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A LOCALLY CONNECTED CONTINUUM WITHOUT CONVERGENT SEQUENCES

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Abstract. We answer a question of Juhász by constructing under CH an example of a locally connected continuum without nontrivial convergent sequences.

1. Introduction

During the Ninth Prague Topological Symposium, Juhász asked whether there is a locally connected continuum without nontrivial convergent sequences. This question arose naturally in his investigation in [6] with Gerlits, Soukup, and Szentmiklóssy on characterizing continuity in terms of the preservation of compactness and connectedness. The aim of this note is to answer this question in the affirmative under the Continuum Hypothesis (abbreviated: CH).

Fedorchuk [5] constructed a consistent example of a compact space of cardinality $\mathfrak{c}$ containing no nontrivial convergent sequences. See also van Douwen and Fleissner [10] for a somewhat simpler construction under the Definable Forcing Axiom. These constructions yield zero-dimensional spaces. As a consequence, our construction has to be somewhat different. As in [5] and [10], we 'kill' all possible nontrivial convergent sequences in a transfinite process of length ω_1. However, our 'killing' is done in the Hilbert cube $Q = \prod_{n=1}^{\infty}[-1, 1]_n$ instead of the Cantor set.

For all undefined notions, see [4] and [11].

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2. The Hilbert cube

A *Hilbert cube* is a space homeomorphic to Q. Let M^Q denote an arbitrary Hilbert cube.

A closed subset A of M^Q is a *Z-set* if for every $\varepsilon > 0$ there is a continuous function $f\colon M^Q \to M^Q \setminus A$ which moves the points less than ε. It is clear that a closed subset of a Z-set is a Z-set. We list some other important properties of Z-sets.

(1) Every singleton subset of M^Q is a Z-set.
(2) A countable union of Z-sets is a Z-set provided it is closed.
(3) A homeomorphism between Z-sets can be extended to a homeomorphism of M^Q.
(4) If X is compact and $f\colon X \to M^Q$ is continuous then f can be approximated arbitrarily closely by an imbedding whose range is a Z-set.

See [11, Chapter 6] for details.

Observe that by (1) and (2), every nontrivial convergent sequence with its limit is a Z-set in M^Q.

A *near homeomorphism* between compacta X and Y is a continuous surjection $f\colon X \to Y$ which can be approximated arbitrarily closely by homeomorphisms. This means that for every $\varepsilon > 0$ there is a homeomorphism $g\colon X \to Y$ such that for every $x \in X$ we have that the distance between $f(x)$ and $g(x)$ is less than ε.

A closed subset $A \subseteq M^Q$ has *trivial shape* if it is contractible in any of its neighborhoods. A continuous surjection f between Hilbert cubes M^Q and N^Q is *cell-like* provided that $f^{-1}(q)$ has trivial shape for every $q \in N^Q$. The following fundamental result is due to Chapman [3] (see also [11, Theorem 7.5.7]).

(5) Let $f\colon M^Q \to N^Q$ be cell-like, where M^Q and N^Q are Hilbert cubes. Then f is a near homeomorphism.

It is easy to see that if $f\colon M^Q \to N^Q$ is a near homeomorphism between Hilbert cubes then f is cell-like. So within the framework of Hilbert cubes the notions 'near homeomorphism' and 'cell-like' are equivalent.

A continuous surjection f between Hilbert cubes M^Q and N^Q is called a Z^*-map provided that for every Z-set $A \subseteq N^Q$ we have that $f^{-1}[A]$ is a Z-set in M^Q.

Lemma 2.1. *Let M^Q and N^Q be Hilbert cubes, and let $f\colon M^Q \to N^Q$ be a continuous surjection for which there is a Z-set $A \subseteq M^Q$ which contains all nondegenerate fibers of f. Then f is a Z^*-map.*

Proof. Let $B \subseteq N^Q$ be an arbitrary Z-set, and put $B_0 = B \setminus f[A]$. Write B_0 as $\bigcup_{n=1}^{\infty} E_n$, where each E_n is compact. It follows from [11, Theorem 7.2.5] that for every n the set $f^{-1}[E_n]$ is a Z-set in M^Q. As a consequence,

$$f^{-1}[B] \subseteq A \cup \bigcup_{n=1}^{\infty} f^{-1}[E_n]$$

is a countable union of Z-sets and hence a Z-set by (2). $\square$

Theorem 2.2. *Let $(Q_n, f_n)_n$ be an inverse sequence of Hilbert cubes such that every f_n is cell-like as well as a Z^*-map. Then*

(A) *$\varprojlim(Q_n, f_n)_n$ is a Hilbert cube.*

(B) *The projection $f_n^\infty \colon \varprojlim(Q_n, f_n)_n \to Q_n$ is a cell-like Z^*-map for every n.*

Proof. It will be convenient to let Q_∞ denote $\varprojlim(Q_n, f_n)_n$.

By (5), every f_n is a near homeomorphism. Hence we get (A) from Brown's Approximation Theorem for inverse limits in [2]. It follows from [11, Theorem 6.7.4] that every projection $f_n^\infty \colon Q_\infty \to Q_n$ is a near homeomorphism, hence is cell-like.

For every n let ϱ_n be an admissible metric for Q_n which is bounded by 1. The formula

$$\varrho(x,y) = \sum_{n=1}^{\infty} 2^{-n} \varrho_n(x_n, y_n)$$

defines an admissible metric for Q_∞. With respect to this metric we have that f_n^∞ is a $2^{-(n-1)}$-mapping ([11, Lemma 6.7.3]).

For (B) it suffices to prove that f_1^∞ is a Z^*-map. To this end, let $A \subseteq Q_1$ be a Z-set, and let $\varepsilon > 0$. Pick $n \in \mathbb{N}$ so large that $2^{-(n-1)} < \varepsilon$. It follows that for every $x \in Q_n$ we have that the diameter of the fiber $(f_n^\infty)^{-1}(x)$ is less than ε. An easy compactness argument gives us an open cover $\mathcal{U}$ of Q_n such that for every $U \in \mathcal{U}$ we have that

$$\operatorname{diam}(f_n^\infty)^{-1}[U] < \varepsilon. \tag{$*$}$$

Let $\gamma > 0$ be a Lebesgue number for this cover ([11, Lemma 1.1.1]). Since f_n^∞ is a near homeomorphism, there is a homeomorphism $\varphi \colon Q_\infty \to Q_n$ such that for every $x \in Q_\infty$ we have

$$\varrho_n(f_n^\infty(x), \varphi(x)) < {}^1\!/_2\gamma.$$

Observe that $A_n = (f_1^n)^{-1}[A]$ is a Z-set in Q_n. There consequently is a continuous function $\xi \colon Q_n \to Q_n \setminus A_n$ which moves the points less than ${}^1\!/_2\gamma$. Now define $\eta \colon Q_\infty \to Q_\infty$ by

$$\eta = \varphi^{-1} \circ \xi \circ f_n^\infty.$$

It is clear that $\eta[Q_\infty]$ misses $(f_1^\infty)^{-1}[A]$. In order to check that η is a 'small' move, pick an arbitrary element $x \in Q_\infty$. By construction,

$$\varrho_n\big(x_n, \xi(x_n)\big) < {}^1\!/_2\gamma.$$

Since $\eta(x) = \varphi^{-1}\big(\xi(x_n)\big)$, clearly

$$\varrho_n\big(\eta(x)_n, \xi(x_n)\big) < {}^1\!/_2\gamma.$$

We conclude that $\varrho_n(\eta(x)_n, x_n) < \gamma$. Pick an element $U \in \mathcal{U}$ which contains both $\eta(x)_n$ and x_n. By (∗) it consequently follows that $\varrho(\eta(x), x) < \varepsilon$, which is as required. □

Theorem 2.3. *If $(A_n)_n$ is a relatively discrete sequence of closed subsets of Q such that $\overline{\bigcup_{n=1}^{\infty} A_n}$ is a Z-set then there are a Hilbert cube M and a continuous surjection $f\colon M \to Q$ such that*

(A) *f is a cell-like Z^*-map.*
(B) *The closures of the sets $\bigcup_{n=1}^{\infty} f^{-1}[A_{2n}]$ and $\bigcup_{n=0}^{\infty} f^{-1}[A_{2n+1}]$ are disjoint.*

Proof. Consider the subspace $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ of Q, and the 'remainder' $R = A \setminus \bigcup_{n=1}^{\infty} A_n$. Observe that R is compact since the sequence $(A_n)_n$ is relatively discrete. Let T denote the product $A \times \mathbb{I}$; put

$$S = (R \times \mathbb{I}) \cup \Big(\bigcup_{n=1}^{\infty} A_{2n} \times \{0\} \Big) \cup \Big(\bigcup_{n=0}^{\infty} A_{2n+1} \times \{1\} \Big).$$

Then S is evidently a closed subspace of T. Let $\pi\colon R \times \mathbb{I} \to R$ denote the projection. It is clear that the adjunction space (cf., [12, Page 507]) $S \cup_\pi (R \times \mathbb{I})$ is homeomorphic to A. By (4), any constant function $S \to Q$ can be approximated by an imbedding whose range is a Z-set. So we may assume without loss of generality that S is a Z-subset of some Hilbert cube M^Q. Now consider the space $N = M^Q \cup_\pi (R \times \mathbb{I})$ with natural decomposition map f. It is clear that f is cell-like, each non-degenerate fiber of f being an arc ([11, Corollary 7.1.2]). We will prove below that $N \approx Q$. Once we know that, we also get by Lemma 2.1 that f is a Z^*-map. Observe that the projection $\pi\colon R \times \mathbb{I} \to R$ is a hereditary shape equivalence. So by a result of Kozlowski [7] (see also [1]), it follows that N is an AR. Since S is a Z-set in M^Q it consequently follows from [11, Proposition 7.2.12] that $f[S] \approx A$ is a Z-set in N. But $N \setminus f[S]$ is obviously a Q-manifold, and consequently has the disjoint-cells property. But this implies that N has the disjoint-cells property, i.e., $N \approx Q$ by Toruńczyk's topological characterization of Q in [9] (see also [11, Corollary 7.8.4]). So we conclude that $f[S] \approx A$ is a Z-set in the Hilbert cube N. By (3) there is a homeomorphism of pairs $(Q, A) \approx (N, f[S])$. This homeomorphism may be chosen to be the 'identity' on A. This shows that we are done by Lemma 2.1 and the obvious fact that the sets

$$\bigcup_{n=1}^{\infty} A_{2n} \times \{0\}, \quad \bigcup_{n=0}^{\infty} A_{2n+1} \times \{1\}$$

have disjoint closures in M^Q. □

3. The construction

We will now construct our example under CH. After the preparatory work in §2, the construction is very similar to known constructions in the literature (see e.g., Kunen [8]).

Consider the 'cube' Q^{ω_1}. For every $1 \le \alpha < \omega_1$ let $\{S^\alpha_\xi : \xi < \omega_1\}$ list all nontrivial convergent sequences in Q^α that do not contain their limits. For all $\alpha, \xi < \omega_1$ pick disjoint complementary infinite subsets A^α_ξ and B^α_ξ of S^α_ξ.

We shall construct for $1 \le \alpha \le \omega_1$ a closed subspace $M_\alpha \subseteq Q^\alpha$. The space we are after will be M_{ω_1}.

Let $\tau\colon \omega_1 \to \omega_1 \times \omega_1$ be a surjection such that $\tau(\beta) = \langle \alpha, \xi\rangle$ implies $\alpha \le \beta$.

For $\alpha \le \beta \le \omega_1$ let π^β_α be the natural projection from Q^β onto Q^α. The following conditions will be satisfied:

(A) $M_\alpha \approx Q$ for every $1 \le \alpha < \omega_1$, and if $\alpha \le \beta$ then $\pi^\beta_\alpha[M_\beta] = M_\alpha$. We put $\rho^\beta_\alpha = \pi^\beta_\alpha \restriction M_\beta\colon M_\beta \to M_\alpha$.

(B) If $\alpha \le \beta$ then $\rho^\beta_\alpha\colon M_\beta \to M_\alpha$ is a cell-like Z^*-map.

(C) If $\beta < \omega_1$, $\tau(\beta) = \langle \alpha, \xi\rangle$, and $S^\alpha_\xi \subseteq M_\alpha$ then $(\rho^{\beta+1}_\alpha)^{-1}[A^\alpha_\xi]$ and $(\rho^{\beta+1}_\alpha)^{-1}[B^\alpha_\xi]$ have disjoint closures in $M^{\beta+1}$.

Observe that the construction is determined at all limit ordinals γ. By compactness and (A) we must have

$$M_\gamma = \{x \in Q^\gamma : (\forall \alpha < \gamma)(\pi^\gamma_\alpha(x) \in M_\alpha)\}.$$

Also, if $(\gamma_n)_n$ is any strictly increasing sequence of ordinals with $\gamma_n \nearrow \gamma$ then M_γ is canonically homeomorphic to

$$\varprojlim \left(M_{\gamma_n}, \rho^{\gamma_{n+1}}_{\gamma_n}\right)_n.$$

By Theorem 2.2 this implies that $M_\gamma \approx Q$ and also that $\rho^\gamma_{\gamma_n}$ is a cell-like Z^*-map for every n. Since γ_1 can be *any* ordinal smaller than γ, the same argument yields that ρ^γ_α is a cell-like Z^*-map for *every* $\alpha < \gamma$. So in our construction we need only worry about successor steps.

Put $M_1 = Q^{\{0\}}$, and let $1 \le \beta < \omega_1$ be arbitrary. We shall construct $M_{\beta+1}$ assuming that M_β has been constructed. To this end, let $\tau(\beta) = \langle \alpha, \xi\rangle$. We make the obvious identification of $Q^{\beta+1}$ with $Q^\beta \times Q$. If $S^\alpha_\xi \not\subseteq M_\alpha$ then there is nothing to do. We then fix any element $q \in Q$, and put

$$M_{\beta+1} = M_\beta \times \{q\}.$$

So assume that $S^\alpha_\xi \subseteq M_\alpha$. By Theorem 2.3 there exists a cell-like Z^*-map $f\colon Q \to M_\beta$ such that

$$f^{-1}\left[(\rho^\beta_\alpha)^{-1}[A^\alpha_\xi]\right], \quad f^{-1}\left[(\rho^\beta_\alpha)^{-1}[B^\alpha_\xi]\right]$$

have disjoint closures in Q. Put

$$M_{\beta+1} = \{\langle f(x), x\rangle \in Q^\beta \times Q : x \in Q\}.$$

So $M_{\beta+1}$ is nothing but the graph of f. It is clear that $M_{\beta+1}$ is as required.

Now put $M = M_{\omega_1}$. Observe that M is a locally connected continuum, being the inverse limit of an inverse system of locally continua with monotone surjective bonding maps (see e.g., [4, 6.3.16 and 6.1.28]). Assume that T is a nontrivial convergent sequence with its limit x in M. Since $T \cup \{x\}$ is countable, there exists $\alpha < \omega_1$ such that $\rho^{\omega_1}_\beta \restriction (T \cup \{x\})$ is one-to-one and hence a homeomorphism for every $\beta \geq \alpha$. Pick $\xi < \omega_1$ such that $S^\alpha_\xi = \rho^{\omega_1}_\alpha[T]$, and $\beta \geq \alpha$ such that $\tau(\beta) = \langle \alpha, \xi \rangle$. Then $\rho^{\omega_1}_{\beta+1}[T \cup \{x\}]$ is a nontrivial convergent sequence with its limit in $M_{\beta+1}$ which is mapped by $\rho^{\beta+1}_\alpha$ onto S^α_ξ with its limit. But this is clearly in conflict with (C).

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