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ORBITS OF TURNING POINTS FOR MAPS OF FINITE GRAPHS AND INVERSE LIMIT SPACES

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Abstract. In this paper we examine the topology of inverse limit spaces generated by maps of finite graphs. In particular we explore the way in which the structure of the orbits of the turning points affects the inverse limit. We show that if f has finitely many turning points each on a finite orbit then the inverse limit of f is determined by the number of elements in the ω-limit set of each turning point. We go on to identify the local structure of the inverse limit space at the points that correspond to points in the ω-limit set of f when the turning points of f are not necessarily on a finite orbit. This leads to a new result regarding inverse limits of maps of the interval.

1. Introduction

Every one-dimensional continuum is an inverse limit on finite graphs, and many, though not all, are homeomorphic to an inverse limit on a finite graph with a single bonding map. These spaces also naturally appear in dynamical systems. R.F. Williams showed that if a manifold diffeomorphism F has a one-dimensional hyperbolic attractor Λ (with associated stable manifold structure)then F restricted to Λ is topologically conjugate with the shift homeomorphism on an inverse limit of a piecewise monotone map f of some finite graph, [11], and Barge and Diamond, [2], remark that for any map $f : G \to G$ of a finite graph there is a homeomorphism $F : \mathbb{R}^3 \to \mathbb{R}^3$ with an attractor on which F is conjugate to the shift homeomorphism on $\varprojlim\{G, f\}$. More recently, Anderson and Putnam, [1], have shown that the dynamics arising from a substitution tiling is often conjugate to the action of a shift-map on an inverse limit of a branched d-manifold. They then demonstrate how to use knowledge about the inverse limit space to compute the cohomology and K-theory of a space of tilings. Extending these ideas, Barge, Jacklitch and Vago, [4], use inverse limits induced by certain Markov

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maps on wedges of circles to analyze one-dimensional substitution tiling spaces and one-dimensional unstable manifolds of hyperbolic sets. Many of their results rely on showing that certain pairs of these inverse limit spaces are not homeomorphic.

It is often quite difficult to distinguish between inverse limit spaces, even when the dynamics of the bonding maps are very different. Many papers have been written to this end, [3], [5], [6], [8], and [10]. However most of the techniques have been focused on maps of the interval. Perhaps one of the easiest ways to decide if two inverse limit spaces are not homeomorphic is to count their endpoints. Barge and Martin have shown that the number of endpoints of $\varprojlim\{[0,1], f\}$ is the same as the number of elements in the ω-limit set of the turning points of the bonding map, f, when f has a dense orbit and finitely many turning points, [5].

In this paper we distinguish between these inverse limit spaces by showing that many of the points have neighborhoods homeomorphic to the product of a zero-dimensional set and $(0,1)$, and we show that the exceptional points are those that always project onto the ω-limit set of the turning points. We do this for inverse limits on graphs, but of course, the result holds for inverse limits on the interval. In the case of the interval, our theorem is still an extension of Barge and Martin's result, because there are many bonding maps that give rise to a three-endpoint indecomposable continua that have more than three points in the ω-limit set of their turning points. Our theorem can be used to easily distinguish between these inverse limit spaces.

2. Preliminaries

By a *continuum* we mean a compact, connected, metric space, and by a *mapping* we mean a continuous function. We will say a mapping, f, is *monotone* on A if, and only if, $f^{-1}(x)$ is connected for all $x \in A$. The *inverse limit* induced by a single bonding map, f, on a continuum M is defined as follows:

$$\varprojlim\{M, f\} = \{(x_0, x_1, \dots) | x_i \in M \quad \text{and} \quad f(x_{i+1}) = x_i\}.$$

Since M is metric and f is a mapping, $\varprojlim\{M, f\}$ is a continuum with the metric:

$$d(x,y) = \sum_{i=0}^{\infty} \frac{d_M(x_i - y_i)}{2^i},$$

where d_M is the metric on M and we assume that $d_M(x,y) < 1$ for all $x, y \in M$. Define the projection maps $\pi_n : \varprojlim\{M, f\} \to M$ by $\pi_n(x) = x_n$, where $x = (x_1, x_2, \dots) \in \varprojlim\{M, f\}$. Also, define the shift homeomorphism $h : \varprojlim\{M, f\} \to \varprojlim\{M, f\}$ by

$$h(x) = (f(x_0), f(x_1), f(x_2), \dots) = (f(x_0), x_0, x_1, \dots).$$

A *linear chaining*, or just *chaining*, of a continuum M is a finite sequence, $L_1, L_2, L_3, \ldots, L_n$ of open subsets of M such that L_i intersects L_j if and only if $|i-j|<2$. The open sets comprising the chain are called the *links* of the chain. The *mesh* of a chain is the largest of the diameters of its links. A continuum M is said to be *chainable* provided that for each positive number ϵ there is a chaining of M with mesh less than ϵ, such a chain is called an ϵ-*chain*. It is a well-known fact that an inverse limit of chainable continua is a chainable continuum. A *closed chain* is a chain whose links are closed sets and if $i \neq j$, then $L_i \cap L_j = \mathrm{Bd}(L_i) \cap \mathrm{Bd}(L_j)$ if, and only if, $|i-j|<2$. We lose no generality in assuming that all of the chains in this paper are taut, i.e. if $L_i \cap L_j = \emptyset$ then $\overline{L_i} \cap \overline{L_j} = \emptyset$, [9]. Notice that if $\mathcal{L}$ is a taut chaining of an inverse limit space and $L_i \cap L_j = \emptyset$ then it is possible to find a positive integer, q, large enough so that $\pi_q(\overline{L_i}) \cap \pi_q(\overline{L_j}) = \emptyset$, which implies that $\pi_q(\mathcal{L})$ is a chain.

A *finite graph*, G, is a continuum that can be written as the union of finitely many arcs any two of which are either disjoint or intersect at only one of their endpoints. For any finite graph, G, there is a finite set of points called *vertices*, $V = \{v_1, v_2, \ldots, v_n\}$, and a set of arcs, E, with endpoints from V called *edges*, with the property that if $v_k \in e_{ij} \in E$ then either $k=i$ or $k=j$, and if two edges meet, they meet only at a single common vertex. For simplicity, we adopt the convention that, since $e_{ij} = e_{ji}$, if we label an edge e_{ij} then $i<j$. For every point, $x \in G$, define the *degree of* x, $\deg(x)$, to be the number of edges in G that have x as an endpoint. Let $V' \subseteq V$ be the set of all points, x, with $\deg(x) \geq 3$.

Let $a, b \in G$. We will denote an arc between a and b by $\overline{ab}$. Clearly this arc is not uniquely determined. However if $a, b \in e_{ij}$ then there is a unique arc with endpoints a and b that is contained in e_{ij}. We will denote this arc by $[a,b]$, assuming that a is closer to v_i in the linear ordering of e_{ij} that has v_i as its least element, otherwise we denote it $[b,a]$.

We will now extend the idea of linear-chains to graph-chains. Let n be a positive integer and let R be a relation on $\{1, 2 \ldots n\}$ with the property that if $(i,j) \in R$ then $i<j$. For every $(i,j) \in R$, let $\mathcal{C}_{i,j} = \{C_1^{i,j}, C_2^{i,j}, \ldots, C_{n_{i,j}}^{i,j}\}$ be a taut chain with the closure of every link of $\mathcal{C}_{i,j}$ being disjoint from the closure of every link of $\mathcal{C}_{k,l}$ whenever $(k,l) \neq (i,j)$, except $C_1^{i,j}$ which meets every link of the form $C_1^{i,k}$ and every $C_{n_{m,i}}^{m,i}$ and also except for $C_{n_{i,j}}^{i,j}$ which meets every link of the form $C_1^{j,k}$ and every $C_{n_{m,j}}^{m,j}$. Call $\mathcal{C}_{i,j}$ an *edge-chain*. Let

$$\mathcal{C} = \bigcup_{(i,j) \in R} \mathcal{C}_{i,j}.$$

Call $\mathcal{C}$ a *graph-chain*. Let G be a finite graph with vertex set, V, and edge set E. A *graph-chaining of* G is a graph-chain with $R = \{(i,j) | e_{ij} \in E\}$ such that each vertex, v_i, is only in links of the form $C_1^{i,j}$ or $C_{n_{m,i}}^{m,i}$, and each edge-chain, $\mathcal{C}_{i,j}$, is a chaining of the corresponding edge, e_{ij}.

For a given graph-chain $\mathcal{C}$, call the set $E' = R$ the *edge index set.* For notational convenience we will often denote a graph-chain by

$$\mathcal{C} = \{C_1^i, C_2^i, \ldots C_{n_i}^i | i \in E'\}$$

using i to represent an ordered pair in E'.

A continuum, M, is said to be *graph-chainable* provided that for each positive number ϵ there exists a graph-chaining of M with mesh less than ϵ. By a *closed graph-chain* we will mean a graph-chain $\mathcal{C}$ such that every link of $\mathcal{C}$ is closed and if A and B are different links in $\mathcal{C}$ with $A \cap B \neq \emptyset$, then $A \cap B = \text{Bd}(A) \cap \text{Bd}(B)$. Notice that this implies that the only point in common to links of the form $C_1^{i,j}$ and $C_{n_{k,i}}^{k,i}$ is the vertex v_i.

It is easy to show that the inverse limit induced by maps on graph-chainable continua is itself a graph-chainable continuum.

3. Markov Graph-Maps

First, we extend the definition of a Markov map of the interval (see [3] or [7])to a Markov map of a graph. Let f be a mapping of a finite graph, G, with vertex set V and edge set E and define a *Markov graph-chaining*, $\mathcal{T}^f$, of G with respect to f to be a closed graph-chaining of G where, for every $i \in E'$, $|\mathcal{T}_i^f| = n_i$, f restricted to each link is monotone but not constant, and for every $i \in E'$ and $k \leq n_i$ there exists a subset of $E' \times \mathbb{N}$, $A_{i,k}$, such that

$$f(T_k^i) = \bigcup_{(p,r) \in A_{i,k}} T_r^p.$$

We will call a set of the form $A_{i,k}$ the *index set* for (i,k) under f, and we will call a map that admits such a Markov graph-chain a *Markov graph-map* or simply a *Markov map.* The endpoints of each link of $\mathcal{T}^f$ determine a Markov partition of each edge. We define the *Markov partition* of the graph to be the set

$$B_f = \{v_i = c_0^{i,j} < c_1^{i,j} < \cdots < c_{n_{i,j}}^{i,j} = v_j | (i,j) \in E'\}$$

where $c_k^{i,j}$ and $c_{k+1}^{i,j}$ are the endpoints of $T_{k+1}^{i,j}$. Notice that $f(B_f) \subseteq B_f$, f is not constant on $[c_{k-1}^{i,j}, c_k^{i,j}]$, and f restricted to each such arc is monotone.

Also define the set $S_{i,k} \subset E' \times \mathbb{N}$ such that $(p,r) \in S_{i,k}$ if and only if $[f^{-1}(T_k^i)]^\circ \cap T_r^p \neq \emptyset$ i.e. $S_{i,k}$ is the collection of indices of links of $\mathcal{T}_f$ that are mapped onto T_k^i by f. We will call the set $S_{i,k}$ the *inverse index set* of (i,k) under f.

Let $f : G_1 \to G_2$ be a a map between finite graphs G_1 and G_2. Then $x \in G_1$ is a *turning point* of f if there is an arc, $\overline{ab} \subseteq G_1$, containing x in its interior, such that $f[\overline{ab}] = \overline{zf(x)}$ where $z \in \{f(a), f(b)\}$ and both of $f|_A$ and $f|_B$ are monotone, where $A = \overline{ax} \subseteq \overline{ab}$ and $B = \overline{xb} \subseteq \overline{ab}$. Denote the set of turning points of f by P_f.

Generally there is much freedom in determining the Markov chain; however we assume that the Markov chains used in this paper are "natural" in the sense that elements of the Markov partition are either vertices, turning points, or in the orbit of a turning point or vertex. In the next section we will need to assume that $f^{-1}(x)$ consists of only isolated points, for all $x \in G$. The next theorem demonstrates that we lose no generality in assuming this when f is Markov.

Theorem 3.1. *Let each of f and g be a Markov mapping of G, a finite graph, with associated Markov partitions, $B_f = \{c_0^i < c_1^i < \cdots < c_{n_i}^i | i \in E'\}$ and $B_g = \{d_0^i < d_1^i < \cdots < d_{n_i}^i | i \in E'\}$. Suppose that for every $i \in E'$ and $k \leq n_i$ there is a $p \in E'$ and a $r \leq n_p$ such that $f(c_k^i) = c_r^p$ if and only if $g(d_k^i) = d_r^p$, then $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G, g\}$.*

Before presenting the proof of this theorem we will present a few useful facts about graph-chainable continua.

Let $\mathcal{C}$ be a closed graph-chaining of a continuum, M, with edge index set R, and let $\mathcal{C}'$ be a refinement of $\mathcal{C}$ with edge index set R'. Let h be a function such that for every link, $C_k^{\prime i} \in \mathcal{C}'$, let $h(i, k) = (p, r)$ if and only if $C_k^{\prime i}$ is a subset of C_r^p in $\mathcal{C}$. In this case we shall say that $\mathcal{C}'$ *follows pattern* h in $\mathcal{C}$. The proof of the next theorem is quite obvious, and so it has been omitted.

Theorem 3.2. *Let A be a graph-chainable continuum, and let $\{\mathcal{C}_i\}_{i=1}^{\infty}$ be a sequence of refining graph-chainings of A such that*

$$\lim_{i \to \infty} \operatorname{mesh}(\mathcal{C}_i) = 0$$

and $\mathcal{C}_i$ follows pattern h_i in $\mathcal{C}_{i-1}$. If B is also a graph-chainable continuum with a sequence of refining graph-chainings, $\{\mathcal{D}_i\}_{i=1}^{\infty}$, such that

$$\lim_{i \to \infty} \operatorname{mesh}(\mathcal{D}_i) = 0$$

and $\mathcal{D}_i$ follows pattern h_i in $\mathcal{D}_{i-1}$ then A is homeomorphic to B.

Suppose that $f : G \to G$ is a Markov mapping of a finite graph, G, with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E, and let $\mathcal{T}^f$ be a Markov chain of G for f where, for every $i \in E'$, $|\mathcal{T}_i^f| = n_i$. Suppose that $\mathcal{C}$ is a closed refinement of $\mathcal{T}^f$ with $|\mathcal{C}_i| = m_i$, such that $\mathcal{C}$ follows pattern h in $\mathcal{T}^f$. We define the *Markov graph-chain function*, $\hat{f}$, on the elements of $\mathcal{C}$ by the following. (We denote the lexicographical ordering on E' by $\ll$.)

First let j be the least integer, k, such that $(1, k) \in E'$ and define

$$\hat{f}_{p,r}(C_1^{1,j}) = f^{-1}(C_1^{1,j}) \cap T_r^p$$

where $(p, r) \in S_{h((1,j),1)}$. For $(k, \ell) \in E'$ and $(p, r) \in S_{h((k,\ell),1)}$ let

$$\hat{f}_{p,r}(C_1^{k,\ell}) = \left[f^{-1}(C_1^{k,\ell}) \cap T_r^p\right] - \left[\bigcup_{(q,s) \in E', (q,s) \ll (k,\ell)} \hat{f}_{p,r}(C_1^{q,s})\right]^{\circ}.$$

For $(p,r) \in S_{h((1,j),n_{i,j})}$ define

$$\hat{f}_{p,r}(C^{1,j}_{n_{1,j}}) \quad = \quad \left[f^{-1}(C^{1,j}_{n_{1,j}}) \cap T^p_r\right] - \left[\bigcup_{(j,\ell)\in E'} \hat{f}_{p,r}(C^{j,\ell}_1)\right]^{\circ}.$$

For $(k,\ell) \in E'$ and $(p,r) \in S_{h((k,\ell),n_{k,\ell})}$, let

$$\begin{aligned}\hat{f}_{p,r}(C^{k,\ell}_{n_{k,\ell}}) &= \\ &f^{-1}(C^{k,\ell}_{n_{k,\ell}}) \cap T^p_r \\ &- \left(\left[\bigcup_{(\ell,m)\in E'} \hat{f}_{p,r}(C^{\ell,m}_1)\right]^{\circ} \cup \left[\bigcup_{(q,s)\in E',(q,s)\ll(k,\ell)} \hat{f}_{p,r}(C^{q,s}_{n_{q,s}})\right]^{\circ}\right).\end{aligned}$$

Finally for any $(j,k) \in E'$, $1 < m < n_{j,k}$ and $(p,r) \in S_{h((j,k),m)}$ let

$$\begin{aligned}\hat{f}_{p,r}(C^{j,k}_m) \quad &= \quad f^{-1}(C^{j,k}_m) \cap T^p_r \\ &- \left(\left[\hat{f}_{p,r}(C^{j,k}_{m-1})\right]^{\circ} \cup \left[\hat{f}_{p,r}(C^{j,k}_1)\right]^{\circ} \cup \left[\hat{f}_{p,r}(C^{j,k}_{n_{j,k}})\right]^{\circ}\right).\end{aligned}$$

Define

$$\hat{f}(\mathcal{C}) = \left\{\hat{f}_{p,r}(C^{i,j}_k) | (i,j) \in E',\ k \le n_i \quad \text{and} \quad (p,r) \in S_{h((i,j),k)}\right\}.$$

Notice that since f restricted to each link of $\mathcal{T}^f$ is monotone, f restricted to each link of $\mathcal{C}$ is monotone. So each element of $\hat{f}(\mathcal{C})$ is connected.

Lemma 3.1. *If $\mathcal{C}$ is a closed refinement of $\mathcal{T}^f$ then $\hat{f}(\mathcal{C})$ is a closed graph-chain and the components of $\hat{f}(\mathcal{C})$ refine $\mathcal{T}^f$.*

Proof. For every $i \in E'$, let $m_i = |\mathcal{C}_i|$. Every element of $\hat{f}(\mathcal{C})$ is closed and $\hat{f}(\mathcal{C})$ covers G. Suppose now that $C^i_k \cap C^p_r = \emptyset$, but $\hat{f}(C^i_k) \cap \hat{f}(C^p_r) \neq \emptyset$. Let $x \in \hat{f}(C^i_k) \cap \hat{f}(C^p_r)$. This implies that $f(x) \in C^i_k \cap C^p_r$, a contradiction. So the only elements of $\hat{f}(\mathcal{C})$ which intersect are images of links of $\mathcal{C}$ which intersected, and since $\mathcal{C}$ is a closed graph-chaining of G, $\hat{f}(\mathcal{C})$ is also a closed graph-chaining of G. By definition, links of $\hat{f}(\mathcal{C})$ intersect only on their boundary and the components of $\hat{f}(\mathcal{C})$ refine $\mathcal{T}^f$. □

Now suppose that g is another Markov mapping of G and let

$$\mathcal{S}^g = \{S^i_0, \dots S^i_{n_i} | i \in E'\}$$

be the Markov graph-chain associated with g where, for every $i \in E'$, $|\mathcal{S}_i| = n_i$. Denote the corresponding Markov partition by

$$B_g = \{d^{i,j}_0 < d^{i,j}_1 < \cdots < d^{i,j}_{n_{i,j}} | (i,j) \in E'\}.$$

We are now ready to prove Theorem 3.1.

Proof of 3.1. Choose a positive number δ_1 such that if $\mathcal{H}$ is a closed graph-chaining of G with mesh less than δ_1 which refines $\mathcal{T}^f$ or $\mathcal{S}^g$ then $\pi_1^{-1}(\mathcal{H}) \cap \varprojlim\{G,f\}$ and $\pi_1^{-1}(\mathcal{H}) \cap \varprojlim\{G,g\}$ both have mesh less than $\frac{1}{2}$. For every

$i \in E'$ and $j \leq n_i$, let Q^i_j be a positive integer such that $\delta_1 \cdot Q^i_j > \mathrm{diam}(T^i_j)$ and $\delta_1 \cdot Q^i_j > \mathrm{diam}(S^i_j)$.

Let $\mathcal{C}_1$ be a closed graph-chaining of G with mesh less than δ_1 which refines $\mathcal{T}^f$ such that, for every $i \in E'$ and $j \leq n_i$, there are Q^i_j links of $\mathcal{C}_1$ contained in T^i_j. Let $\mathcal{J}_1$ be defined similarly with respect to g and $\mathcal{S}^g$.

Let $\mathcal{D}_1 = \pi_1^{-1}(\mathcal{C}_1) \cap \varprojlim\{G, f\}$ and let $\mathcal{K}_1 = \pi_1^{-1}(\mathcal{J}_1) \cap \varprojlim\{G, g\}$. It is easy to see that both of $\mathcal{D}_1$ and $\mathcal{K}_1$ are closed graph-chainings of $\varprojlim\{G, f\}$ and $\varprojlim\{G, g\}$ respectively with mesh less than $\frac{1}{2}$.

By lemma 3.1 both of $\hat{f}(\mathcal{C}_1)$ and $\hat{g}(\mathcal{J}_1)$ are closed graph-chainings of G which refine $\mathcal{T}^f$ and $\mathcal{S}^g$ respectively. Let δ_2 be a positive number so that any closed graph-chaining of G, $\mathcal{H}$, with mesh less than δ_2 has $\pi_2^{-1}(\mathcal{H}) \cap \varprojlim\{G, f\}$ and $\pi_2^{-1}(\mathcal{H}) \cap \varprojlim\{G, g\}$ both have mesh less than $\frac{1}{4}$.

By the hypothesis of the theorem, $\hat{f}_{p,r}(C^i_j)$ is defined if and only if $\hat{g}_{p,r}(J^i_j)$ is defined. Notice that by the construction of $\mathcal{C}_1$ and $\mathcal{J}_1$, there is a function, ℓ, such that $\mathcal{C}_1$ follows pattern ℓ in $\mathcal{T}^f$ and $\mathcal{J}_1$ also follows pattern ℓ in $\mathcal{S}^g$. So for every $i \in E'$, $j \leq n_i$, and $(p, r) \in S_{\ell(i,j)}$, let $Q^{p,r}_{i,j}$ be a positive integer such that $Q^{p,r}_{i,j} \cdot \delta_2 > \mathrm{diam}(\hat{f}_{p,r}(C^i_j))$ and $Q^{p,r}_{i,j} \cdot \delta_2 > \mathrm{diam}(\hat{g}_{p,r}(J^i_j))$. Let $\mathcal{C}_2$ be a refinement of $\hat{f}(\mathcal{C}_1)$ such that there are $Q^{p,r}_{i,j}$ links of $\mathcal{C}_2$ inside each $\hat{f}_{p,r}(C^i_j)$. Let $\mathcal{J}_2$ be a refinement of $\hat{g}(\mathcal{J}_2)$ defined similarly. Define $\mathcal{D}_2$ to be $\pi_2^{-1}(\mathcal{C}_2) \cap \varprojlim\{G, f\}$ and define $\mathcal{K}_2$ to be $\pi_2^{-1}(\mathcal{J}_2) \cap \varprojlim\{G, g\}$.

Notice that if A is a subset of $\hat{f}_{p,r}(C^i_j)$ then

$$\pi_2^{-1}(A) \cap \varprojlim\{G, f\} \subseteq \pi_1^{-1}(C^i_j) \cap \varprojlim\{G, f\},$$

and similarly if A is a subset of $\hat{g}_{p,r}(J^i_j)$ then

$$\pi_2^{-1}(A) \cap \varprojlim\{G, g\} \subseteq \pi_1^{-1}(J^i_j) \cap \varprojlim\{G, g\}.$$

So, since we have exactly $Q^{p,r}_{i,j}$ links of $\mathcal{C}_2$ and $\mathcal{J}_2$ in $\hat{f}_{p,r}(C^i_j)$ and $\hat{g}_{p,r}(J^i_j)$ respectively, $\mathcal{D}_2$ follows the same pattern, h_2, in $\mathcal{D}_1$ that $\mathcal{K}_2$ follows in $\mathcal{K}_1$.

Clearly, chains of $\varprojlim\{G, f\}$ and $\varprojlim\{G, g\}$, $\mathcal{D}_3$ and $\mathcal{K}_3$ can be constructed such that $\mathrm{mesh}(\mathcal{D}_3) < \frac{1}{8}$, $\mathrm{mesh}(\mathcal{K}_3) < \frac{1}{8}$, and both $\mathcal{D}_3$ and $\mathcal{K}_3$ follow pattern h_3 in $\mathcal{D}_2$ and $\mathcal{K}_2$ respectively.

So it is easy to see that we can build a sequence of refining chainings, $\{\mathcal{D}_i\}_{i=1}^{\infty}$, of $\varprojlim\{G, f\}$ such that $\mathcal{D}_i$ follows pattern h_i in $\mathcal{D}_{i-1}$ and

$$\lim_{i\to\infty} \mathrm{mesh}(\mathcal{D}_i) = 0,$$

and we can build a sequence of refining chainings, $\{\mathcal{K}_i\}_{i=1}^{\infty}$, of $\varprojlim\{G, g\}$ such that $\mathcal{K}_i$ follows pattern h_i in $\mathcal{K}_{i-1}$ and

$$\lim_{i\to\infty} \mathrm{mesh}(\mathcal{K}_i) = 0.$$

Thus, by Theorem 3.2, $\varprojlim\{G,f\}$ is homeomorphic to $\varprojlim\{G,g\}$. $\square$

This theorem provides some justification for the assumption that we make in the next section that $f^{-1}(x)$ is completely disconnected for all $x \in G$. It shows that for a bonding map with finitely many turning points each on a finite orbit that we lose no generality in making this assumption. It also has an interesting, and immediate, corollary which is an extension of a theorem of Holte ([7], Theorem 3.2).

Corollary 3.2.1. *Let each of f and g be Markov maps of the interval with associated Markov partitions, $B_f = \{0 = c_0 < c_1 < \cdots < c_n = 1\}$, and $B_g = \{0 = d_0 < d_1 < \cdots < d_n = 1\}$. Suppose that $f(c_i) = c_j$ if, and only if, $g(d_i) = d_j$, then $\varprojlim\{[0,1,f\}$ is homeomorphic to $\varprojlim\{[0,1],g\}$.*

This corollary extends Holte's theorem in the sense that the Markov partitions for f and g do not need to be the same set of points in the interval. So it allows for not only eliminating flat spots, but also dramatically changing slopes and shifting turning points around.

4. Inverse Limit Spaces

In this section we will assume that $f : G \to G$ is continuous, has finitely many turning points and, for every $x \in G$, $f^{-1}(x)$ is completely disconnected. We will also assume that for every $y \in G$ there exists a positive integer n such that $f^{-n}(y)$ consists of more than one point and if $C \subseteq G$ is connected then $\mathrm{diam}(C') \leq \mathrm{diam}(C)$ for every component, C' of $f^{-1}(C)$.

Let $E_G \subseteq G$ be the set of endpoints of G. The *ω-limit set of a point x under a mapping f*, $\omega_f(x)$ or simply $\omega(x)$, is given by

$$\omega(x) = \bigcap_{N\in\mathbb{N}} \overline{\{f^n(x)|n \geq N\}},$$

and the ω-limit set of a set, X, is given by $\omega(X) = \bigcup_{x\in X} \omega(x)$.

Denote the elements of P_f, the turning points of f, in the edge $e_{i,j}$ by $t^{i,j}_{1,0} < t^{i,j}_{2,0} \cdots < t^{i,j}_{m,0}$, where $<$ is the linear ordering of the arc $e_{i,j}$ that has v_i as its least element, and for every $t^{i,j}_{k,0}$ denote the orbit of $t^{i,j}_{k,0}$ by

$$\mathrm{orb}(t^{i,j}_{k,0}) = \{t^{i,j}_{k,\ell} = f^{\ell}(t^{i,j}_{k,0})|\ell \in \mathbb{N}\}.$$

Let

$$\mathrm{orb}(P_f) = \bigcup_{t^{i,j}_{k,0}\in P_f} \mathrm{orb}(t^{i,j}_{k,0}).$$

Theorem 4.1. *Let $x \in \varprojlim\{G,f\}$ have the property that for some $N \in \mathbb{N}$, if $n \geq N$ then $deg(x_n) \in \{0,2\}$. Then there is a positive number ϵ and a zero-dimensional set S with the property that $B_\epsilon(x)$ is homeomorphic to $(0,1) \times S$ if, and only if:*

(i) *there is a positive number, δ, and a positive integer, s, such that $B_\delta(x_s) \cap f^p(E_G) = \emptyset$ for every $p \geq 0$, and*
(ii) *there exists a positive integer, m, such that $x_m \notin \omega(P_f)$.*

Proof. Let $N \in \mathbb{N}$ such that if $n \geq N$ then $\deg(x_n) < 3$, and let $m \geq N$ be such that $x_m \notin \omega(P_f)$. Since $f[\omega(P_f)] \subseteq \omega(P_f)$, if $n \geq m$ then $x_n \notin \omega(P_f)$. Let $n \geq m$. Since $x_n \notin \omega(P_f)$, $x_n \notin \omega(t^{i,j}_{k,0})$ for every $t^{i,j}_{k,0} \in P_f$. By the definition of the ω-limit set, for every $t^{i,j}_{k,0} \in P_f$, there is a positive integer, $p^{i,j}_{k,0}$ with the property that $x_n \notin \overline{\{t^{i,j}_{k,r} | r \geq p^{i,j}_{k,0}\}}$. Since there are only finitely many turning points for f, there exists a positive integer, q, such that $x_n \notin \overline{\{t^{i,j}_{k,r} | r \geq q\}}$ for all $t^{i,j}_{k,0} \in P_f$. So x_{n+q+1} is not in $\overline{\{t^{i,j}_{k,r} | r \in \mathbb{N}\}}$ for all $t^{i,j}_{k,0} \in P_f$. Let $p = n+q+1$. There is a positive number, $\lambda_p < \delta$, such that

$$B_{\lambda_p}(x_p) \cap \left[\bigcup_{t^{i,j}_{k,0} \in P_f} \overline{\{t^{i,j}_{k,r} | r \in \mathbb{N}\}} \right] = \emptyset,$$

and if $y \in B_{\lambda_p}(x_p)$ then $\deg(y)$ is either 0 or 2. So $B_{\lambda_p}(x_p)$ is homeomorphic to $(0,1)$, and if $q \in \mathbb{N}$ and A is any component of $f^{-q}[B_{\lambda_p}(x_p)]$ then A is homeomorphic to $(0,1)$. Let ϵ be a positive number such that $\pi_p[B_\epsilon(x)] \subseteq B_{\lambda_p}(x_p)$. Let $A = \pi_p[B_\epsilon(x)]$. Let $A_1, A_2, \dots A_n$ be the components of $f^{-1}(A)$. For each $i \leq n$, $f|_{A_i}$ is monotone and $A_i \cap f^m(E) = \emptyset$ for all $m \geq 0$. For each $i \leq n$, let $A^i_1, A^i_2, \dots A^i_{n_i}$ be the components of $f^{-1}(A_i)$. Again, for each $j \leq n_i$, $f|_{A^i_j}$ is monotone and $A^i_j \cap f^m(E) = \emptyset$ for all $m \geq 0$. Assuming that $A^{i,j,\dots p,t}_k$ is a component of $f^{-1}[A^{i,j,\dots p}_t]$, define $A^{i,j,\dots k}_1, A^{i,j\dots k}_2 \dots A^{i,j,\dots k}_{n_{i,j,\dots k}}$ to be the components of $f^{-1}[A^{i,j,\dots t}_k]$.

Let S be a collection of sequences of positive integers such that $\langle y_i \rangle \in S$ if, and only if, A_{y_1} is defined and for every other $i \in \mathbb{N}$, $A^{y_1,\dots y_i}_{y_{i+1}}$ is defined. Clearly S is zero-dimensional. Each sequence, $\langle y_i \rangle$ in S defines a sequence of open arcs, $A_{y_1}, A^{y_1}_{y_2}, \dots$, with the property that f maps $A^{y_1,y_2,\dots y_i}_{y_{i+1}}$ onto $A^{y_1,y_2\dots y_{i-1}}_{y_i}$ monotonically. So $\varprojlim\{A^{y_1,y_2,\dots y_i}_{y_{i+1}}, f\}$ is homeomorphic to $(0,1)$, and

$$B_\epsilon(x) = \bigcup_{\langle y_i \rangle \in S} \varprojlim\{A^{y_1,y_2,\dots y_i}_{y_{i+1}}, f\}$$

is homeomorphic to $S \times (0,1)$.

Now assume that x does not satisfy either (i) or (ii) in the theorem. First let ϵ be a positive number, and consider the ϵ-neighborhood around x, $B_\epsilon(x)$. Let n be a positive integer and let γ be a positive number such that $\pi_n^{-1}[B_\gamma(x_n)] \subseteq B_\epsilon(x)$, $B_\gamma(x_n)$ meets P_f at at most one point, and if $y \in B_\gamma(x_n)$ then $\deg(y) < 3$.

If for every positive number $\delta < \gamma$ and every positive integer M, there exists a positive integer, $m > M$, such that every image under f^{-m} of $B_\delta(x_n)$ meets the set of endpoints for G, E_G, then clearly for every positive number,

λ, we can λ-chain $\varprojlim\{G, f\}$ with a linear subchain that starts at x. Thus there is a chainable endcontinuum in G having x as an endpoint, and $B_\epsilon(x)$ cannot be homeomorphic to $(0,1) \times S$ where S is a zero-dimensional set.

Instead, now suppose that γ is small enough and n is large enough so that $f^{-m}(B_\gamma(x_n))$ misses E_G for every positive integer m. Let $A = B_\gamma(x_n)$, and as above, enumerate the components of the preimages of A, $A_1, A_2, \dots A_n$. Continuing as previously, let S be the set of sequences of positive integers, $\langle y_i \rangle$, where $\langle y_i \rangle \in S$ if, and only if, for every positive integer i, $A_{y_{i+1}}^{y_1, \dots y_i}$ is a component of $f^{-1}(A_{y_i}^{y_1, \dots y_{i-1}})$. Let $t_{k,0}^{i,j} \in P_f$ with $x_n \in \omega(t_{k,0}^{i,j})$. Then for infinitely many positive integers, m, $t_{k,m}^{i,j} \in A$. So there are infinitely many connected subsets of G, $A_{y_{r+1}}^{y_1 \dots y_r}$ that contain $t_{k,0}^{i,j}$. If there exists one of these subsets, $A_{y_{r+1}}^{y_1, \dots y_r}$ that only meet B_f at the singleton $t_{k,0}^{i,j}$ then, since these components do not contain any endpoints or vertices of G, every component of the preimages of $A_{y_{r+1}}^{y_1, \dots y_r}$ is mapped with a single fold across $A_{y_{r+1}}^{y_1, \dots y_r}$. Thus $B_\epsilon(x)$ contains a subspace homeomorphic to a neighborhood of $(0,1)$ in the $\sin(1/x)$-continuum, and it cannot be homeomorphic to $(0,1) \times T$, where T is a zero-dimensional set. If instead, for some $A_{y_{r+1}}^{y_1, \dots, y_r}$ we have $A_{y_{r+1}}^{y_1, \dots y_r} \cap B_f$ is a finite set then clearly we can restrict the size of $A_{y_{r+1}}^{y_1, \dots, y_r}$ in order to make the subset meet B_f at a singleton and produce a similar subspace.

So suppose that each subset, $A_{y_{r+1}}^{y_1, \dots y_r}$ that contains a turning point meets B_f on an infinite set. Pick one of these subsets and call it A_1. Let A_2 be a connected subset of G such that $f^{n_1}(A_2) = A_1$ and $A_2 \cap P_f \neq \emptyset$. Given A_i define A_{i+1} to be a connected subset of G with the property that $f^{n_i}(A_{i+1}) = A_i$ and A_{i+1} meets P_f. Then since A is small enough to not meet P_f at two points and since all of these subsets are preimages of A, they must all meet P_f at a single point. Similarly, they must all miss V' and the set of endpoints of G. Thus, for every positive integer, i, $f|_{A_i}$ is a two-pass map, and $f^{n_i}|_{A_{i+1}}$ is at least a two-pass map. Thus $\varprojlim\{A_i, f|_{A_i}\}$ is an indecomposable subcontinuum, and $B_\epsilon(x)$ contains an indecomposable subcontinuum. Hence, $B_\epsilon(x)$ is not homeomorphic to $(0,1) \times T$ where T is a zero-dimensional set. ☐

Corollary 4.1.1. *Suppose that f and g are maps of $[0,1]$ with the properties listed above. Further suppose that $|\omega(P_f)| = n$ and $|\omega(P_g)| = m$. If $n \neq m$ then $\varprojlim\{[0,1], f\}$ is not homeomorphic to $\varprojlim\{[0,1], g\}$.*

Proof. Notice that for every point in $\omega(P_f)$ there is a point in the inverse limit that either is an endpoint or is in a neighborhood homeomorphic to a neighborhood of $(0,1)$ in the $\sin(1/x)$-continuum. Also notice that any other point in the inverse limit has a neighborhood homeomorphic to the product of $(0,1)$ with a zero-dimensional set, S. These properties are preserved by homeomorphism, and so any space homeomorphic to it must have the same number of points in the ω-limit set of its turning points. ☐

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