[a, b]$ such that

(1) the image of $V(t)$ by f is contained in X_α,

(2) f is continuous at t as a mapping from $V(t)$ into the component space $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$, more precisely, if $\lim t_i = t$ in $V(t)$, then $\lim f(t_i) = f(t)$ in the countably semi-normed space X_α topologized by $\{p_m^\alpha\}_{m=0}^\infty$.

An X-valued function f is said to be *continuous on* $[a, b]$ if it is continuous at every point of $[a, b]$.

In general, we do not know whether or not t he following statement (UCs-N,c) is true:

(UCs-N,c) If a (UCs-N) space valued function f is continuous on $[a, b]$, then there exists an α such that the image of $[a, b]$ by f is contained in X_α and f is continuous on $[a, b]$ as a function taking values in the space X_α topologized by $\{p_m^\alpha\}_{m=0}^\infty$.

We have characterized (UCs-N) spaces for which the statement (UCs-N,c) is true (see [14, Propositions 15 and 16, pp. 116, 117]). In particular, we have:

Example 2

In the $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{D}$ and $\mathcal{D}'$-valued cases, the statement (UCs-N,c) is true ([14, p. 117]).

3. (UCs-N) space valued countably additive measures. Let X be a separated (UCs-N) space with component spaces $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$ and $\mathcal{M}$ a ring of sets. A function $F : \mathcal{M} \to X$ is called a *countably additive vector measure* if $F(\emptyset) = 0$ and $F(\cup_{i=1}^\infty A_i) = \lim_{n\to\infty} \sum_{i=1}^n F(A_i)$ for every sequence $\{A_i\}$ of mutually disjoint sets of $\mathcal{M}$ such that $\cup_{i=1}^\infty A_i \in \mathcal{M}$ (cf. [12, p. 807]). In general, we do not know whether or not the following statement (UCs-N,m) is true.

(UCs-N,m) Let $\mathcal{M}$ be a σ-ring of sets and let $F : \mathcal{M} \to X$ be a countably additive vector measure. Then, there exists an α such that the image of $[a, b]$ by F is contained in X_α and F is countably additive as a function taking values in the space X_α topologized by $\{p_m^\alpha\}_{m=0}^\infty$.

It is pointed out that the statement (UCs-N,m) is true if the (UCs-N) space has the property (†) indicated in [12, p. 807] (see [12, Theorem 2, p. 808]). In particular, we have:

Example 3

In the $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{D}$ and $\mathcal{D}'$-valued cases, the statement (UCs-N,m) is true ([12, p. 808]).

The statements in Sections 2 and 3 motivate us to define Bochner, McShane and Henstock-Kurzweil integrals as in Definitions 6 and 7 of Chapter IV below for the functions taking values in the (UCs-N) spaces.

II. Nuclear Hilbertian (UCs-N) spaces

@**Definition 2.** Let X_α and X_β be Hilbert spaces with scalar products $(\ ,\ )_\alpha$ and $(\ ,\ )_\beta$, and p_α and p_β the norms associated with $(\ ,\ )_\alpha$ and $(\ ,\ )_\beta$, respectively. A continuous linear mapping T^α_β of the Hilbert space X_α into the Hilbert space X_β is called a *nuclear mapping* if it has the following form.

$$N(\alpha,\beta): \quad T^\alpha_\beta(x) = \sum_{k=1}^{\infty} \lambda^\alpha_{\beta k}(x, e^\alpha_k)_\alpha e^\beta_k \quad \text{ for } x \in X_\alpha,$$

where $\{e^\alpha_k\}_{k=1}^\infty$ and $\{e^\beta_k\}_{k=1}^\infty$ are orthonormal systems of vectors in X_α and X_β, respectively, $\lambda^\alpha_{\beta k} > 0$ and $\sum_{k=1}^\infty \lambda^\alpha_{\beta k} < \infty$.

Next, we define a (UCs-N) space endowed with nuclearity. We begin by defining a mapping.

Let X be a *separated* (UCs-N) space with complete component spaces $(X_\alpha, \{p^\alpha_m\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$ such that, on each X_α, for $m = 0, 1, 2, \ldots$ a positive hermitian form $(\ ,\ )^\alpha_m$ is defined and p^α_m is the semi-norm associated with $(\ ,\ )^\alpha_m$.

Let $\alpha \leq \beta$ and $m \geq n$, and let T^α_β be the *embedding mapping* of X_α into X_β, i.e., $T^\alpha_\beta(x) = x$ for $x \in X_\alpha$. Since then X is a (UCs-N) space, we have $X_\alpha \subset X_\beta$ and $p^\alpha_m(x) \geq p^\beta_n(x)$ for $x \in X_\alpha$. Therefore

$$p^\alpha_m(x) \geq p^\beta_n(T^\alpha_\beta(x)) \text{ for } x \in X_\alpha.$$

Now, consider the quotient spaces $X_\alpha/N(\alpha, m)$ and $X_\beta/N(\beta, n)$, where $N(\alpha, m) = \{x \in X_\alpha : p^\alpha_m(x) = 0\}$ and $N(\beta, n) = \{x \in X_\beta : p^\beta_n(x) = 0\}$. Then, the quotient spaces $X/N(\alpha, m)$ and $X/N(\beta, n)$, denoted by X^α_m and X^β_n, become prehilbert spaces with non-degenerate positive hermitian forms, still denoted by $(\ ,\ )^\alpha_m$ and $(\ ,\ )^\beta_n$. We denote the norm associated with $(\ ,\ )^\alpha_m$ (resp. $(\ ,\ )^\beta_n$) by p^α_m (resp. p^β_n) and the element of X^α_m (resp. X^β_n) having $x \in X_\alpha$ (resp. X_β) as a representative by $[x]^\alpha_m$ (resp. $[x]^\beta_n$). Let $T^{\alpha m}_{\beta n}$ be the mapping of the prehilbert space X^α_m into the prehilbert space X^β_n defined as follows.

$$T^{\alpha n}_{\beta n}([x]^\alpha_m) = [x]^\beta_n.$$

Then

$$p^\alpha_m([x]^\alpha_m) \geq p^\beta_n(T^{\alpha m}_{\beta n}([x]^\alpha_m)) \text{ for } x^\alpha_m \in X^\alpha_m,$$

and the mapping $T^{\alpha m}_{\beta n}$ is a continuous linear mapping of X^α_m into X^β_n. Next, we denote the completions of prehilbert spaces X^α_m and X^β_n with respect to p_m^α and p_n^β by $\hat{X}_m^\alpha$ and $\hat{X}_n^\beta$, respectively. Then, they become Hilbert spaces. If $\{[x_i]_m^\alpha\}_{i=1}^\infty$ is a Cauchy sequence in X_m^α, then $\{[x_i]_n^\beta\}_{i=1}^\infty$ is a Cauchy sequence in X_n^β. Hence, the element of $\hat{X}_n^\beta$ having the Cauchy sequence $\{[x_i]_n^\beta\}_{i=1}^\infty$ as a representative is uniquely determined by the element of $\hat{X}_m^\alpha$ having the Cauchy sequence $\{[x_i]_m^\alpha\}_{i=1}^\infty$ as a representative. We denote the correspondence by $\hat{T}_{\beta n}^{\alpha m}$. Then, $\hat{T}_{\beta n}^{\alpha m}$ is a continuous linear mapping of $\hat{X}_m^\alpha$ into $\hat{X}_n^\beta$ such that

$$\hat{p}_m^\alpha(\hat{x}_m^\alpha) \geq \hat{p}_n^\beta(\hat{T}_{\beta n}^{\alpha m}(\hat{x}_m^\alpha)) \text{ for } \hat{x}_m^\alpha \in \hat{X}_m^\alpha,$$

where $\hat{p}_m^\alpha$ and $\hat{p}_n^\beta$ are the norms associated with the scalar products on $\hat{X}_m^\alpha$ and $\hat{X}_n^\beta$, respectively.

Definition 3. Let X be a *separated* (UCs-N) space with *complete* component spaces $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$ such that, on each X_α, for $m = 0, 1, 2, \ldots$, a positive hermitian form $(\ , \ )_m^\alpha$ is defined and p_m^α is the semi-norm associated with $(\ , \ )_m^\alpha$. Suppose that, corresponding to each $\alpha \in \Sigma$, we can find

(†) a $\beta \in \Sigma$ and increasing sequences of non-negative integers $\{m(0) < m(1) < \cdots\}$ and $\{n(0) < n(1) < \cdots\}$ such that:

(1) $\alpha \leq \beta$,

(2) $m(i) \geq n(i)$ for $i = 0, 1, 2, \ldots$, and

(3) $\hat{T}_{\beta n(i)}^{\alpha m(i)}$ is nuclear for $i = 0, 1, 2, \ldots$, where $\hat{T}_{\beta n(i)}^{\alpha m(i)}$ is the continuous linear mapping of $\hat{X}_{m(i)}^\alpha$ into $\hat{X}_{n(i)}^\beta$ defined in the above.

Then we call such a space X a *nuclear Hilbertian* (UCs-N) *space*. β, $\{m(i)\}$, $\{n(i)\}$ is a system associated with α. We have:

Proposition 1. *If $m \geq m(i)$ and $n(i) \geq n$ for some i, then $\hat{T}_{\beta n}^{\alpha m}$ is a nuclear mapping of $\hat{X}_m^\alpha$ into $\hat{X}_n^\beta$.*

Example 4

(a) *Nuclear Hilbertian* (CN) *space*. Let X be a vector space endowed with a sequence of norms $\{p_m\}_{m=0}^\infty$. Suppose that the sequence $\{p_m\}_{m=0}^\infty$ is non-decreasing and all of the norms p_m are compatible (see [2, p. 13] for the definition). Completing the space X by p_m , we obtain a Banach space. We denote the Banach space by $\hat{X}_m$. In this case, the space $\hat{X}_m$ $(m = 0, 1, 2, \ldots)$ can be considered to have the following relationship with each other and with X.

$$\hat{X}_0 \supset \hat{X}_1 \supset \cdots \supset X.$$

The topological space X topologized by such a $\{p_m\}_{m=0}^{\infty}$ has been called a *countably normed space* [2, p. 15]. If the space is complete, then we have $X = \cap_{m=0}^{\infty} \hat{X}_m$ ([2, p. 17]).

Now, consider a vector space X endowed with a sequence of nondegenerate positive hermitian forms $(\ ,\ )_m$ $(m = 0, 1, 2, \dots)$. Let p_m be the norm associated with $(\ ,\ )_m$. In this case, the space $\hat{X}_m$ defined as in the above becomes a Hilbert space. Suppose that the sequence $\{p_m\}_{m=0}^{\infty}$ is non-decreasing and compatible (so the space $(X, \{p_m\}_{m=0}^{\infty})$ is a countably normed space) and the space $(X, \{p_m\}_{m=0}^{\infty})$ is complete. Then, the countably normed space $(X, \{p_m\}_{m=0}^{\infty})$ has been called a *nuclear countably Hilbert space* if for $m > n$, the embedding $\hat{T}_n^m$ of the Hilbert space $\hat{X}_m$ to the Hilbert space $\hat{X}_n$ is a nuclear mapping [3, p. 62]. A nuclear countably Hilbert space is an example of vector space which can be considered as a nuclear Hilbertian (CN) space. The space $\mathcal{S}$ is such a typical space ([18, p. 331]).

(b) *Nuclear Hilbertian* (CUCN) *space such that each component space is a nuclear Hilbertian* (CN) *space.* The space $\mathcal{D}$ is a nuclear Hilbertian (CUCN) space such that each component space is a nuclear countably Hilbert space. Further, in the case of space $\mathcal{D}$, there is a sequence of non-degenerate positive hermitian forms $(\ ,\ )_m$ $(m = 0, 1, 2, \dots)$ defined on $\mathcal{D}$ such that in each component space X_n $(n = 1, 2, \dots)$, we have $(x, y)_m^n = (x, y)_m$ for $x, y \in X_n$ and $m = 0, 1, 2, \dots$ ([18, p. 332]).

(c) *Nuclear Hilbertian* (CUN) *space such that each component space is a Hilbert space.* Each component space X_n $(n = 1, 2, \dots)$ is a Hilbert space endowed with a non-degenerate positive hermitian form $(\ ,\ )_n$ for each n. In this case, for every n there exists an $m > n$ such that the embedding of X_n into X_m is a nuclear mapping. The space $\mathcal{S}'$ is such a typical space ([18, p. 333]).

(d) *Nuclear Hilbertian* (UCs-N) *space such that* Σ *is uncountable and each component space is not a nuclear Hilbertian* (Cs-N) *space.* The space $\mathcal{D}'$ is such a typical space ([18, p. 334]).

III. Bochner, McShane and Henstock-Kurzweil Integrals and differentiations of (Cs-N) space valued functions

Consider a (Cs-N) space $(X, \{p_m\}_{m=0}^{\infty})$ (a (UCs-N) space with a single component space). Then, $\{p_m\}_{m=0}^{\infty}$ is a sequence of seminorms such that $p_0(x) \le p_1(x) \le \cdots$ for $x \in X$. Put $N(m) = \{x \in X : p_m(x) = 0\}$. Then, the quotient space $X/N(m)$ is a normed space. We denote the element of the quotient space with $x \in X$ as a representative by $[x]_m$. We denote the completion of the normed space $X/N(m)$ by $(\hat{X}_m, \hat{p}_m)$, where $\hat{p}_m$ denotes the norm on $\hat{X}_m$. In particular, we denote the element of $\hat{X}_m$ with a Cauchy sequence $\{[x]_m, [x]_m, \ldots\}$ $(x \in X)$ as a representative by $\{[x]_m\}\hat{}$. For an X-valued function f, we define $\hat{X}_m$-valued function $\hat{f}_m$ by $\hat{f}_m(t) = \{[f(t)]_m\}\hat{}$.

1. Bochner integral of (Cs-N) space valued function.

Let $(X, \{p_m\}_{m=0}^{\infty})$ be a separated (Cs-N) space, and let $(S, \mathcal{M}, \mu)$ be a non-negative finite valued measure space such that S is an abstract set, $\mathcal{M}$ is a σ-algebra of subsets of S, and μ is a real valued non-negative countably additive measure on $\mathcal{M}$. An X-valued function f on S is called *simple* if there exist $x(1), x(2), \ldots, x(n) \in X$ and $A(1), A(2), \ldots, A(n) \in \mathcal{M}$ such that $f = \sum_{i=1}^{n} x(i)\chi_{A(i)}$, where $\chi_{A(i)}(x) = 1$ if $x \in A(i)$ and $\chi_{A(i)}(x) = 0$ if $x \notin A(i)$. An X-valued function f on S is called *μ-measurable* if there exists a sequence of X-valued simple functions $\{f_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ μ-almost everywhere on S.

Definition 4.1. Let X be a separated (Cs-N) space $(X, \{p_m\}_{m=0}^{\infty})$. An X-valued function f on S is said to be *Bochner integrable* with respect to μ on S if f is μ-measurable and there exists a sequence of X-valued simple functions $\{f_n\}_{n=1}^{\infty}$ such that

(1) $\lim_{n\to\infty} \int_S p_m(f_n(t) - f(t))d\mu = 0$ for $m = 0, 1, 2, \ldots$,

(2) $\lim_{n\to\infty} \int_S f_n(t)d\mu$ exists in $(X, \{p_m\}_{m=0}^{\infty})$.

In this case, the integral of f on S, written $\int_S f(t)d\mu$, is defined by $\lim_{n\to\infty} \int_S f_n(t)d\mu$.

In the definition above, if the space $(X, \{p_m\}_{m=0}^{\infty})$ is complete, $\lim_{n\to\infty} \int_S f_n(t)d\mu$ necessarily exists. Hence, we obtain the following definition.

Definition 4.2. Let X be a separated and complete (Cs-N) space $(X, \{p_m\}_{m=0}^{\infty})$. An X-valued function f on S is said to be *Bochner integrable* with respect to μ on S if f is μ-measurable and there exists a sequence of X-valued simple functions $\{f_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \int_S p_m(f_n(t) - f(t))d\mu = 0$ for $m = 0, 1, 2, \ldots$. In this case, the integral of f with respect to μ on S is defined by $\lim_{n\to\infty} \int_S f_n(t)d\mu$.

Proposition 2. ([19, (0.20), p. 373]). *Let $(X, \{p_m\}_{m=0}^{\infty})$ be a separated* *complete* (Cs-N) *space. Let f be an X-valued function on S. Then, the function f is Bochner integrable with respect to μ on S as an $(X, \{p_m\}_{m=0}^{\infty})$-valued function if and only if for $m = 0, 1, 2, \ldots$ the function $\hat{f}_m$ is Bochner integrable with respect to μ on S as an $(\hat{X}_m, \hat{p}_m)$-valued function. In this case, $\{[\int_a^b f(t)d\mu]_m\}\hat{} = \int_a^b \hat{f}_m(t)d\mu$ for $m = 0, 1, 2, \ldots$.*

2. McShane and Henstock-Kurzweil integrals of (Cs-N) space valued function.

Definition 5. Let X be a separated (Cs-N) space $(X, \{p_m\}_{m=0}^{\infty})$, ν a non-decreasing real valued function on $[a, b]$ and f an X-valued function on $[a, b]$. The function f is said to be *McShane* (resp. *Henstock-Kurzweil*) *integrable with respect to* ν to $z(\in X)$ on $[a, b]$ if for $m = 0, 1, 2, \ldots$ there exists a positive function δ_m on $[a, b]$ such that for any δ_m-fine Lebesgue (resp. Perron) partition P of $[a, b]$: $\{([t_0, t_1], x_1), \ldots, ([t_{p-1}, t_p], x_p)\}$, we have

$$p_m(\sum_{i=1}^{p} f(x_i)(\nu(t_i) - \nu(t_{i-1})) - z) < 1/2^m.$$

Such a z is uniquely determined ([18, Lemma 4, p. 320]). In this case, the integral of f with respect to ν on [a,b], written $\int_a^b f(t)d\nu$, is z. Simply, we set

$$\sigma(f, P; \nu) = \sum_{i=1}^{p} f(x_i)(\nu(t_i) - \nu(t_{i-1})).$$

Proposition 3. *Let X be a separated complete* (Cs-N) *space $(X, \{p_m\}_{m=0}^{\infty})$. Let f be an X-valued function on $[a, b]$. Then, the function f is McShane* (*resp. Henstock-Kurzweil*) *integrable with respect to ν on $[a, b]$ as an $(X, \{p_m\}_{m=0}^{\infty})$-valued function if and only if for $m = 0, 1, 2, \ldots$, the function $\hat{f}_m$ is McShane* (*resp. Henstock-Kurzweil*) *integrable with respect to ν on $[a, b]$ as an $(\hat{X}_m, \hat{p}_m)$-valued function. In this case, $\{[\int_a^b f(t)d\nu]_m\}\hat{} = \int_a^b \hat{f}_m(t)d\nu$ for $m = 0, 1, 2, \ldots$.*

Proof. We prove only for the case of Henstock-Kurzweil integral. The "only if" part follows from [18, Lemma 6, p. 322]. The "if" part: Suppose that $\hat{f}_m$ is Henstock-Kurzweil integrable with respect to ν to an element of $\hat{X}_m$, say $\hat{z}_m$, on $[a, b]$ as an $(\hat{X}_m, \hat{p}_m)$-valued function. Then, there exists a positive function δ_m on $[a, b]$ such that for all δ_m-fine Perron partitions P of $[a, b]$ we have

$$\hat{p}_m(\sigma(\hat{f}_m, P; \nu) - \hat{z}_m) < 1/2^{m+1}.$$

Since the set $\{\{[x]_m\}\hat{}: x \in X\}$ is dense in $(\hat{X}_m, \hat{p}_m)$, there exists an $x_m \in X$ such that

$$\hat{p}_m(\hat{z}_m - \{[x_m]_m\}\hat{}\,) < 1/2^{m+1}.$$

Therefore, for all δ_m-fine Perron partition P of $[a,b]$ we have

$$\hat{p}_m(\sigma(\hat{f}_m, P; \nu) - \{[x_m]_m\}\hat{}\,) < 1/2^m.$$

Now, fix a non-negative integer n and let $m \geq n$. Then, we have

$$\hat{p}_n(\sigma(\hat{f}_n, P; \nu) - \{[x_m]_n\}\hat{}\,) \leq \hat{p}_m(\sigma(\hat{f}_m, P; \nu) - \{[x_m]_m\}\hat{}\,) < 1/2^m$$

for all δ_m-fine Perron partitions P of $[a,b]$. Hence, by Cousin's lemma, if $k, m \geq n$,

$$\hat{p}_n(\{[x_k]_n\}\hat{} - \{[x_m]_n\}\hat{}\,) < 1/2^k + 1/2^m,$$

and therefore

$$p_n(x_k - x_m) < 1/2^k + 1/2^m.$$

Hence, the sequence $\{x_m\}_{m=0}^{\infty}$ in X is a Cauchy sequence under the semi-norm p_n for $n = 0, 1, 2, \ldots$. Therefore, by (C$_2$) the sequence is a Cauchy sequence in the space $(X, \{p_m\}_{m=0}^{\infty})$. Since the space is complete, the limit $\lim_{m\to\infty} x_m = x$ exists in $(X, \{p_m\}_{m=0}^{\infty})$. Now, we have

$$p_n(x_{n+1} - x) \leq 1/2^{n+1}.$$

On the other hand, for all δ_{n+1}-fine Perron partitions P of $[a,b]$, where δ_{n+1} is the one chosen in the above, we have

$$p_n(\sigma(f, P; \nu) - x_{n+1}) < 1/2^{n+1}.$$

Consequently, for all δ_{n+1}-fine Perron partitions P of $[a,b]$ we have

$$p_n(\sigma(f, P; \nu) - x) < 1/2^n.$$

Thus, f is Henstock-Kurzweil integrable with respect to ν on $[a,b]$ to x as an $(X, \{p_m\}_{m=0}^{\infty})$-valued function.

3. Differentiation of (Cs-N) space valued function. Let X be a separated (Cs-N) space $(X, \{p_m\}_{m=0}^{\infty})$ and f an X-valued function on $[a,b]$. Let $t \in [a,b]$. If there exists an element $x \in X$ such that for $m = 0, 1, 2, \ldots$, there exists $\delta_m > 0$ such that $p_m((f(s) - f(t))/(s-t) - x) < 1/2^m$ whenever

$0 < |s-t| < \delta_m$ $(s \in [a,b])$, then the function f is said to be *differentiable at* t . Since X is separated, such an element x is uniquely determined, and we denote the element x by $f'(t)$.

Proposition 4. *Let X be a separated complete* (Cs-N) *space* $(X, \{p_m\}_{m=0}^{\infty})$ *and f an X-valued function defined on $[a,b]$. Then, the function f is differentiable at $t \in [a,b]$ if and only if, for $m = 0,1,2,\ldots$, $\hat{f}_m$ is differentiable at t as an $(\hat{X}_m, \hat{p}_m)$-valued function. In this case, we have $(\hat{f}_m)'(t) = \{[f'(t)]_m\}\hat{}$ for $m = 0,1,2,\ldots$.*

Proof. The "only if" part is clear. The "if" part: For a given non-negative integer m, there exists $\hat{x}_m \in \hat{X}_m$ such that we can find a $\delta(m) > 0$ so that $\hat{p}_m((\hat{f}_m(s) - \hat{f}_m(t))/(s-t) - \hat{x}_m) < 1/2^m$ whenever $0 < |s-t| < \delta(m)$ $(s \in [a,b])$. Since the set $\{\{[x]_m\}\hat{}: x \in X\}$ is dense in $(\hat{X}_m, \hat{p}_m)$, there exists an $x_m \in X$ such that

$$\hat{p}_m((\hat{f}_m(s)-\hat{f}_m(t))/(s-t)-\{[x_m]_m\}\hat{}) < 1/2^m \text{ if } 0 < |s-t| < \delta(m)\ (s \in [a,b]).$$

Hence, we have

$$p_m((f(s)-f(t))/(s-t)-x_m) < 1/2^m \text{ if } 0 < |s-t| < \delta(m)\ (s \in [a,b]).$$

Therefore, for any $n \geq m$, we have

$$p_m((f(s)-f(t))/(s-t)-x_n) < 1/2^n \text{ if } 0 < |s-t| < \delta(n)\ (s \in [a,b]),$$

because $p_n(x) \geq p_m(x)$ in X. Fix a non-negative integer m. Given $\varepsilon > 0$, take an $n(\varepsilon)$ with $n(\varepsilon) > m$ and $1/2^{n(\varepsilon)} < \varepsilon/2$. Then, for h and k with $h,k \geq n(\varepsilon)$ we have $p_m(x_h - x_k) < \varepsilon$. Hence, $\{x_k\}_{k=0}^{\infty}$ is a fundamental sequence in (X, p_m) for $m = 0,1,2,\ldots$, and therefore in $(X, \{p_m\}_{m=0}^{\infty})$. Since the space $(X, \{p_m\}_{m=0}^{\infty})$ is complete, the limit $\lim x_k = x$ exists in $(X, \{p_m\}_{m=0}^{\infty})$. Now, for $m = 0,1,2,\ldots$, take an $n = n(m)$ such that $n > m$ and $p_m(x_n - x) < 1/2^{m+1}$, then

$$\begin{aligned} & p_m((f(s)-f(t))/(s-t)-x) \\ \leq\ & p_m((f(s)-f(t))/(s-t)-x_n) + p_m(x_n - x) < 1/2^m \end{aligned}$$

if $0 < |s-t| < \delta(n)$ $(s \in [a,b])$. Thus, f is differentiable at t as an $(X, \{p_m\}_{m=0}^{\infty})$-valued function, and $f'(t) = x$. The last part of the proposition is clear from the demonstration above.

IV. (UCs-N) space valued Bochner, McShane and Henstock-Kurzweil integrals

1. (UCs-N) space valued Bochner integrals.

Definition 6. Let X be a separated (UCs-N) space with component spaces $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$ and let $(S, \mathcal{M}, \mu)$ be a non-negative finite valued measure space. We say an X-valued function f on S is *Bochner integrable with respect to* μ to z on S if there exists a component space X_α such that:

(1) The image of S by f is contained in X_α and $z \in X_\alpha$.

(2) The function f is Bochner integrable with respect to μ on S to z as an $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$-valued function.

When it is necessary to indicate explicitly such a space X_α , we say the function f is *Bochner integrable*(X_α) *with respect to* μ. Definition 6 is well defined by the facts that Σ is upward directed, and if f is Bochner integrable(X_α) with respect to μ to z on S, then f is Bochner integrable(X_β) with respect to μ to the same element z on S for every β with $\beta \geq \alpha$. In particular, we say an X-valued function f on S is μ*-measurable*(X_α) on S if the image of S by f is contained in X_α and f is μ-measurable on S as an $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$-valued function. (Cf. [26, Definition 5, p. 75] for example.)

When $([a,b], \mathcal{M}, \mu)$ is a measure space such that $\mathcal{M}$ is the family of Lebesgue measurable subsets of $[a,b]$ and μ is Lebesgue measure on $\mathcal{M}$, we say f is *Bochner integrable with respect to Lebesgue measure*, or simply, *Bochner integrable*, or *measurable*, etc. omitting "with respect to Lebesgue measure".

Proposition 5. *Let X be a separated* (UCs-N) *space. Then, any simple function is Bochner integrable with respect to μ on S.*

Proposition 6. ([14, Proposition 21, p. 120], [18, p. 324]). *If X is a complete separated* (Cs-N) *space, then any continuous X-valued function on $[a,b]$ is Bochner integrable on $[a,b]$.*

Proposition 7. *Let X be a separated* (UCs-N) *space with complete component spaces for which the statement* (UCs-N,c) *is true. Then, any continuous X-valued function on $[a,b]$ is Bochner integrable on $[a,b]$.*

2. (UCs-N) space valued McShane and Henstock-Kurzweil integrals.

Definition 7. Let X be a separated (UCs-N) space with component spaces $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$ and ν a non-decreasing real valued function on $[a, b]$. An X- valued function f on $[a, b]$ is said to be *McShane* (resp. *Henstock-Kurzweil*) *integrable with respect to* ν to z on $[a, b]$ if there exists a component space X_α such that:

(1) The image of $[a, b]$ by f is contained in X_α and $z \in X_\alpha$.

(2) The function f is McShane (resp. Henstock-Kurzweil) integrable with respect to ν to z on $[a, b]$ as an $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$-valued function.

When it is necessary to indicate explicitly such an X_α, we say f is *McShane* (resp. *Henstock-Kurzweil*) *integrable*(X_α) *with respect to* ν. Definition 7 is well defined by the same reason as in Definition 6. In particular, when ν is the identity function on the real space R, i.e., $\nu(t) = t$ on R, we simply say f is *McShane* (resp. *Henstock-Kurzweil*) *integrable*, omitting "with respect to ν".

Proposition 8. ([22, Proposition 13, p. 268]). *Let X be a separated* (UCs-N) *space. If an X-valued function f is Bochner integrable on $[a, b]$, then it is McShane integrable on $[a, b]$, and both integrals coincide.*

Proposition 9. *Let X be a separated* (UCs-N) *space. A simple X-valued function on $[a, b]$ is Bochner and McShane integrable on $[a, b]$ and both integrals coincide.*

Proposition 10. *Let X be a complete separated* (Cs-N) *space. Any continuous X-valued function on $[a, b]$ is Bochner and McShane integrable on $[a, b]$ and both integrals coincide.*

Proposition 11. *Let X be a separated* (UCs-N) *space with complete component spaces for which the statement* (UCs-N,c) *is true. Then, any continuous X-valued function on $[a, b]$ is Bochner and McShane integrable on $[a, b]$ and both integrals coincide.*

Example 5

By Example 2, $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{D}$ and $\mathcal{D}'$-valued continuous functions are Bochner and McShane integrable on $[a, b]$ and both integrals coincide.

V. McShane and Henstock-Kurzweil integrals for functions taking values in a nuclear Hilbertian (UCs-N) space

We first show the following two key lemmas which are pivotal in our theory.

Key Lemma 1. *Let X be a Hilbert space with a scalar product $(\ ,\ )$, and p the norm associated with $(\ ,\ )$. Let $z_i \in X$ and let a_i be real numbers for $i = 1, 2, \ldots, q$, and $x \in X$. Then*

$$|\sum_{i=1}^{q}(a_i z_i, x)| \leq p(\sum_{i=1}^{q} a_i z_i)p(x).$$

For the converse of Key Lemma 1, we have the following idea.

Key Lemma 2. *Let X_α and X_β be Hilbert spaces with scalar products $(\ ,\ )_\alpha$ and $(\ ,\ )_\beta$, and p_α and p_β the norms associated with $(\ ,\ )_\alpha$ and $(\ ,\ )_\beta$, respectively. Let T_β^α be a nuclear mapping of the space X_α into X_β , i.e., a mapping having the form:*

$$N(\alpha, \beta): \quad T_\beta^\alpha(x) = \sum_{k=1}^{\infty} \lambda_{\beta k}^\alpha (x, e_k^\alpha)_\alpha e_k^\beta \quad \text{for } x \in X_\alpha,$$

where $\{e_k^\alpha\}_{k=1}^\infty$ and $\{e_k^\beta\}_{k=1}^\infty$ are orthonormal systems of vectors in X_α and X_β, respectively, $\lambda_{\beta k}^\alpha > 0$, and $\sum_{k=1}^\infty \lambda_{\beta k}^\alpha < \infty$.
Then, the following inequality holds for any finite sequence $z_i \in X_\alpha$ $(i = 1, 2, \ldots, p)$.

$$\sum_{i=1}^{p} p_\beta(T_\beta^\alpha(z_i)) \leq \sum_{k=1}^{\infty} \lambda_{\beta k}^\alpha \{\sum_{i=1}^{p} |(z_i, e_k^\alpha)_\alpha|\}.$$

For example, we sketch here the proof of the following lemma, because it is a typical way of proofs of propositions in the nuclear space valued case.

Lemma 1. *Under the same assumption as in Key Lemma 2, the following holds. Let f be an X_α -valued McShane (resp. Henstock-Kurzweil) integrable function on $[a, b]$, and let F be an indefinite integral of f. If δ is a positive function on $[a, b]$ such that*

$$p_\alpha(\sum_{i=1}^{p} f(x_i)(t_i - t_{i-1}) - F([t_{i-1}, t_i])) < \varepsilon/(3\sum_{k=1}^{\infty} \lambda_{\beta k}^\alpha)$$

for any δ-fine Lebesgue (resp. Perron) partition

$$\{([t_0, t_1], x_1), \ldots, ([t_{p-1}, t_p], x_p)\}$$

of $[a,b]$, then for the same δ we have

$$\sum_{i=1}^{p} p_\beta(T^\alpha_\beta(f(x_i))(t_i - t_{i-1}) - T^\alpha_\beta(F([t_{i-1}, t_i]))) < \varepsilon$$

for any δ-fine Lebesgue (resp. Perron) partition

$$\{([t_0, t_1], x_1), \ldots, ([t_{p-1}, t_p], x_p)\}$$

of $[a,b]$.

Proof. First we use the idea of *Key Lemma 1.* Put

$$z_i = f(x_i)(t_i - t_{i-1}) - F([t_{i-1}, t_i]) \text{ for } i = 1, 2, \ldots, p.$$

For $k = 1, 2, \ldots$, we have

$$\begin{array}{ll} & |\sum_{i=1}^{p}((f(x_i), e^\alpha_k)_\alpha(t_i - t_{i-1}) - (F([t_{i-1}, t_i]), e^\alpha_k)_\alpha| \\ < & p_\alpha(\sum_{i=1}^{p} f(x_i)(t_i - t_{i-1}) - F([t_{i-1}, t_i]))\ (\textit{by Key Lemma 1}) \\ < & \varepsilon/(3\sum_{k=1}^{\infty}\lambda^\alpha_{\beta k}). \end{array}$$

Using the property in the real valued case (which appears in the proof of *Saks-Henstock lemma in the real valued case*), we have

$$\sum_{i=1}^{p} |(f(x_i), e^\alpha_k)_\alpha(t_i - t_{i-1}) - (F([t_{i-1}, t_i]), e^\alpha_k)_\alpha| < \varepsilon/(\sum_{k=1}^{\infty}\lambda^\alpha_{\beta k}).$$

Hence, using *Key Lemma 2*, we have

$$\begin{array}{ll} & \sum_{i=1}^{p} p_\beta(T^\alpha_\beta(f(x_i))(t_i - t_{i-1}) - T^\alpha_\beta(F([t_{i-1}, t_i]))) \\ \leq & \sum_{k=1}^{\infty}\lambda^\alpha_{\beta k}\{\sum_{i=1}^{p}|(f(x_i), e^\alpha_k)_\alpha(t_i - t_{i-1}) - (F([t_{i-1}, t_i]), e^\alpha_k)_\alpha|\} \\ < & \varepsilon. \end{array}$$

VI. Nuclear Hilbertian (UCs-N) space valued McShane and Henstock-Kurzweil integrals

In what follows, unless mentioned otherwise X denotes a nuclear Hilbertian (UCs-N) space with component spaces $(X_\alpha, \{p^\alpha_m\}_{m=0}^{\infty})$ $(\alpha \in \Sigma)$ which is defined in Definition 3. The following propositions are proved. The idea of the proof is analogous to that of the corresponding proposition in Section 3 of [19] except Proposition 25. The idea of the proof of Proposition 25 is analogous to that of Proposition 12 of [21].

Proposition 13 (Saks-Henstock lemma). *Let f be an X-valued McShane (resp. Henstock-Kurzweil) integrable function on $[a,b]$, and F an indefinite integral of f. Then, there exists an $\alpha \in \Sigma$ such that the function f is McShane (resp. Henstock-Kurzweil) integrable(X_α) and for $m = 0, 1, 2, \dots$, there exists a positive function δ_m on $[a,b]$ such that for any δ_m-fine Lebesgue (resp. Perron) partition $\{([c_1,d_1],x_1),\dots,([c_p,d_p],x_p)\}$ in $[a,b]$ we have*

$$\sum_{i=1}^{p} p_m^\alpha(f(x_i)(d_i - c_i) - F([c_i,d_i])) < 1/2^m.$$

Proposition 14. *If an X-valued function f is Henstock-Kurzweil integrable on $[a,b]$, then there exists an $\alpha \in \Sigma$ such that f is measurable(X_α) on $[a,b]$.*

Proposition 15. *An X-valued function f is McShane integrable on $[a,b]$ if and only if it is Bochner integrable on $[a,b]$, and both integrals coincide.*

Proposition 16. *Let f be an X-valued Henstock-Kurzweil integrable function on $[a,b]$. Then, there exists an $\alpha \in \Sigma$ such that if $\beta \geq \alpha$, for $m = 0, 1, 2, \dots$, $p_m^\beta(f(t))$ is measurable on $[a,b]$ and*

$$p_m^\beta\Big(\int_c^d f(t)dt\Big) \leq \int_c^d p_m^\beta(f(t))dt \quad \textit{for any } [c,d] \subset [a,b].$$

A *division* of an interval $[a,b]$ is a finite collection of non-overlapping intervals whose union is $[a,b]$. Let X be a semi-normed space (X,p) with a semi-norm p, and let F be an X-valued function. The *variation* of F on $[a,b]$ is the extended real number

$$V(F,[a,b]) = \sup\Big(\sum_{i=1}^{p} p(F([t_{i-1},t_i]))\Big),$$

where the supremum is taken over all divisions $\{[t_0,t_1],[t_1,t_2],\dots,[t_{p-1},t_p]\}$ of $[a,b]$. If $V(F,[a,b]) < \infty$, we say that F is of *bounded variation* on $[a,b]$. When the space X is a (Cs-N) space $(X,\{p_m\}_{m=0}^\infty)$, we say that an X-valued function F is of *bounded variation* on $[a,b]$ if F is of bounded variation on $[a,b]$ as an (X,p_m)-valued function for $m = 0,1,2,\dots$. When X is a (UCs-N) space with component spaces $(X_\alpha,\{p_m^\alpha\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$, we say that an X-valued function F is of *bounded variation* on $[a,b]$ if there exists an α such that the image of $[a,b]$ by F is contained in X_α and F is of bounded variation on $[a,b]$ as an $(X_\alpha,\{p_m^\alpha\}_{m=0}^\infty)$-valued function.

Proposition 17. *Let f be an X-valued Henstock-Kurzweil integrable function on $[a,b]$ and F an indefinite integral of f. Then, f is McShane integrable on $[a,b]$ if and only if F is of bounded variation on $[a,b]$. In this case, there exists an α such that for $m = 0,1,2,\dots$, $p_m^\alpha(f(t))$ is McShane integrable on $[a,b]$ and $\int_a^b p_m^\alpha(f(t))dt$ coincides with the variation on $[a,b]$ of F treated as an (X_α, p_m^α)-valued function.*

1. Fundamental theorem of calculus

Let X be a separated (UCs-N) space with component spaces $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$ and let f be an X-valued function defined on $[a,b]$. Let $t \in [a,b]$. If there are an α and a neighborhood $V(t)$ of t in $[a,b]$ such that the image of $V(t)$ by f is contained in X_α and f is differentiable at t as an $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$-valued function, then f is said to be *differentiable at* t, or *differentiable*(X_α) *at* t if necessary(see Chapter III, 3).

Proposition 18. *Let f be an X-valued Henstock-Kurzweil integrable function on $[a,b]$ and let F be an indefinite integral of f. Then, there exists an α such that F is differentiable(X_α) almost everywhere on $[a,b]$ and $F'(t) = f(t)$.*

Proposition 19. *Let F be an X-valued function on $[a,b]$. Suppose that there exists an α such that the image of $[a,b]$ by F is contained in X_α, and F is continuous on $[a,b]$ as an $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$-valued function and differentiable(X_α) except possibly at a countable set $D \subset [a,b]$. Then, the function $F'(t)$ defined to be t he derivative of F at t if $t \in [a,b] - D$ and an arbitrary element of X_α if $t \in D$ is Henstock-Kurzweil integrable on $[a,b]$ and $\int_a^b F'(t)dt = F(b) - F(a)$.*

2. Descriptive definition

Let X be a (Cs-N) space $(X, \{p_m\}_{m=0}^\infty)$. An X-valued function F on $[a,b]$ is called AC_* if, given a negligible set E in $[a,b]$ (i.e. a set E of measure zero) and a non-negative integer m, there exists a positive function δ_m on E such that $\sum_{i=1}^p p_m(F([c_i,d_i]) < 1/2^m$ for any Perron partition $\{([c_1,d_1],x_1),\dots,([c_p,d_p],x_p)\}$ in $[a,b]$ (anchored in E) that is δ_m-fine. When X is a (UCs-N) space with component spaces $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$ $(\alpha \in \Sigma)$, an X-valued function F is called AC_*, or $AC_*(X_\alpha)$ if necessary, if there exists an α such that the image of $[a,b]$ by F is contained in X_α and F is AC$_*$ on $[a,b]$ as an $(X_\alpha, \{p_m^\alpha\}_{m=0}^\infty)$-valued function.

Proposition 20. *Let F be an X-valued function on $[a, b]$. Suppose that there exists an α such that the image of $[a, b]$ by F is contained in X_α and F is differentiable(X_α) almost everywhere on $[a, b]$. Let F' be an X_α-valued function on $[a, b]$ defined to be the derivative of F except for a set A of measure zero. Then, the following holds.*

(1) *If F' is Henstock-Kurzweil integrable on $[a, b]$ and F is an indefinite integral of F', then the function F is AC_* on $[a, b]$.*

(2) *If F is AC_* on $[a, b]$, then F' is Henstock-Kurzweil integrable on $[a, b]$ and F is an indefinite integral of F'.*

Proposition 21 (Descriptive definition)**.** *Let f be an X-valued function on $[a, b]$. The function f is Henstock-Kurzweil integrable on $[a, b]$ if and only if there exists an X-valued function F on $[a, b]$ such that*

(1) *there exists an α such that F is differentiable(X_α) almost everywhere on $[a, b]$ and $F'(t) = f(t)$,*

(2) *F is AC_* on $[a, b]$.*

In this case, F is an indefinite integral of f.

This follows from Propositions 18 and 20.

3. Convergence theorem

Proposition 22. *Let f be an X-valued function on $[a, b]$ for which there exists an α such that f is Henstock-Kurzweil integrable(X_α) on $[a, c]$ for every $c \in (a, b)$. Suppose that $\lim_{c\to b-} \int_a^c f dt$ exists in $(X_\beta, \{p_m^\beta\}_{m=0}^\infty)$ for some β and the limit is z. Then, f is Henstock-Kurzweil integrable to z on $[a, b]$.*

Proposition 23. *Let $\{f_j\}_{j=1}^\infty$ be a sequence of X-valued functions such that for some α each f_j is McShane (resp. Henstock-Kurzweil) integrable(X_α) on $[a, b]$. Suppose that there exists a β such that:*

(1) *The image of $[a, b]$ by f_j is contained in X_β for every j, and there exists an X_β-valued function f such that $\lim_{j\to\infty} f_j(t) = f(t)$ as $(X_\beta, \{p_m^\beta\}_{m=0}^\infty)$-valued functions almost everywhere on $[a, b]$.*

(2) *$\int_a^b p_m^\beta(f_j(t) - f_k(t))dt \to 0$ as $j, k \to \infty$ for $m = 0, 1, 2, \ldots$.*

Then, for some γ, f is McShane (resp.Henstock-Kurzweil) integrable(X_γ) on $[a, b]$ and for any figure $B \subset [a, b]$,

$$\lim_{j\to\infty} \int_B f_j(t)dt = \int_B f(t)dt \quad \text{in } (X_\gamma, \{p_m^\gamma\}_{m=0}^\infty),$$

where the convergence is uniform in B.

Proposition 24. *Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of X-valued functions on $[a,b]$. Suppose that:*

(1) *For some α, f_j is McShane (resp. Henstock-Kurzweil) integrable(X_α) on $[a,b]$ for $j=1,2,\ldots$.*

(2) *There exists a β' such that for any $\beta \geq \beta'$ and any $e \in X_\beta$ with $p_m^\beta(e)=1$, the following $(*)$ holds for $m=0,1,2,\ldots$.*

$$(*) \quad (f_1(t),e)_m^\beta \leq (f_2(t),e)_m^\beta \leq \cdots \quad \textit{almost everywhere on } [a,b].$$

(3) *The limit $\lim_{j\to\infty}\int_a^b f_j(t)dt$ exists in X.*

(4) *For some γ, the limit $\lim_{j\to\infty} f_j(t) = f(t)$ exists in $(X_\gamma, \{p_m^\gamma\}_{m=0}^{\infty})$ for almost all $t\in[a,b]$.*

Then, there exists a κ such that f is McShane (resp. Henstock-Kurzweil) integrable (X_κ) on $[a,b]$ and for any figure B in $[a,b]$

$$\lim_{j\to\infty}\int_B f_j(t)dt = \int_B f(t)dt \quad \textit{in } (X_\kappa, \{p_m^\kappa\}_{m=0}^{\infty}),$$

where the convergence is uniform in B.

Proposition 25 (Convergence theorem [4, Theorem 9.1])**.** *Let f_j $(j=1,2,\ldots)$ and f be X-valued functions on $[a,b]$. Suppose that for some α:*

(1) *f_j is Henstock-Kurzweil integrable(X_α) on $[a,b]$ for $j=1,2,\ldots$.*

(2) *$\lim_{j\to\infty} f_j(t) = f(t)$ almost everywhere on $[a,b]$ as $(X_\alpha, \{p_m^\alpha\}_{m=0}^{\infty})$-valued functions.*

(3) *For $m=0,1,2,\ldots$ there are an $M_m\geq 0$ and a positive function δ_m on $[a,b]$ such that*

$$p_m^\alpha\Big(\sum_{i=1}^{p} f_{j(x(i))}(x(i))(t_i - t_{i-1})\Big) \leq M_m$$

for all δ_m-fine Perron partitions $\{([t_0,t_1],x(1)),\ldots,([t_{p-1},t_p],x(p))\}$ of $[a,b]$ and all choices of positive integer valued functions $j(t)$ on $[a,b]$.
Then for some β with $\beta\geq\alpha$, f is Henstock-Kurzweil integrable(X_β) on $[a,b]$ and for any figure $B\subset[a,b]$

$$\lim_{j\to\infty}\int_B f_j dt = \int_B f dt \quad \textit{as } (X_\beta, \{p_m^\beta\}_{m=0}^{\infty})\textit{-valued functions},$$

where the convergence is uniform in B.

4. Integration by parts

Proposition 26. *Let f be a Henstock-Kurzweil integrable X-valued function on $[a, b]$ and G a non-decreasing real valued function on $[a, b]$. Let $F(t) = \int_a^t f dt$ $(t \in [a, b])$. Then, fG is Henstock-Kurzweil integrable on $[a, b]$, F is McShane integrable with respect to G on $[a, b]$ and*

$$\int_a^b fGdt = F(b)G(b) - \int_a^b FdG.$$

Proposition 27. *Let f be a McShane (resp. Henstock-Kurzweil) integrable X-valued function on $[a, b]$ and g a McShane (resp. Henstock-Kurzweil) integrable real valued function on $[a, b]$. Let $F(t) = \int_a^t f dt$ and $G(t) = \int_a^t g dt$. Then, $fG + Fg$ is McShane (resp. Henstock-Kurzweil) integrable on $[a, b]$, and*

$$\int_a^b (fG + Fg)dt = F(b)G(b).$$

5. Ordinary differential equations

Let

$$X \text{ be a vector space,}$$

$$g : [a, b] \times E \to X, \text{ where } E \subset X,$$

$$x' = g(t, x).$$

The methods of the Carathéodory and Henstock solutions for the ordinary differential equations of the type above shown in the case when X is the real space can be extended to the case when X is a nuclear countably Hilbert space, i.e., nuclear Hilbertian (CN) space. See [20] and [21] in detail.

Finally we propose the following problem.

Problem. *Develop the theory analogous to our theory to the case when the space X is treated as a topological vector space.*

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