Existence conditions for Feynman integrals

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Abstract

For the Feynman path integral to exist in the sense of Henstock, the existence of certain finite-dimensional Henstock integrals involving the potential function V is sufficient.

1 Background

The integration system and other preliminaries required for this problem are given in [8]. We start with a review of some features of Henstock's theory of integration in function spaces [1, 2, 3].

The primary space in which integration is required is $\mathbf{R}^{[\tau',\tau[}$. We integrate expressions $f(x)Q(I)$, where f and Q are, respectively, functions of points and subsets of $\mathbf{R}^{[\tau',\tau[}$. In fact these subsets are cylindrical intervals $I[N]$, restricted only in finite subsets N of dimensions in $[\tau',\tau[$, and the integrand may depend explicitly on the elements of N, which are variable in the Riemann sums, just as x and I are. The integrand is given more generally as $h(x,N,I)$, and the Riemann sums have the form $\sum h(x,N,I)$, where the cyindrical intervals $I = I[N]$ in the sum form a disjoint cover for the domain of integration.

In Henstock's theory of integration, the elements (x,I), (or (x,N,I) in the case of infinite dimensional integrals) may appear in a Riemann sum provided they are *attached* or *associated*. In the finite-dimensional situation involving only (x,I), this may mean that x must be a vertex of I. In the infinite-dimensional case, attachment or association of the components of (x,N,I) means that $x(N)$, as an element of $\mathbf{R}^N$, is a vertex of $I(N)$, which is a parallelepiped in $\mathbf{R}^N$.

Attachment or association may take other forms. For instance, when the Lebesgue integral in a one-dimensional space is constructed by the gauge integral method, the associated elements in the Riemann sums are (x, I), where x need not be contained in I, nor be a vertex of I, nor even be in the closure of I, but must only satisfy [4]

$$I \subseteq (x - \delta(x), x + \delta(x)).$$

Generally, the theory does not restrict us to two or three associated elements, as in (x, I), or (x, N, I). [5] gives an integration involving four associated elements $(x, N, N^*, I[N])$.

A gauge γ is a rule which determines a set S_γ of associated elements (x, I) (or (x, N, I), (x, N, N^*, I) and so on) from which Riemann sums may be formed.

Thus, in one dimension we may form Riemann sums $\sum f(x)Q(I)$ where each associated (x, I) belongs to the set S_δ satisfying $|I| < \delta$. If the family $\mathcal{A}$ of all such S_δ is formed from the various choices of δ (a positive constant), the result is Riemann integration.

Thus Henstock's theory of integration is a way of classifying systems of integration and generating new systems of integration. It does this by means of the family $\mathcal{A}$ which determines an integration by:

1. Specifying some rule of association between the integration elements x, $I, \ldots$; and

2. Specifying the collections S_γ from which the terms of each γ-fine Riemann sum are produced.

In infinite-dimensional generalised Riemann integration, one of the most useful methods of integration is obtained by the following determination of $\mathcal{A}$.

The integration elements x, N, I are *associated* or *attached* in the manner already described. A collection S_γ is specified as follows. Let A be any countable subset of $[\tau', \tau[$. For each x in $\overline{\mathbf{R}}^{[\tau', \tau[}$ let $L_A(x)$ be a finite subset of A. For every finite subset N of $[\tau', \tau[$ containing $L_A(x)$ let

$$\delta_N : \overline{\mathbf{R}}^N \mapsto \mathbf{R}^+$$

be a gauge in $\mathbf{R}^N$ in the usual sense of finite-dimensional Henstock integration.[1] These selections give us a gauge or rule γ in $\mathbf{R}^{[\tau', \tau[}$ which determines

[1] In [7, 8] these finite-dimensional gauges were said to be determined by the same $\delta(x)$ for all N. This was an error, in that the resulting system is not sufficient to enable us to integrate the functionals which arise in Brownian motion, quantum mechanics and other areas.

membership of the set S_γ, so that $(x, I[N])$ is a member of S_γ (or $(x, I[N])$ is *γ-fine*) if and only if N contains $L_A(x)$ and $|I(N)| < \delta_N(x(N)$. Each different selection of the countable set A and the mappings L_A and δ_N gives us a different gauge γ and a different set S_γ. The family $\mathcal{A}$ of all such S_γ determines the integration.[2]

We define a relation $\prec$ on the gauges $\{\gamma\}$ as follows. Suppose, for $j = 1, 2$, γ_j is defined by $(A_j, L^{(j)}_{A_j}, \delta^{(j)}_N)$. Then $\gamma_1 \prec \gamma_2$ if and only if $S_{\gamma_1} \supseteq S_{\gamma_2}$; in other words, if and only if $A_1 \subseteq A_2$, $L^{(1)}_{A_1}(x) \subseteq L^{(2)}_{A_2}(x)$ for all x, and $\delta^{(1)}_N(x(N)) \geq \delta^{(2)}_N(x(N))$ for all x and for all $N \supseteq L^{(2)}_{A_2}(x) \supseteq L^{(1)}_{A_1}(x)$.

The family $\mathcal{A}$ is a net, in that the sets S_γ satisfy the conditions:

- $S_{\gamma_1} \supseteq S_{\gamma_2}$, $S_{\gamma_2} \supseteq S_{\gamma_3}$ implies $S_{\gamma_1} \supseteq S_{\gamma_3}$; and,
- Given S_{γ_1} and S_{γ_2}, there exists S_{γ_3} with $S_{\gamma_1} \supseteq S_{\gamma_3}$ and $S_{\gamma_2} \supseteq S_{\gamma_3}$.

Equivalently we can say that the set of rules or gauges γ form a net, so:

- $\gamma_1 \prec \gamma_2$, $\gamma_2 \prec \gamma_3$ imply $\gamma_1 \prec \gamma_3$; and
- Given γ_1 and γ_2, there exists γ_3 with $\gamma_1 \prec \gamma_3$, $\gamma_2 \prec \gamma_3$.

To illustrate the reasoning involved in the latter step, let γ_j be defined by $(A_j, L^j_{A_j}, \delta^j_N)$ for $j = 1, 2$. We define a binary operation $\wedge$ on gauges as follows. Let $\gamma_1 \wedge \gamma_2$ be γ_3, where γ_3 is defined by $(A_3, L^3_{A_3}, \delta^3_N)$, with $A_3 = A_1 \cup A_2$, $L^3_{A_3}(x) = L^1_{A_1}(x) \cup L^2_{A_2}(x)$, and $\delta^3_N(x) = \min\{\delta^1_N(x), \delta^2_N(x)\}$. This is equivalent to taking $S_{\gamma_3} = S_{\gamma_1} \cap S_{\gamma_2}$ in the alternative formulation.

2 Limits Relative To Nets

Let $\{c_\alpha\}$ be a set of real or complex numbers indexed by elements α of a net $\{\alpha, \succ\}$ ordered by a relation $\succ$. Then $\{c_\alpha\}$ converges (relative to $\{\alpha, \succ\}$) to c if, given $\varepsilon > 0$, there exists α_ε so that $\alpha \succ \alpha_\varepsilon$ implies $|c_\alpha - c| < \varepsilon$. Write $c = \lim_\alpha c_\alpha$ in this case.

Let h_α be a family of functions indexed by a net $\{\alpha, \succ\}$. We say that $\{h_\alpha\}$ *converges boundedly* to h if there exists a positive function $\beta(x, N, I)$ defined for associated (x, N, I), a gauge γ_0 and an element $\alpha_0 = \alpha_0(x, N, I)$ of the net defined for each γ_0-fine pair $(x, I[N])$ so that

[2] In [3], Henstock gives an elegant construction which does not use the countable set A.

- $(\mathcal{E}_{\gamma_0}) \sum \beta(x, N, I)$ is bounded for all $\mathcal{E}_{\gamma_0}$, and
- $\alpha \succ \alpha_0$ implies $|h_\alpha(x, N, I) - h(x, N, I)| < \varepsilon\beta(x, N, I)$ for all γ_0-fine $(x, I[N])$.

Henstock [2] gave criteria for taking limits of sequences of functions under the integral sign. We give the corresponding results and proofs here with nets instead of sequences. There is little difference in the wordings of either the statements or the proofs of Henstock's results when they are re-written for nets instead of sequences.

Proposition 1 *Suppose $h_\alpha(x, N, I)$ converges boundedly to h.*

1. *h is integrable in E if and only if there exist a ball B_1 of arbitrarily small radius and, correspondingly, a gauge γ_1, and elements $\alpha_1 = \alpha_1(x, N, I)$, so that if $\alpha = \alpha(x, N, I)$ satisfies $\alpha \succ \alpha_1(x, N, I)$ and $\mathcal{E}_{\gamma_1}$ is γ_1-fine, then*
$$(\mathcal{E}_{\gamma_1}) \sum h_{\alpha(x,N,I)}(x, N, I) \in B_1.$$
2. *$\int_E h_\alpha$ converges relative to the net $\{\alpha, \succ\}$ if and only if there exist a ball B_2 of arbitrarily small radius, a corresponding element α_2, and, for each α with $\alpha \succ \alpha_2$, a gauge γ_α so that for every γ_α-fine $\mathcal{E}_{\gamma_\alpha}$ we have*
$$(\mathcal{E}_{\gamma_\alpha}) \sum h_\alpha(x, N, I) \in B_2.$$
3. *Given the existence of $\int_E h$ and $\lim_\alpha \int_E h_\alpha$, the two are equal if and only if $B_1 \cap B_2 \neq \emptyset$.*

Proof

1. If h is integrable, with $\int_E h = a_1$, then let $\varepsilon > 0$ be given, take $\gamma_1 = \gamma_0$, take $\alpha_1(x, N, I) = \alpha_0(x, N, I)$, and let K be an upper bound for $(\mathcal{E}_\gamma) \sum \beta(x, N, I)$ for all $\mathcal{E}_\gamma$, so that
$$(\mathcal{E}_\gamma) \sum h_{\alpha(x,N,I}(x, N, I) \in B(a_1, K\varepsilon).$$

   Conversely, if there exist B_1, γ and $\alpha_1(x, N, I)$ so that
$$(\mathcal{E}_\gamma) \sum h_{\alpha(x,N,I)}(x, N, I) \in B_1$$

for all α satisfying $\alpha \succ \alpha_1(x, N, I)$ and all $\mathcal{E}_\gamma$, then, with K as before, take $\gamma' = \gamma_0 \wedge \gamma$, and let $\alpha = \alpha(x, N, I)$ satisfy $\alpha \succ \alpha_j(x, N, I)$ for $j = 0, 1$ and all γ'-fine $(x, I[N])$. Then, for every $\mathcal{E}_{\gamma'}$, we have

$$(\mathcal{E}_{\gamma'}) \sum h(x, N, I) \in B_1 + B(0, K\varepsilon),$$

giving the integrability of h.

2. If $\lim_\alpha \int_E h_\alpha, = a_2$, exists, then there exists α_2 so that every α satisfying $\alpha \succ \alpha_2$ gives $\int_E h_\alpha \in B(a_2, \varepsilon)$. In other words, there exists γ_α so that, for all $\mathcal{E}_{\gamma_\alpha}$, we have

$$(\mathcal{E}_{\gamma_\alpha}) \sum h_\alpha(x, N, I) \in B(a_2, 2\varepsilon).$$

Conversely, suppose there exist B_2, α_2 and γ_α, with $\alpha_2 \succ \alpha$, so that every $(\mathcal{E}_{\gamma_\alpha}) \sum h_\alpha(x, N, I) \in B_2$. Then, since each h_α is integrable there exist gauges γ'_α so that for all $\mathcal{E}_{\gamma'_\alpha}$,

$$|(\mathcal{E}_{\gamma'_\alpha}) \sum h_\alpha - \int_E h_\alpha| < \varepsilon.$$

Now take $\gamma''_\alpha = \gamma_\alpha \wedge \gamma'_\alpha$, and consider γ''_α-fine divisions, so that for α satisfying $\alpha \succ \alpha_2$,

$$\int_E h_\alpha \in B_2 + B(0, \varepsilon).$$

Therefore $\lim_\alpha \int_E h_\alpha$ exists.

3. If ε is an arbitrary positive number, then, given the existence of $a_1 = \int_E h$ and $a_2 = \lim_\alpha \int_E h_\alpha$, we have $a_1 = a_2$ if and only if $B_1 = B(a_1, K\varepsilon)$ and $B_2 = B(a_2, 2\varepsilon)$ have non-empty intersection. □

It is sometimes useful to have Cauchy versions of these criteria. So we give details of a Cauchy version of the first one.

Proposition 2 *h is integrable in E if and only if, for every $\varepsilon > 0$, there exists a gauge γ_1, and elements $\alpha_1 = \alpha_1(x, N, I)$ so that for $\alpha = \alpha(x, N, I) \succ \alpha_1(x, N, I)$ and $\alpha' \succ \alpha_1(x, N, I)$, and for each $\mathcal{E}_{\gamma_1}$, we have*

$$|(\mathcal{E}_{\gamma_1}) h_{\alpha(x,N,I)}(x, N, I) - (\mathcal{E}_{\gamma_1}) h_{\alpha'(x,N,I)}(x, N, I)| < \varepsilon.$$

Proof Assuming the latter inequality, let

$$S_{\alpha_1} = \{(\mathcal{E}_\gamma) \sum h_{\alpha(x,N,I)}(x,N,I) : \alpha \succ \alpha_1(x,N,I)\}.$$

By the assumption, we can choose $\alpha_1, = \alpha_1(\varepsilon)$, so that

$$\text{diam}(S_{\alpha_1}) < \varepsilon.$$

For $j = 1, 2, 3, \ldots$, choose $\alpha_1 = \alpha_{1j}$ so that $\text{diam}(S_{\alpha_{1j}}) < j^{-1}$. Let $R_n = \cap_{j=1}^n S_{\alpha_{1j}}$. For each n, choose $s_n \in R_n$. Then s_n is a Cauchy sequence, so there exists a_1 with $s_n \to a_1$ as $n \to \infty$. With $\varepsilon > 0$ given, choose $n > \varepsilon^{-1}$, so if $s \in S_{\alpha_1(\varepsilon)}$ we have

$$|s - a_1| \leq |s - s_n| + |s_n - a_1|,$$

and $s \in B(a_1, 2\varepsilon)$. The converse is straightforward. $\square$

3 Application to Feynman integrals

The notation is that of [8]. Let M denote a finite subset of $[\tau, \tau[$, and let $\mathcal{M}$ be the family of all M. $\mathcal{M}$ is a net under the relation of $\subseteq$; because, given $M_1, M_2 \in \mathcal{M}$, there exists $M_3 \in \mathcal{M}$ with $M_1 \subseteq M_3$, $M_2 \subseteq M_3$; and, given $M_1 \subseteq M_2$ and $M_2 \subseteq M_3$, we have $M_1 \subseteq M_3$.

Suppose $\mathcal{E}$ is a division of $[\tau', \tau[$. Suppose the partition points of the division are $\tau' = t_0 < t_1 < \cdots < t_n = \tau$, and that $(t, [t_{j-1}, t_j))$ is the jth term of $\mathcal{E}$, with $t = t_{j-1}$ or t_j. (If this is δ-fine then $t_j - t_{j-1} < \delta(t)$.) Now consider a division $\mathcal{D}^{(j)}$ of $[t_{j-1}, t_j[$. Let $\mathcal{D}_\mathcal{E} := \cup_{j=1}^n \mathcal{D}^{(j)}$. Then $\mathcal{D}_\mathcal{E}$ is a division of $[\tau', \tau[$, and we call it a *refinement* of $\mathcal{E}$.

If $h(t, I)$ is defined on the points t and subintervals I of $[\tau', \tau[$, we say that h is *refinement-Henstock* integrable in $[\tau', \tau[$, with integral a, if, given $\varepsilon > 0$, there exists a gauge $\delta(t) > 0$ defined for $\tau' \leq t \leq \tau$, so that, for every δ-fine division $\mathcal{E}$ of $[\tau', \tau[$, and every refinement $\mathcal{D}_\mathcal{E}$, we have

$$|(\mathcal{D}_\mathcal{E}) \sum h(t, I) - a| < \varepsilon.$$

Since $\mathcal{D}_\mathcal{E}$ need not be δ-fine, this is less than Henstock integrability, in the sense that there may be functions which are Henstock integrable, but not refinement-Hentock integrable.

Following the notation of [8], let

$$v(x, M) = \exp\left(\lambda \sum_{j=1}^{n} V(x_{j-1})(t_j - t_{j-1})\right), \text{ and}$$

$$v(x) = \exp\left(\lambda \int_{\tau'}^{\tau} V(x(t))dt\right), \text{ whenever the integral exists.}$$

First we review some preliminary results.

Proposition 3 *If c is a complex number with $\Re(c) \leq 0$, $\Im(c) \geq 0$, $c \neq 0$, then $\int_0^\infty \exp(cy^2)dy$ exists and equals $\frac{1}{2}\sqrt{\frac{\pi}{-c}}$.*

This is Fresnel's integral, and the evaluation is given in [5], Proposition 63, page 81.

Let D be the set of x in T which are discontinuous at some t in $[\tau', \tau]$, and let F_0 be the "continuous modification" of G_0 as described in [8], Proposition 4, page 135. We make use of the notion of the *variation* of a function in a set, as described in [5, 9].

Proposition 4 *F_0 has variation 0 in D.*

This follows from [6], Proposition 4, page 135.

Proposition 5 *If c is a complex number with $\Re(c) \leq 0$, $\Im(c) \geq 0$, $c \neq 0$, then $F_0(x, N, I)$ is integrable in every elementary set E of $\mathbf{R}^{[\tau', \tau[}$, and*

$$\int_{\mathbf{R}^{[\tau', \tau[}} F_0(x, N, I) = \left(\frac{\pi}{-c}(\tau - \tau')\right)^{-\frac{1}{2}} \exp \frac{c(\xi - \xi')^2}{\tau - \tau'}.$$

Let $\mathcal{E}$ be a division of $\mathbf{R}^{[\tau', \tau[}$. Let M be the union of those dimension sets N for which $(x, I[N]) \in \mathcal{E}$. For each such $I[N]$ let $I(M)$ denote the projection of $I[N]$ into $\mathbf{R}^M$, so the projection of $\mathcal{E}$ into $\mathbf{R}^M$ is a division $\mathcal{E}(M) = \{(x(M), I(M))\}$ of $\mathbf{R}^M$. Then the Riemann sum $(\mathcal{E}) \sum F_0(x, N, I)$ can be interpreted as

$$(\mathcal{E}(M)) \sum F_0(x(M), I(M)).$$

Except for the fact that we have excluded from the Riemann sums those terms for which $x(t) = \pm\infty$, $t \in N$, the result would now follow from Fresnel's integral above. But, since these points form a subset of D, by the previous result we can choose a gauge γ so that, for all $\mathcal{E}_\gamma$, we have $|(\mathcal{E}_\gamma)\sum F_0(x,N,I) - \int_{\mathbf{R}^M} F_0(y(M)), I(M))dy(M)| < \varepsilon$. So

$$\left|(\mathcal{E})\sum F_0(x,N,I) - \left(\frac{\pi}{-c}(\tau-\tau')\right)^{-\frac{1}{2}}\exp\frac{c(\xi-\xi')^2}{\tau-\tau'}\right| < \varepsilon,$$

giving the required result. □

Proposition 6 *If $V(x(t))$ is refinement-Henstock integrable in $[\tau', \tau[$, then $\lim_M v(x,M)$ exists and equals $v(x)$.*

Proof Let $\varepsilon > 0$ be given. Choose a δ-fine division $\mathcal{E}$ of $[\tau', \tau[$ for which

$$\left|\exp\left(\lambda(\mathcal{E})\sum V(x_j)(t_j - t_{j-1})\right) - \exp\left(\int_{\tau'}^{\tau}\lambda V(x(t))dt\right)\right| < \varepsilon.$$

Letting $M_{\mathcal{E}}$ be the partition points of $\mathcal{E}$, this gives

$$|v(x, M_{\mathcal{E}}) - v(x)| < \varepsilon.$$

Let M be any finite subset of $[\tau', \tau[$ containing $M_{\mathcal{E}}$. For every such M we can take a corresponding refinement division $\mathcal{D}_{\mathcal{E}}$ of $\mathcal{E}$, and

$$\left|\exp\left(\lambda(\mathcal{D}_{\mathcal{E}})\sum V(x_j)(t_j - t_{j-1})\right) - \exp\left(\int_{\tau'}^{\tau}\lambda V(x(t))dt\right)\right| < \varepsilon, \text{ or}$$

$$|v(x, M) - v(x)| < \varepsilon,$$

giving the result. □

Proposition 7 *Suppose V satisfies the following pair of (finite-dimensional) integrability conditions. Firstly, $V(x(t))$ is refinement-Henstock integrable in $[\tau', \tau[$ for each $x \in C$. Secondly, $v(x(M), M)$ is Henstock integrable in $\mathbf{R}^M$ for each M. Then $v(x)$ is F_0-integrable in $\mathbf{R}^{[\tau',\tau[}$.*

Proof By [5], Proposition 69, page 87, assumption (2) implies the cylinder functional $v(x,M)$ is F_0-integrable in $\mathbf{R}^{[\tau',\tau[}$ for each fixed M. Let $\varepsilon > 0$ be given. Let γ_1, X_k and α_k, $k = 1, 2, \ldots$, be as they are defined in [8], (14), page 15. Choose $\gamma \prec \gamma_1$ so that, for each $x \in X_k$ and each $L_A^{(\gamma)}(x)$, if we have $M \supseteq L_A^{(\gamma)}(x)$, then

$$|v(x, M) - v(x)| < \varepsilon\alpha_k^{-1}2^{-k}.$$

Then for any γ-fine division $\mathcal{E}_\gamma$ of $\mathbf{R}^{[\tau',\tau[}$, and any choices $M(x)$, $M'(x)$ containing $L_A^{(\gamma)}(x)$, we have

$$\begin{array}{rl} & |(\mathcal{E}_\gamma)\sum v(x,M(x))F_0(I[N]) - (\mathcal{E}_\gamma)\sum v(x,M'(x))F_0(I[N])| \\ \leq & (\mathcal{E}_\gamma)\sum |v(x,M(x)) - v(x,M'(x))||F_0(I[N])| \\ \leq & (\mathcal{E}_\gamma)\sum |v(x,M(x) - v(x)||F_0(I[N])| \\ & \quad -(\mathcal{E}_\gamma)\sum |v(x) - v(x,M'(x)||F_0(I[N])| \\ < & 2\varepsilon\sum_{k=1}^{\infty} 2^{-k} \\ = & 2\varepsilon. \end{array}$$

The result follows from the Cauchy version, given above, of Henstock's first criterion for limits under the integral sign. $\Box$

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