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MORITA EQUIVALENCE IN THE CONTEXT OF HILBERT MODULES

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ABSTRACT. The Morita equivalence of m-regular involutive quantales in the context of the theory of Hilbert A-modules is presented. The corresponding fundamental representation theorems are shown. We also prove that two commutative m-regular involutive quantales are Morita equivalent if and only if they are isomorphic.

In the paper [5] F. Borceux and E.M. Vitale made a first step in extending the theory of Morita equivalence to quantales. They considered unital quantales and the category of all left modules over these unital quantales which are unital in a natural sense. They proved that two such module categories over unital quantales A and B, say, are equivalent if and only if there exist a unital $A-B$ bimodule M and a unital $B-A$ bimodule N such that $M \otimes_B N \simeq A$ and $N \otimes_A M \simeq B$.

The aim of this paper is to extend this theory in the following way: to cover also the case of m-regular (generally non-unital) involutive quantales and Hilbert modules over them. Our motivation to work in this setting comes from the theory of operator algebras, where there is a theory of Morita equivalence for C*-algebras for the non-unital case (see [4], [8] and [12]). Our presentation is a combination of those in [1], [4] and [5].

This paper is closely related to the papers [9] and [11] where the interested reader can find unexplained terms and notation concerning the subject. For facts concerning quantales and quantale modules in general we refer to [13]. The algebraic background may be found in any account of Morita theory for rings, such as [1] or [3].

The paper is organized as follows. First, we recall the notion of a right Hilbert A-module and related notions. In Section 1 the necessary basic properties of right Hilbert modules are established. Moreover, a categorical

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characterization of surjective module maps in the category of m-regular right Hilbert modules is given. In section 2 the key result is the Eilenberg-Watts theorem which states that colimit preserving $*$-functors between categories of Hilbert modules correspond to right Hilbert bimodules. Our second result is the fundamental Morita theorem for Hilbert modules. As a consequence we get that two m-regular involutive quantales are strongly Morita equivalent if and only if they are Morita equivalent. Moreover, two commutative m-regular involutive quantales are Morita equivalent if and only if they are isomorphic.

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Preliminaries

Let us begin by establishing the common symbols and notations in this paper.

In what follows, a complete lattice will be called *sup-lattice*. *Sup-lattice homomorphisms* are maps between sup-lattices preserving arbitrary joins. We shall denote, for S, T sup-lattices, $SUP(S,T)$ the sup-lattice of all sup-lattice homomorphisms from S to T, with the supremum given by the pointwise ordering of mappings. If $S = T$ we put $\mathcal{Q}(S) = SUP(S,S)$. Recall that a *quantale* is a sup-lattice A with an associative binary multiplication satisfying

$$x \cdot \bigvee_{i\in I} x_i = \bigvee_{i\in I} x \cdot x_i \quad \text{and} \quad (\bigvee_{i\in I} x_i) \cdot x = \bigvee_{i\in I} x_i \cdot x$$

for all $x, x_i \in A$, $i \in I$ (I is a set). 1 denotes the greatest element of A, 0 is the smallest element of A. A quantale A is said to be *unital* if there is an element $e \in A$ such that $e \cdot a = a = a \cdot e$ for all $a \in A$. An *opposite quantale* A^d to A is a sup-lattice with the same join operation as A and with the multiplication $a \circ b = b \cdot a$. A *subquantale* A' of a quantale A is a subset of A closed under $\bigvee$ and $\cdot$. Since the operators $a \cdot -$ and $- \cdot b : A \to A$, $a, b \in A$ preserve arbitrary joins, they have right adjoints. Explicitly, they are given by

$$a \to_r c = \bigvee\{s \in A | a \cdot s \leq c\} \quad \text{and} \quad b \to_l d = \bigvee\{t \in A | t \cdot b \leq d\}$$

respectively.

An *involution* on a sup-lattice S is a unary operation such that

$$\begin{aligned} a^{**} &= a, \\ (\bigvee a_i)^* &= \bigvee a_i^* \end{aligned}$$

for all $a, a_i \in S$. An *involution* on a quantale A is an involution on the sup-lattice A such that

$$(a \cdot b)^* = b^* \cdot a^*$$

for all $a, b \in A$. A sup-lattice (quantale) with the involution is said to be *involutive*.

By a *morphism of* (*involutive*) *quantales* will be meant a $\bigvee$- (*-) and $\cdot$-preserving mapping $f : A \to A'$. If a morphism preserves the unital element we say that it is *unital*.

Let A be a quantale. A *right module over* A (shortly a right A-module) is a sup-lattice M, together with a *module action*

$$-\diamond- : M \times A \to M$$

satisfying

(M1) $m \diamond (a \cdot b) = (m \diamond a) \diamond b$
(M2) $(\bigvee X) \diamond a = \bigvee\{x \diamond a : x \in X\}$
(M3) $m \diamond \bigvee S = \bigvee\{m \diamond s : s \in S\}$

for all $a, b \in A$, $m \in M$, $S \subseteq A$, $X \subseteq M$. So we have two maps

$$- \to_L - : M \times A \to M, \quad - \to_R - : M \times M \to A$$

such that, for all $a \in A$, $m, n \in M$,

$$m \diamond a \leq n \quad \text{iff} \quad a \leq m \to_R n \quad \text{iff} \quad m \leq a \to_L n.$$

M is called a *unital* A-module if A is a unital quantale with the unit e and $m \diamond e = m$ for all $m \in M$.

Let M and N be modules over A and let $f : M \to N$ be a sup-lattice homomorphism. f is a *module homomorphism* if $f(m \diamond a) = f(m) \diamond a$ for all $a \in A, m \in M$. We shall denote by MOD_A the category of right $A-$modules and module homomorphisms.

For a module X in MOD_A the submodule $\mathrm{ess}(X) = X \diamond A$ generated by the elements $x \diamond a$ is called the *essential part* of X. If $\mathrm{ess}(X) = X$ we say that X is *essential*. The full subcategory of essential A-modules is denoted $\mathrm{ess} - MOD_A$. We shall say that A is *right separating* for the A-module M and that M is (*right*) *separated* by A if $m \diamond (-) = n \diamond (-)$ implies $m = n$. We say that M is *m-regular* if it is both separated by A and essential. An involutive quantale A is called *m-regular* if it is m-regular as an A-module. Then, evidently $1 \cdot 1 = 1$ in A and $a \cdot (-) = b \cdot (-)$ implies $a = b$.

Note that we may dually define the notion of a (unital) left A-module with a left multiplication $\bullet$. We then have two maps

$$- \to_l - : M \times M \to A, \quad - \to_r - : A \times M \to M$$

such that, for all $a \in A$, $m, n \in M$,

$$a \bullet m \leq n \quad \text{iff} \quad a \leq m \to_l n \quad \text{iff} \quad m \leq a \to_r n.$$

The theory of Hilbert A-modules (we refer the reader to [9] for details and examples) is a generalization of the theory of complete semilattices with a duality and it is the natural framework for the study of modules over an involutive quantale A endowed with A-valued inner products.

Let A be an involutive quantale, M a right (left) A-module. We say that M is a *right* (*left*) Hilbert A-module (*right* (*left*) *strict Hilbert A-module),* right (left) *pre-Hilbert A-module* if M is equipped with a map

$$\langle -,-\rangle : M\times M\to A$$

called the *inner product*, such that for all $a\in A$, $m,n\in M$ and $m_i\in M$, where $i\in I$, the conditions (0.1)–(0.5) ((0.1)–(0.6), (0.1)–(0.4)) are satisfied.

$$\langle m,n\rangle\cdot a=\langle m,n\diamond a\rangle \quad (a\cdot\langle m,n\rangle=\langle a\bullet m,n\rangle); \tag{0.1}$$

$$\bigvee_{i\in I}\langle m_i,n\rangle=\langle\bigvee_{i\in I}m_i,n\rangle; \tag{0.2}$$

$$\bigvee_{i\in I}\langle m,m_i\rangle=\langle m,\bigvee_{i\in I}m_i\rangle; \tag{0.3}$$

$$\langle m,n\rangle^*=\langle n,m\rangle; \tag{0.4}$$

$$\langle -,m\rangle=\langle -,n\rangle \ (\langle m,-\rangle=\langle n,-\rangle) \text{ implies } m=n; \tag{0.5}$$

$$\langle m,m\rangle=0 \text{ implies } m=0. \tag{0.6}$$

If A is an involutive quantale, let $HMOD_A$ be the category of right Hilbert A-modules with morphisms the usual A-module maps. The full subcategory of m-regular right Hilbert A-modules is denoted $\mathrm{mreg}-HMOD_A$.

Let A be an involutive quantale, $f : M\to N$ a map between right (left) pre-Hilbert A-modules. We say that a map $g : N\to M$ is a **-adjoint to f* and f is *adjointable* if

$$\langle f(m),n\rangle=\langle m,g(n)\rangle$$

for all $m\in M$, $n\in N$. Evidently, any adjointable map is a module homomorphism. If f is adjointable we put

$$f^*=\bigvee\{g : N\to M; g \text{ is a }^*\text{-adjoint to } f\}.$$

Note that $f\le f^{**}$ and $f^*=f^{***}$. If M and N are Hilbert A-modules the *-adjoint to f is uniquely determined by property (0.5) i.e. $f=f^{**}$. The set of all adjointable maps from M to N is denoted by $\mathcal{A}_A(M,N)$. We shall denote by $\mathrm{mreg}-Hilb_A$ the category of m-regular right Hilbert A-modules and adjointable mappings. We say that an adjointable map $f : M\to N$ is an *isometry* if, for all $m_1,m_2\in M$, $\langle m_1,m_2\rangle=\langle f(m_1),f(m_2)\rangle$. This is equivalent to $f^*\circ f=\mathrm{id}_M$. Similarly, an adjointable map $f : M\to N$ is *unitary* if $f^*\circ f=\mathrm{id}_M$ and $f\circ f^*=\mathrm{id}_N$. Note that any surjective isometry is necessarily unitary.

Recall that from [9] we know that, for any right Hilbert A-module M and for all $m\in M$, the map $m^\sim : A\to M$ defined by $a\mapsto m\diamond a$ has a *-adjoint $m^\star : M\to A$ defined by $n\mapsto\langle m,n\rangle$.

Similarly, let A and B be involutive quantales, and let M and X be right Hilbert B-modules. We say that M is a *right Hilbert $A-B$ bimodule* if it is a left A-module satisfying

$$a \bullet (x \diamond b) = (a \bullet x) \diamond b \text{ and } \langle a \bullet x, y\rangle_B = \langle x, a^* \bullet y\rangle_B \tag{0.7}$$

for all $a \in A$, $x, y \in M$, and $b \in B$. We say that F is an m-regular right Hilbert $A - B$ bimodule if it is both an m-regular left A-module and an m-regular right B-module. An *isomorphism* of right Hilbert bimodules is a bijective $\bigvee$-preserving map $\Phi\colon M \to N$ such that

(i) $\Phi(a \bullet x) = a \bullet \Phi(x)$,
(ii) $\Phi(x \diamond b) = \Phi(x) \diamond b$, and
(iii) $\langle \Phi(x), \Phi(y)\rangle_B = \langle x, y\rangle_B$

i.e. Φ is an isometric surjective bimodule homomorphism. In particular, Φ is a unitary B-module map.

We say that a (full m-regular) right Hilbert $A - B$ bimodule X is an (*imprimitivity*) $A - B$ *bimodule* if X is also a (full m-regular) left Hilbert A-module in such a way that

$${}_A\langle x, y\rangle \bullet z = x \diamond \langle y, z\rangle_B \text{ and } {}_A\langle x \diamond b, y\rangle = {}_A\langle x, y \diamond b^*\rangle\,.$$

Suppose that M is a right Hilbert A-module, N is a right Hilbert $A - B$ bimodule. Then the sup-semilattice tensor product $M \otimes N$ is a pre-Hilbert B-module under the pre-inner product given on simple tensors by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_{M\otimes N} = \langle y_1, \langle x_1, x_2\rangle \bullet y_2\rangle$$

for all $x_1, x_2 \in M$ and $y_1, y_2 \in N$. We then denote by $M\dot{\otimes}_A N$ the corresponding Hilbert A-module $(M \otimes N)_{R_H}$ and we say that $M\dot{\otimes}_A N$ is a *interior tensor product* of M and N over A. It is shown in [11] that our interior tensor product has similar properties as its Hilbert C*-module counterpart. Note only that a right Hilbert A-module is m-regular if and only if $M\dot{\otimes}_A A \simeq M$ via the standard isomorphism $m\dot{\otimes}_A a \mapsto m \diamond a$.

We shall sometimes use $\mathbf{C}_A$ for some of the following categories: $\text{mreg} - HMOD_A$, $\text{mreg} - Hilb_A$. Similarly, we have the categories of left modules ${}_AMOD$, ${}_AHMOD$, $\text{mreg} - {}_AHMOD$, $\text{mreg} - {}_AHilb$).

In this paper we are concerned with functors between categories of modules. Such functors, e.g. $F : \text{mreg}-Hilb_A \to \text{mreg}-Hilb_B$ are assumed to be join-preserving (SUP-functors) on $\bigvee$-semilattices of morphisms. Thus the map $T \mapsto F(T)$ from $\text{mreg} - Hilb_A(X, W)$ to $\text{mreg} - Hilb_B(F(X), F(W))$ is join-preserving, for all pairs of objects $X, W \in \mathbf{C}_A$.

In what follows let us assume that A and B are m-regular involutive quantales. We say that a SUP-functor

$$F : \text{mreg} - Hilb_A \to \text{mreg} - Hilb_B$$

is a $*$-*functor* if $F(T^*) = F(T)^*$ for all adjointable A-module maps T. In particular, from [11] we know that the function

$$(-)\dot{\otimes}_A N : \text{mreg} - Hilb_A \to \text{mreg} - Hilb_B$$

which assigns to each right Hilbert A-module M the inner tensor product $M\dot{\otimes}_A N$ and to each adjointable map f between right Hilbert A-modules the adjointable map $f\dot{\otimes}_A\mathrm{id}_F$ between right Hilbert B-modules is a *-functor preserving biproducts.

We say two $*$-functors

$$F_1, F_2 : \mathrm{mreg} - Hilb_A \to \mathrm{mreg} - Hilb_B$$

are (naturally) unitarilly isomorphic, if they are naturally isomorphic via a natural transformation τ in the sense of category theory [1], with the natural transformations being unitaries i.e. $\tau^*_{F_1(M)} \circ \tau_{F_1(M)} = \mathrm{id}_{F(M_1)}$ and $\tau_{F(M_1)} \circ \tau^*_{F(M_1)} = \mathrm{id}_{F(M_2)}$. In this case we write $F_1 \cong F_2$ *unitarilly.* Similarly, we say that a $*$-functor $F : \mathrm{mreg} - Hilb_A \to \mathrm{mreg} - Hilb_B$ is a *unitary equivalence functor* if there is a $*$-functor $G : \mathrm{mreg} - Hilb_B \to \mathrm{mreg} - Hilb_A$ such that we have natural unitary isomorphisms $\eta : GF \to \mathrm{Id}_{\mathrm{mreg}-Hilb_A}$ and $\zeta : FG \to \mathrm{Id}_{\mathrm{mreg}-Hilb_B}$.

1. Hilbert modules

In this section we develop the basic properties of Hilbert modules not mentioned in [9]. Let us begin with a small observation about Hilbert modules.

Lemma 1.1. *For any involutive quantale $A \in HMOD_A$ and any right Hilbert A-module M, A is right separating for M.*

Proof. Assume that $m \diamond (-) = n \diamond (-)$ i.e. $m \diamond a = n \diamond a$ for all $a \in A$. Hence, for all $p \in M$ and all $a \in A$ we have that $\langle p, m \diamond a\rangle = \langle p, n \diamond a\rangle$ i.e. $\langle p, m\rangle \cdot a = \langle p, n\rangle \cdot a$ i.e. $\langle p, m\rangle = \langle p, n\rangle$. So we get that $m = n$. $\square$

The preceding observation then yields

Corollary 1.2. *Let A be an involutive quantale, $A \in HMOD_A$. Then a morphism f in $\mathbf{C}_A \in \{\mathrm{mreg} - HMOD_A, \mathrm{mreg} - Hilb_A\}$ is mono iff the map f is one-to-one.*

Proof. Evidently, any injective morphism in $\mathbf{C_A}$ is a monomorphism. Conversely, let $f : M \to N$ be a monomorphism in $\mathbf{C}_A$, $f(m_1) = f(m_2)$. Then $f \circ (m_1 \diamond (-)) = f \circ (m_2 \diamond (-))$ i.e. $m_1 \diamond (-) = m_2 \diamond (-)$. Hence $m_1 = m_2$ and f is injective. $\square$

Note that in the category $Hilb_A$ over $A \in HMOD_A$ an adjointable map is mono iff f^* is epi. This immediately yields that for any surjective adjointable map its adjoint is one-to-one.

Lemma 1.3. *The category $\mathbf{C}_A \in \{\mathrm{mreg} - HMOD_A, \mathrm{mreg} - Hilb_A\}$ has biproducts.*

Proof. We know from [9] that $Hilb_A$ has biproducts (with injections i_j and projections π_j) and these are exactly cartesian products. The lemma follows then from the fact that evidently any cartesian product of m-regular modules is m-regular. $\square$

As usual, for any $A \in HMOD_A$, A^J will be viewed as a Hilbert A-module equipped with the inner-product

$$\langle (a_j)_{j\in J}, (b_j)_{j\in J}\rangle = \bigvee_{j\in J} a_j^* b_j.$$

Let us observe that A^J will often stand for the set of column matrices over A of the type $S \times 1$. In that way, the above inner-product can be expressed, for $v = (a_j)_{j\in J}$ and $w = (b_j)_{j\in J}$, as $\langle v, w\rangle = v^* w$. Note that v^* refers to the conjugate-transpose matrix. We shall denote, for any $a \in A$ and all $j \in J$, by $\mathbf{a}_j$ an element of A^J defined as follows:

$$\pi_k(\mathbf{a}_j) = \begin{cases} a & \text{if } k = j \\ 0 & \text{otherwise.} \end{cases}$$

For each J-tuple $\mu = (m_j)_{j\in J}$ in the Hilbert A-module M^J, we denote by Ω_μ the operator in $\mathcal{A}_A(A^J, M)$ defined by

$$\Omega_\mu\big((a_j)_{j\in J}\big) = \bigvee_{j\in J} m_j \diamond a_j, \quad (a_j)_{j\in J} \in A^J.$$

It is easy to see that Ω_μ^* is given by

$$\Omega_\mu^*(x) = \big(\langle m_j, x\rangle\big)_j, \quad x \in M.$$

If $\nu = (n_j)_{j\in J}$ is an J-tuple of elements of N, then the operator $T = \Omega_\nu \circ \Omega_\mu^*$ is in $\mathcal{A}_A(M, N)$. In particular, for all $m \in M$ and $n \in N$, $\Omega_{(n)} = n^\sim$, $\Omega_{(m)}^* = m^\star$ and defining

$$\Theta_{n,m}(x) = (\Omega_{(n)} \circ \Omega_{(m)}^*)(x) = n \diamond \langle m, x\langle$$

we have

$$T(x) = \bigvee_{j\in J} \Theta_{n_j,m_j}(x), \quad x \in M.$$

Maps such as T are exactly *compact* operators in the sense of [9] and the set of all compact maps will be denoted $\mathcal{K}_A(M, N)$ or just $\mathcal{K}_A(M)$ in case $M = N$. Note that a composition of a compact operator with an adjointable one is again compact i.e. $\mathcal{K}_A(N, P) \circ \mathcal{A}_A(M, N) \subseteq \mathcal{K}_A(M, P)$ and $\mathcal{A}_A(N, P) \circ \mathcal{K}_A(M, N) \subseteq \mathcal{K}_A(M, P)$. An expository treatment of compact operators on Hilbert A-modules may be found in [9].

Lemma 1.4. *Let A be an m-regular involutive quantale and let $M \in \mathrm{mreg}-HMOD_A$, M is full. Then $\mathcal{K}_A(M)$ is an m-regular involutive quantale and M is an m-regular full left Hilbert $\mathcal{K}_A(M)$-module.*

Proof. Evidently, $\mathcal{K}_A(M)$ is an involutive quantale (see [9]). Let $f, g \in \mathcal{K}_A(M)$. Assume that $f \circ \Theta_{y,x} = g \circ \Theta_{y,x}$ for all $y, x \in X$. This gives us $\Theta_{f(y),x} = \Theta_{g(y),x}$ and since M is full then $f(y) \diamond a = g(y) \diamond a$ for all $a \in A$

i.e. $f=g$. Similarly,

$$\begin{array}{rcl}\Theta_{y,x} & = & \Theta_{\bigvee_{i\in I} y_i\diamond a_i, x}\\ & = & \bigvee_{i\in I}\Theta_{y_i\diamond a_i,x}\\ & = & \bigvee_{i\in I}\Theta_{y_i\diamond\bigvee_{j\in J}\langle v_{ij},u_{ij}\rangle,x}\\ & = & \bigvee_{i\in I,j\in J}\Theta_{y_i\diamond\langle v_{ij},u_{ij}\rangle,x}\\ & = & \bigvee_{i\in I,j\in J}\Theta_{y_i,v_{ij}}\circ\Theta_{u_{ij},x}\end{array}$$

for suitable elements y_i, u_{ij}, v_{ij}, $i\in I, j\in J$. This gives us that $\mathcal{K}_A(M)$ is m-regular.

We shall define the module action on $\bullet : \mathcal{K}_A(M)\times M\to M$ by

$$f\bullet m = f(m)$$

and the inner product ${}_{\mathcal{K}_A(M)}\langle -,-\rangle : M\times M\to\mathcal{K}_A(M)$ by

$${}_{\mathcal{K}_A(M)}\langle y,x\rangle = \Theta_{y,x}.$$

Then evidently the pre-Hilbert module conditions are satisfied,

$$M=\mathcal{K}_A(M)\bullet M,\ {}_{\mathcal{K}_A(M)}\langle M,M\rangle=\mathcal{K}_A(M).$$

Assume that ${}_{\mathcal{K}_A(M)}\langle y_1,-\rangle = {}_{\mathcal{K}_A(M)}\langle y_2,-\rangle$ i.e. $y_1\diamond a=y_2\diamond a$ for all $a\in A$. Hence $y_1=y_2$. □

Proposition 1.5. *Let A be an m-regular involutive quantale and let $M\in$ mreg $-$ $HMOD_A$. For each $\mu=(\mu_j)_{j\in J}$ in M^J one has that Ω_μ is in $\mathcal{K}_A(A^J,M)$ and hence also that Ω_μ^* is in $\mathcal{K}_A(M,A^J)$. Moreover, $\mathcal{K}_A(A^J,M)$ $\simeq M^J$ as sup-lattices.*

Proof. It is obviously enough to consider the case $|J|=1$. Let $\mu=(m)$ and $\bigvee_{\lambda\in\Lambda} m_\lambda\diamond a_\lambda=m$. Therefore we have for all a in A

$$\begin{array}{rcl}\Omega_\mu(a) & = & m\diamond a\\ & = & (\bigvee_{\lambda\in\Lambda} m_\lambda\diamond a_\lambda)\diamond a\\ & = & \Omega_{(m_\lambda)_{\lambda\in\Lambda}}\Omega^*_{({a_\lambda}^*)_{\lambda\in\Lambda}}(a).\end{array}$$

We have that

$$\Omega_{(m)}\circ\Omega^*_{(a)}(b)=\Omega_{(m\diamond a^*)}(b)$$

i.e. any generator of $\mathcal{K}_A(A^J,M)$ has the form $\Omega_{(m)}$. Since $\Omega_{(m)}$ is compact for all m we have that

$$\mathcal{K}_A(A^J,M)\simeq\{\Omega_{(m)} : m\in M\}\simeq M.$$

□

Lemma 1.6. *Let A be a unital involutive quantale and let $M\in$ mreg $-$ $HMOD_A$. Then $HMOD_A(A^J,M)=\mathcal{K}_A(A^J,M)\simeq M$.*

Proof. Let $f: A^J\to M$ be any module map, $x=(x_j)_{j\in J}$. Then

$$f(x)=\bigvee_{j\in J} f(\mathbf{e}_j)\diamond x_j=\bigvee_{j\in J}\left(\Omega_{f(\mathbf{e}_j)}\circ i_j\circ\Omega^*_{\mathbf{e}_j}\right)(x).$$

Hence,

$$f = \bigvee_{j \in J} \Omega_{f(\mathbf{e}_j)} \circ i_j \circ \Omega^*_{\mathbf{e}_j} \in \mathcal{K}_A(A^J, M).$$

□

Corollary 1.7. *Let A be a unital involutive quantale. Then id_{A^J} is in $\mathcal{K}_A(A^J)$.*

Definition 1.8. *Let A be an involutive quantale. A Hilbert A-module M will be said to be a* nuclear *module if the identity operator id_M is in $\mathcal{K}_A(M)$.*

Note that any m-regular involutive quantale $A \simeq \mathcal{K}_A(A)$ is nuclear if and only if it is unital. We shall now give the complete characterization of nuclear Hilbert modules over unital involutive quantales.

Proposition 1.9. *Let A be a unital involutive quantale and let $M \in \mathrm{ess} - HMOD_A$. The following conditions are equivalent:*

(1) *M is nuclear.*
(2) *M is a retract of A^J in $Hilb_A$.*

Proof. (1) $\Longrightarrow$ (2). Assume M to be nuclear. Then $\mathrm{id}_M = \Omega_\nu \Omega^*_\mu$ where μ and ν are in M^J.

(2) $\Longrightarrow$ (1). Let M be a retract of A^J in $Hilb_A$, $r : A^J \to M$ the retraction and $i : M \to A^J$ the embedding such that $r \circ i = \mathrm{id}_M$. Since $\mathrm{id}_{A^J} \in \mathcal{K}_A(A^J)$ we have that $\mathrm{id}_M = r \circ \mathrm{id}_{A^J} \circ i \in \mathcal{K}_A(M)$. □

Sometimes we shall need the following lemma, a part of which is contained in [9].

Lemma 1.10. *Let A be an involutive quantale and let M be a left (right) pre-Hilbert A-module. Then the factor module M_{R_H} defined by the equivalence relation*

$$R_H = \{(m,n) \in M \times M : \langle m,p \rangle = \langle n,p \rangle \text{ for all } p \in M\}$$

is a left (right) Hilbert A-module. Moreover, $j_H : M \to M_{R_H}$ is an adjointable map and if $f : N \to M$ is a module (adjointable) map then there is a unique module (adjointable) map $\overline{f} : N \to M_{R_H}$ such that $\overline{f} = j_H \circ f$; here $j_H(m) = \bigvee\{n \in M : (m,n) \in R_H\}$. Similarly, if $g : M \to P$ is an adjointable map then there is a unique adjointable map $\hat{g} : M_{R_H} \to P$ such that $\hat{g} = g \circ {j_H}^*$.

Proof. The main part of this lemma was proved in [9] and the adjointability of j_H follows from its definition. So let $f : N \to M$ be a module (adjointable) map. We shall put $\overline{f}(n) = j_H(f(n))$. Evidently, $\overline{f}$ is a module map. Let us check that $\overline{f}$ is adjointable. Assume $n \in N, j_H(m) \in M_{R_H}$. Then

$$\left\langle \overline{f}(n), j_H(m) \right\rangle = \langle f(n), j_H(m) \rangle = \langle n, f^*(j_H(m)) \rangle .$$

Clearly, such $\overline{f}$ is uniquely determined.

Similarly, let $g : M \to P$ be an adjointable map. We define

$$\hat{g}(m) = g({j_H}^*(m)) = g(m).$$

Since $\hat{g}$ is a composition of adjointable maps it is adjointable. The uniqueness is evident. $\square$

Theorem 1.11. *Let A be an involutive quantale. Then the category of pre-Hilbert A-modules has limits of arbitrary diagrams.*

Proof. The proof follows general category theoretic principles. We describe the limit of the diagram

$$\langle (M_i)_{i\in O}, (f_j \colon M_{d(j)} \to M_{c(j)})_{j\in J}\rangle$$

as a set of particular elements of the product of all M_i's, the so-called *commuting tuples.*

$$M = \{(x_i)_{i\in O} \in \prod_{i\in O} M_i : \forall j \in J\ x_{c(j)} = f_j(x_{d(j)})\}.$$

Evidently, M is a A-submodule of the product, that is, M is a pre-Hilbert A-module because the coordinatewise supremum of commuting tuples is commuting as all f_j are module maps. This also proves that the projections $\pi_j \colon \prod_{i\in O} M_i \to M_j$ restricted to M are module maps. They give us the maps needed to complement M to a cone.

Given any other cone $\langle E, (g_i \colon E \to M_i)_{i\in O}\rangle$, we define the mediating morphism $h \colon E \to M$ by $h(x) = (g_i(x))_{i\in O}$. Again, it is obvious that this is well-defined and a module map, and that it is the only possible choice. $\square$

We also have the dual:

Theorem 1.12. *Let A be an involutive quantale. Then the category of pre-Hilbert A-modules has colimits of arbitrary diagrams.*

Definition 1.13. *Let $\mathbf{C}$ be a subcategory of MOD_A and let M be a module in $\mathbf{C}$.*

1. *M is called* faithful *if $m \diamond a = m \diamond b$ for all $m \in M$ implies that $a = b$.*
2. *M* generates *$X \in \mathbf{C}$ if there is a direct sum $\coprod_{\gamma\in\Gamma} M$ of copies of M and an epimorphism $\varphi : \coprod_{\gamma\in\Gamma} M \to X$ in $\mathbf{C}$. M is a* generator *for $\mathbf{C}$, if M generates all modules $X \in \mathbf{C}$.*
3. *M* cogenerates *$X \in \mathbf{C}$ if there is a direct product $\prod_{\gamma\in\Gamma} M$ of copies of M and a monomorphism $\psi : X \to \prod_{\gamma\in\Gamma} M$ in $\mathbf{C}$. M is a* cogenerator *for $\mathbf{C}$, if M cogenerates all modules $X \in \mathbf{C}$.*

Lemma 1.14. *Let A be a right separating involutive quantale, M be a full right Hilbert A-module. Then M is faithful.*

Proof. Let $a, b \in A$ such that $m \diamond a = m \diamond b$ for all $m \in M$. Then $\langle n, m \diamond a\rangle = \langle n, m \diamond b\rangle$ for all $m, n \in M$ i.e. $\langle n, m\rangle \cdot a = \langle n, m\rangle \cdot b$ for all $m, n \in M$ i.e. $a = b$. $\square$

Lemma 1.15. *Let A be an m-regular involutive quantale. Then A is both a generator and a cogenerator for $\mathbf{C}_A \in \{\mathrm{mreg} - HMOD_A, \mathrm{mreg} - Hilb_A\}$. More exactly, for every M in $\mathbf{C}_A$, M is both a quotient module and a submodule of the sum $\coprod_{m\in M} A \cong \prod_{m\in M} A = A^M$.*

Proof. We shall use the compact maps $m^\sim : A \to M$ and $m^\star : M \to A$, $m \in M$.

$$\begin{tikzcd}
A^M \arrow[r, "\mathbf{p}_M"] & M \arrow[r, "\mathbf{i}_M"] \arrow[d, shift left, "m^\star"] & A^M \arrow[dl, "\pi_m"] \\
& A \arrow[ul, "i_m"] \arrow[u, shift left, "m^\sim"] &
\end{tikzcd}$$

They give us compact maps $\mathbf{p}_M : A^M \to M$, $\mathbf{p}_M = \Omega_{(m)_{m\in M}}$ and $\mathbf{i}_M : M \to A^M$, $\mathbf{i}_M = \Omega^*_{(m)_{m\in M}}$; here $\pi_m \circ \mathbf{i}_M = m^\star$ and $\mathbf{p}_M \circ \iota_m = m^\sim$. Since M is a Hilbert A-module $\mathbf{i}_M$ is injective and since M is essential $\mathbf{p}_M$ is surjective. Namely, for any $n \in M$, there is an element $u_n \in A^M$ defined by

$$p_m(u_n) = \bigvee \{a \in A : m \diamond a \le n\}$$

such that $n = \bigvee_{m\in M} m \diamond p_m(u_n) = \mathbf{p}_M(u_n)$. □

Note that the surjective module map $\mathbf{p}_M : A^M \to M$ may be defined for any essential A-module M. Similarly as in [6] we have the following proposition.

Proposition 1.16. *Let A be an m-regular involutive quantale. Then $U \in \mathrm{mreg} - HMOD_A$ ($U \in \mathrm{mreg} - Hilb_A$) is faithful if and only if it cogenerates a generator.*

Proof. Suppose that U is faithful. Then the module homomorphism (adjointable map) $f : A \to U^U$ defined by $f(a)(u) = u \diamond a$ is monic and U cogenerates the generator A. Conversely, let M be a generator for $\mathrm{mreg} - HMOD_A$ and U a cogenerator of M. Let $u \diamond a = u \diamond b$ for all $u \in U$. Then $m \diamond a = m \diamond b$ since M is embeddable into the product of copies of U. Since we have an epimorphism $g : M^J \to A$ such that $a^\sim \circ g = b^\sim \circ g$ we obtain that $c \cdot a = c \cdot b$ for all $c \in A$ i.e. $a = b$. □

Proposition 1.17. *Let A be an m-regular involutive quantale. Then, for every surjective module homomorphism $p : P \to M$ in $\mathrm{mreg} - HMOD_A$, p is the coker of a pair (u, v) of compact arrows from the following diagram.*

$$A^J \underset{v}{\overset{u}{\rightrightarrows}} P \xrightarrow{p} M. \tag{1.1}$$

Proof. Note that we know e.g. from [5] or [7] that in the category MOD_A of modules over A the diagram

$$\begin{tikzcd} D \arrow[r, shift left, "u'"] \arrow[r, shift right, "v'"'] & P \arrow[r, "p"] & M. \end{tikzcd}$$

exists; here $D = \{(x, y) \in P \times P : p(x) = p(y)\}$ is a right A-module that is right separated, u', v' are the respective projections. Then p is the coker of the pair (u', v') in this category. Let us form the following diagram.

$$\begin{tikzcd} \mathrm{ess}(D) \arrow[r, "\mathrm{in}_D"] & D \arrow[r, shift left, "u'"] \arrow[r, shift right, "v'"'] & P \arrow[r, "p"] \arrow[dr, "f"'] & M \arrow[d, dashed, "g"] \\ & & & Z \end{tikzcd}$$

Here Z is in $\mathrm{mreg} - HMOD_A$ and f is a module map from P to Z and g is defined by $g(m) = f(x)$ for $m = p(x)$. Let us show that our definition is correct. Let $m \in M$, $m = p(x) = p(y)$. Then $f(x) = f(y)$ iff $f(x) \diamond a = f(y) \diamond a$ for all $a \in A$ i.e. $f(x \diamond a) = f(y \diamond a)$ for all $a \in A$. But the last condition evidently holds since $(x, y) \in D$ gives us $(x \diamond a, y \diamond a) \in \mathrm{ess}(D)$ i.e. $f(x \diamond a) = f(y \diamond a)$. Since $\mathrm{ess}(D)$ is m-regular it is a surjective image of $A^{\mathrm{ess}(D)}$ by the module map $\mathbf{p}_{\mathrm{ess}(D)}$. Then we have the diagram

$$\begin{tikzcd} A^{\mathrm{ess}(D)} \arrow[r, "p_{\mathrm{ess}(D)}"] & \mathrm{ess}(D) \arrow[r, "\mathrm{in}_D"] & D \arrow[r, shift left, "u'"] \arrow[r, shift right, "v'"'] & P \arrow[r, "p"] & M. \end{tikzcd}$$

Evidently, p is the coker of the pair

$$(u, v) = (u' \circ in_D \circ p_{\mathrm{ess}(D)}, v' \circ in_D \circ p_{\mathrm{ess}(D)})$$

in the category of m-regular Hilbert A-modules. Recall that

$$u = \bigvee_{(d_1, d_2) \in \mathrm{ess}(D)} u' \circ in_D \circ ((d_1, d_2) \diamond (-)) = \Omega_{(d_1)_{(d_1, d_2) \in \mathrm{ess}(D)}}$$

and similarly

$$v = \Omega_{(d_2)_{(d_1, d_2) \in \mathrm{ess}(D)}}$$

i.e. by Lemma 1.5 u and v are compact. □

Lemma 1.18. *Let A be an unital involutive quantale. Then* $\mathrm{mreg}-HMOD_A$ *and* $\mathrm{mreg} - Hilb_A$ *have free objects.*

Proof. Let X be an arbitrary set. We put $F_A(X) = A^X$. Evidently, $F_A(X)$ is an m-regular right Hilbert A-module. Let us show that it is free over X. Evidently, the set $\{\mathbf{e}_x : x \in X\}$ generates $F_A(X)$ as a submodule and we have an inclusion $\iota_X : X \to F_A(X)$ defined by $x \mapsto \mathbf{e}_x$. By standard considerations we can check that, for any map $f : X \to M$, M being an m-regular right Hilbert A-module, there is a unique A-module map $g : F_A(X) \to M$

such that $f = g \circ i_X$. Moreover, let us define a map $h : M \to F_A(X)$ by $h(n) = (\langle f(x), n\rangle)_{x\in X}$. Then, for all $(a_x)_{x\in X} \in A^X$ and for all $n \in M$, we have

$$\begin{array}{rcl} \langle g(a_x)_{x\in X}, n\rangle & = & \left\langle \bigvee_{x\in X} f(x) \diamond a_x, n\right\rangle \\ & = & \bigvee_{x\in X} \langle f(x) \diamond a_x, n\rangle \\ & = & \bigvee_{x\in X} a_x{}^* \langle f(x), n\rangle \\ & = & \langle (a_x)_{x\in X}, h(n)\rangle . \end{array}$$

Then $h = g^*$ i.e. mreg$-Hilb_A$ has free objects. Note that $g = \Omega_{(f(x))_{x\in X}}$. □

We also have the following.

Lemma 1.19. *Let A be an m-regular involutive quantale. Assume that M, N are m-regular right Hilbert A-modules and that $f : M \to N$ is an epimorphism in $\mathbf{C}_A \in \{\text{mreg} - HMOD_A, \text{mreg} - Hilb_A\}$. Then $f(M)$ is a right Hilbert A-module, $f(M) \in \mathbf{C}_A$ and the induced surjective map $\overline{f} : M \to f(M)$ is adjointable whenever f is. Moreover, we have an inner-product preserving module embedding $\hat{f}$ from $f(M)$ to N such that $f(M)$ separates elements of N.*

Proof. Evidently, $f(M)$ is a pre-Hilbert A-module and, whenever M is essential then also $f(M)$ is essential.

Let us show that $f(M)$ separates N. Let $n_1, n_2 \in N$, $n_1 \neq n_2$. Then $\langle n_1, -\rangle \neq \langle n_2, -\rangle$ i.e. $\langle n_1, -\rangle \circ f \neq \langle n_2, -\rangle \circ f$ i.e. there is an element $p \in M$ such that $\langle n_1, f(p)\rangle \neq \langle n_2, f(p)\rangle$. In particular,

$$\langle f(m_1), -\rangle_{f(M)} = \langle f(m_2), -\rangle_{f(M)} \text{ implies } f(m_1) = f(m_2)$$

i.e. $f(M)$ is an m-regular right Hilbert A-module. Clearly, for an adjointable f, $\overline{f}$ is adjointable since

$$\left\langle \overline{f}(m_1), f(m_2)\right\rangle = \langle f(m_1), f(m_2)\rangle = \langle m_1, f^*(f(m_2))\rangle .$$

Note that the inclusion $\hat{f} : f(M) \to N$, $f(m) \mapsto f(m)$ is evidently a module map preserving inner-product. □

Corollary 1.20. *Let A be an m-regular involutive quantale. Assume that M, N are Hilbert A-modules and $f : M \to N$ is an epimorphism in $\mathbf{C}_A \in \{\text{mreg} - HMOD_A, \text{mreg} - Hilb_A\}$. Then f is a surjective map iff $\hat{f}$ is adjointable.*

Proof. Evidently, if f is surjective then $\hat{f} = \mathrm{id}_N$. Conversely, let $\hat{f}$ be adjointable and assume that $n \notin f(M)$. Then, for any $m \in M$ there is an element $u_m \in M$ such that

$$\langle f(m), f(u_m)\rangle \neq \langle n, f(u_m)\rangle = \left\langle n, \hat{f}(f(u_m))\right\rangle = \left\langle \hat{f}^*(n), f(u_m)\right\rangle .$$

But $\hat{f}^*(n) = f(m)$ for some $m \in M$, a contradiction. □

Corollary 1.21. *Let A be an m-regular involutive quantale. Assume that M, N are m-regular right Hilbert A-modules and f is an epimorphism in $\mathbf{C}_A \in \{\mathrm{mreg} - HMOD_A, \mathrm{mreg} - Hilb_A\}$. Then f is a surjective map iff f is a coker in $\mathbf{C}_A$.*

Proof. Evidently, if f is a surjective map so it is a coker of maps u and v by Proposition 1.17. Conversely, assume that f is a coker of u and v. Then we have the commutative diagram.

$$\begin{tikzcd} P \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & M \arrow[rr, "f"] \arrow[dr, "\overline{f}"'] & & N \arrow[dl, shift right, "s"'] \\ & & f(M) \arrow[ur, shift right, "\hat{f}"'] & \end{tikzcd}$$

Here $s : N \to f(M)$ is the unique module (adjointable) map such that $\overline{f} = s \circ f$. Then

$$\hat{f} \circ (s \circ f) = \hat{f} \circ \overline{f} = f$$

i.e. $\hat{f} \circ s = \mathrm{id}_N$. Hence $\hat{f}$ is onto i.e. $f(M) = N$. $\square$

Definition 1.22. *Let A be an m-regular involutive quantale, $M, N \in \mathbf{C}_A$, $\mathbf{C}_A \in \{\mathrm{mreg} - HMOD_A, \mathrm{mreg} - Hilb_A\}$. Assume that $f : M \to N$ is a morphism in $\mathbf{C}_A$. We shall denote by ${}_f\mathbf{C}_A$ the category with objects all triples $\langle e, X, m\rangle$, $e : M \to X$ an epimorphism in $\mathbf{C}_A$, $m : N \to X$ an injective module map such that $e = m \circ f$, and as arrows $\langle e, X, m\rangle \xrightarrow{g} \langle e', X', m'\rangle$ all module maps $g : X \to X'$ such that $e' = g \circ e$ and $m' = g \circ m$ as maps. In pictures,*

objects $\langle e, X, m\rangle$:
$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "e"'] & & N \arrow[dl, "m"] \\ & X & \end{tikzcd}$$
;

arrows $\langle e, X, m\rangle \xrightarrow{g} \langle e', X', m'\rangle$:

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "e"] \arrow[ddr, "e'"'] & & N \arrow[dl, "m"'] \arrow[ddl, "m'"] \\
& X \arrow[d, "g"] & \\
& X' &
\end{tikzcd},$$

with the diagram commutative.

Proposition 1.23. *Let A be an m-regular involutive quantale, $Y, Z \in \mathbf{C}_A$, $\mathbf{C}_A \in \{\mathrm{mreg} - HMOD_A, \mathrm{mreg} - Hilb_A\}$. Assume that $q : Y \to Z$ is an epimorphism in $\mathbf{C}_A$, $\overline{q} : Y \to q(Y)$ is the induced surjective module (adjointable) map and $\hat{q} : q(Y) \to Z$ is the induced injective module map. Then q is surjective if and only if the triple $\langle q, Z, \hat{q}\rangle$ is couniversal in ${}_{\overline{q}}\mathbf{C}_A$.*

Proof. Let q be surjective and let $\langle e, X, m\rangle$ be in ${}_{\overline{q}}\mathbf{C}_A$. Then from the first diagram we have that $\langle q, Z, \hat{q}\rangle$ is couniversal since $q = \overline{q}$ and $\hat{q} = \mathrm{id}_Z$. Conversely, suppose that $\langle q, Z, \hat{q}\rangle$ is couniversal. From the second diagram we have that $\langle \overline{q}, q(Y), \mathrm{id}_{q(Y)}\rangle$ is couniversal as well. Since there is an isomorphism $h : Z \to q(Y)$ such that $\overline{q} = h \circ q$ we have that $q = h^{-1} \circ \overline{q}$ is surjective.

$$\begin{tikzcd}
Y \arrow[rr, "\overline{q}"] \arrow[dr, "e"] \arrow[ddr, "q"'] & & q(Y) = Z \arrow[dl, "m"'] \arrow[ddl, "\hat{q} = \mathrm{id}_Z"] \\
& X & \\
& Z \arrow[u, "m"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[rr, "\overline{q}"] \arrow[dr, "e"] \arrow[ddr, "\overline{q}"'] & & q(Y) \arrow[dl, "m"'] \arrow[ddl, "\mathrm{id}_{q(Y)}"] \\
& X & \\
& q(Y) \arrow[u, "m"'] &
\end{tikzcd}$$

□

Definition 1.24. *Let A be an m-regular involutive quantale, $M \in \mathbf{C}_A$, $\mathbf{C}_A \in \{\mathrm{mreg}-HMOD_A, \mathrm{mreg}-Hilb_A\}$. Then M is called* weakly projective *in* $\mathbf{C}_A$ *if to every diagram in* $\mathbf{C}_A$

$$\begin{array}{ccc} & & M \\ & & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow[q]{} & Z \end{array}$$

such that q *is a surjective morphism there is a morphism* $\psi : M \to Y$ *making the diagram commutative.*

Proposition 1.25. *Let* A *be an unital involutive quantale. Then* A *is weakly projective in* $\mathbf{C}_A$, $\mathbf{C}_A \in \{\mathrm{mreg} - HMOD_A, \mathrm{mreg} - Hilb_A\}$.

Proof. Assume that we have the diagram from 1.24. Then there is an element $y \in Y$ such that $q(y) = \varphi(e)$. Define $\psi : A \to Y$ by $\psi(a) = y \diamond a$. Then evidently ψ is an adjointable A-module map such that $q \circ \psi$. $\square$

Corollary 1.26. *Let* A *be an unital involutive quantale. Then* A^J *is weakly projective in* $\mathbf{C}_A$, $\mathbf{C}_A \in \{\mathrm{mreg} - HMOD_A, \mathrm{mreg} - Hilb_A\}$ *for any index set* J.

Proposition 1.27. *Let* A *be an unital involutive quantale,* $M \in \mathbf{C}_A$, $\mathbf{C}_A \in \{\mathrm{mreg} - HMOD_A, \mathrm{mreg} - Hilb_A\}$ *Then* M *is weakly projective in* $\mathbf{C}_A$ *if and only if* M *is a retract of a free Hilbert* A*-module.*

Proof. Assume that M is weakly projective. Since M is a surjective image of A^J by some $q : A^J \to M$ we have a morphism $\psi : M \to A^J$ (taking $\varphi = \mathrm{id}_M$) such that $\mathrm{id}_M = q \circ \psi$. Conversely, let M be a retract of a free Hilbert A-module A^J, $r : A^J \to M$ the retraction and $i : M \to A^J$ the embedding such that $r \circ i = \mathrm{id}_M$. Let Y, Z be Hilbert A-modules, $q : Y \to Z$ a surjective morphism, $\varphi : M \to Z$ a morphism. Since A^J is weakly projective there is a morphism $\psi' : A^J \to Y$ such that $q \circ \psi' = \varphi \circ r$. Let us put $\psi = \psi' \circ i$. Then

$$q \circ \psi = q \circ \psi' \circ i = \varphi \circ r \circ i = \varphi.$$

$\square$

2. The Eilenberg-Watts and fundamental Morita theorems for Hilbert modules

Definition 2.1. *We say that two m-regular involutive quantales* A *and* B *are* Morita equivalent *if there exist* $*$*-functors* $F : \mathrm{mreg} - Hilb_A \to \mathrm{mreg} - Hilb_B$ *and* $G : \mathrm{mreg} - Hilb_B \to \mathrm{mreg} - Hilb_A$*, such that* $FG \cong Id$ *and* $GF \cong Id$ *unitarilly. Such* F *and* G *will be called* equivalence functors*.*

Theorem 2.2 (Eilenberg-Watts theorem for m-regular involutive quantales)**.** *Let* A *and* B *be m-regular involutive quantales and let* F *be a colimit-preserving *-functor from* $\mathrm{mreg} - Hilb_A$ *to* $\mathrm{mreg} - Hilb_B$*. Then there is a* *right Hilbert $A - B$ bimodule Z such that $F(-)$ is naturally unitarily isomorphic to the interior tensor product $(-)\dot{\otimes}_A Z$. That is, there is a natural isomorphism between these functors, which implements a unitary isomorphism $F(Y) \cong Y\dot{\otimes}_A Z$ for all $Y \in \mathrm{mreg} - Hilb_A$.*

Proof. Define $Z = F(A)$. Then Z is an m-regular right Hilbert B-module. We make Z into a left A-module by defining $a \bullet z = F(L(a))(z)$, for $a \in A$, $z \in Z$. Here $L(a) : A \to A$ is the adjointable map $b \mapsto ab$. Let us check that Z is a right Hilbert $A - B$ bimodule. We have

$$\begin{aligned} \langle a \bullet x, y\rangle &= \langle F(L(a))(x), y\rangle \\ &= \langle x, F(L(a))^*(y)\rangle \\ &= \langle x, F(L(a)^*)(y)\rangle \\ &= \langle x, F(L(a^*))(y)\rangle \\ &= \langle x, a^* \bullet y\rangle\,. \end{aligned}$$

Now, we shall check that Z is essential with respect to $\bullet$. Note that we know that the map $\mathbf{p} : A^A \to A$, $\mathbf{p} = \bigvee_{a\in A} L(a) \circ \pi_a$ is surjective i.e. it is a coker of adjointable maps in rm $\mathrm{mreg} - Hilb_A$. Then $F(\mathbf{p}) : F(A)^A \to F(A)$ is a coker in $\mathrm{mreg} - Hilb_B$ i.e.

$$F(\mathbf{p}) = \bigvee_{a\in A} F(L(a)) \circ F(\pi_a) = \bigvee_{a\in A} a \bullet F(\pi_a)$$

is surjective by Corollary 1.21.

It follows that, for all $Y \in \mathrm{mreg} - Hilb_A$, the interior tensor product $Y\dot{\otimes}_A Z$ is well defined. We define an adjointable map $\tau_Y : Y\dot{\otimes}_A Z \to F(Y)$ by $\tau_Y(y\dot{\otimes}_A z) = F(L(y))(z)$, where $L(y) : A \to Y$ is the map $L(y)(a) = y \diamond a$. Let us prove that τ_Y is a unitary natural transformation.

First note that for $Y = A^J$ this is easy, in fact τ_{A^J} is the canonical unitary isomorphism from

$$A^J\dot{\otimes}_A F(A) \cong (A\dot{\otimes}_A F(A))^J \cong F(A)^J \cong F(A^J).$$

This is because if $y = (a_j)_{j\in J} \in A^J, z \in F(A)$ then

$$\begin{aligned} &(a_j)_{j\in J}\dot{\otimes}_A z \mapsto (a_j\dot{\otimes}_A z)_{j\in J} \mapsto (a_j \bullet z)_{j\in J} \\ &= (F(L(a_j))(z))_{j\in J} \mapsto \textstyle\bigvee_{j\in J} F(i_j)(F(L(a_j))(z)) \\ &= \textstyle\bigvee_{j\in J} F(i_j \circ L(a_j))(z) \\ &= \textstyle\bigvee_{j\in J} F(L(i_j(a_j)))(z) \\ &= F(\textstyle\bigvee_{j\in J} L(i_j(a_j)))(z) \\ &= F(L(\textstyle\bigvee_{j\in J} i_j(a_j)))(z) \\ &= F(L(y))(z) \\ &= \tau_Y(y\dot{\otimes}_A z). \end{aligned}$$

Now fix $Y \in \mathrm{mreg} - Hilb_A$. We have then the following commutative diagram for a suitable index set J such that $\mathbf{p}_Y$ is the coker of adjointable maps u, v.

$$\begin{tikzcd} A^J \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & A^Y \arrow[r, "\mathbf{p}_Y"] & Y \end{tikzcd} \tag{2.1}$$

Applying F, $(-)\dot{\otimes}_A Z$ and again F we get a diagram.

$$\begin{tikzcd}
F(A^J) \arrow[r, shift left, "F(u)"] \arrow[r, shift right, "F(v)"'] \arrow[d, "\tau^*_{A^J}"'] & F(A^Y) \arrow[r, "F(\mathbf{p}_Y)"] \arrow[d, "\tau^*_{A^Y}"] & F(Y) \arrow[d, "\tau^*_Y"] \\
A^J\dot{\otimes}_A Z \arrow[r, shift left, "u\dot{\otimes}_A \mathrm{id}_Z"] \arrow[r, shift right, "v\dot{\otimes}_A \mathrm{id}_Z"'] \arrow[d, "\tau_{A^J}"'] & A^Y\dot{\otimes}_A Z \arrow[r, "\mathbf{p}_Y\dot{\otimes}_A \mathrm{id}_Z"] \arrow[d, "\tau_{A^Y}"] & Y\dot{\otimes}_A Z \arrow[d, "\tau_Y"] \\
F(A^J) \arrow[r, shift left, "F(u)"] \arrow[r, shift right, "F(v)"'] & F(A^Y) \arrow[r, "F(\mathbf{p}_Y)"] & F(Y)
\end{tikzcd} \tag{2.2}$$

We shall first check that the diagram is commutative.
Note that

$$\begin{aligned}
\tau_Y(\mathbf{p}_Y(w)\dot{\otimes}_A z) &= F(L(\mathbf{p}_Y(w)))(z) \\
&= F(\mathbf{p}_Y)F(L(w))(z) \\
&= F(\mathbf{p}_Y)\tau_{A^Y}(w\dot{\otimes}_A z)
\end{aligned}$$

for $w \in A^Y$, $z \in Z$. Similarly, let $y \in Y$, $w = (a_x)_{x\in Y} \in A^Y$, $z, z' \in Z$. Then

$$\begin{aligned}
\langle \mathbf{p}_Y\dot{\otimes}_A \mathrm{id}_Z((a_x)_{x\in Y}\dot{\otimes}_A z'), y\dot{\otimes}_A z\rangle &= \langle z', \langle \mathbf{p}_Y(a_x)_{x\in Y}, y\rangle \bullet z\rangle \\
&= \langle \langle y, \mathbf{p}_Y(a_x)_{x\in Y}\rangle \bullet z', z\rangle \\
&= \textstyle\bigvee_{x\in Y} \langle \langle y, x \diamond a_x\rangle \bullet z', z\rangle \\
&= \textstyle\bigvee_{x\in Y} \langle F(L(\langle y, x\diamond a_x\rangle))(z'), z\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\langle (\tau^*_Y \circ F(\mathbf{p}_Y) \circ \tau_{A^Y})((a_x)_{x\in Y}\dot{\otimes}_A z'), y\dot{\otimes}_A z\rangle \\
&\quad = \left\langle (F(\mathbf{p}_Y) \circ \tau_{A^Y})((a_x)_{x\in Y}\dot{\otimes}_A z'), \tau_Y(y\dot{\otimes}_A z)\right\rangle \\
&\quad = \left\langle (F(L(y)^*) \circ F(\mathbf{p}_Y) \circ \tau_{A^Y})((a_x)_{x\in Y}\dot{\otimes}_A z'), z\right\rangle \\
&\quad = \left\langle \textstyle\bigvee_{x\in Y}(F(L(y)^* \circ \mathbf{p}_Y \circ i_x \circ L(a_x))(z'), z\right\rangle \\
&\quad = \textstyle\bigvee_{x\in Y} \langle (F((\langle y, -\rangle) \circ (x \diamond (-)) \circ (a_x \cdot (-)))(z'), z\rangle \\
&\quad = \textstyle\bigvee_{x\in Y} \langle (F(L(\langle y, x\diamond a_x\rangle))(z'), z\rangle\,.
\end{aligned}$$

Then, since the upper and lower horizontal lines of the diagram (2.2) produce the respective cokers $F(\mathbf{p}_Y)$, the composite right square of this diagram is a pushout in mreg $-$ $Hilb_B$.

$$\begin{tikzcd}
F(A^Y) \arrow[r, "F(\mathbf{p}_Y)"] \arrow[d, "\mathrm{id}_{F(A^Y)}"'] & F(Y) \arrow[d, "\tau_Y \circ \tau^*_Y"] \\
F(A^Y) \arrow[r, "F(\mathbf{p}_Y)"] & F(Y)
\end{tikzcd} \tag{2.3}$$

i.e. $\tau_Y \circ \tau_Y^* = \mathrm{id}_{F(Y)}$. Similarly, we have this commutative diagram.

$$\begin{tikzcd} A^Y\dot{\otimes}_A Z \arrow[r, "\mathbf{p}_Y\dot{\otimes}_A\mathrm{id}_Z"] \arrow[d, "\mathrm{id}_{A^Y\dot{\otimes}_A Z}"] & Y\dot{\otimes}_A Z \arrow[d, shift right, "\mathrm{id}_{Y\dot{\otimes}_A Z}"'] \arrow[d, shift left, "\tau_Y^*\circ\tau_Y"] \\ A^Y\dot{\otimes}_A Z \arrow[r, "\mathbf{p}_Y\dot{\otimes}_A\mathrm{id}_Z"] & Y\dot{\otimes}_A Z \end{tikzcd} \tag{2.4}$$

Since $\mathbf{p}_Y\dot{\otimes}_A\mathrm{id}_Z$ is a surjective adjointable map it is an epimorphism and we have $\tau_Y^* \circ \tau_Y = \mathrm{id}_{Y\dot{\otimes}_A Z}$.

Let $f : X \to Y$ be a morphism in $\mathrm{mreg} - Hilb_A$, $x \in X$ and $z \in Z$. Then

$$\begin{aligned} (\tau_Y \circ (f\dot{\otimes}_A\mathrm{id}_Z))(x\dot{\otimes}_A z) &= F(L(f(x)))(z) \\ &= F(f \circ L(x))(z) \\ &= F(f)(F(L(x))(z)) \\ &= (F(f) \circ \tau_X)(x\dot{\otimes}_A z) \end{aligned}$$

and since τ_X, τ_Y are unitary also

$$\tau_Y^* \circ F(f) = (f\dot{\otimes}_A\mathrm{id}_Z))\tau_X^*$$

i.e. τ_Y, τ_Y^* are natural transformations. □

Lemma 2.3. *Let A and B be m-regular involutive quantales and let X_1 and X_2 be essential right Hilbert $A - B$ bimodules such that $F_1 = (-)\dot{\otimes}_A X_1$ and $F_2 = (-)\dot{\otimes}_A X_2$ are functors from* $\mathrm{mreg} - Hilb_A$ *to* $\mathrm{mreg} - Hilb_B$. *Then F_1 and F_2 are equivalent if and only if $X_1 \cong X_2$ unitarily and as bimodules.*

Proof. Assume that $\alpha : F_1 \to F_2$ is a natural isomorphism such that $\alpha_M : F_1(M) \to F_2(M)$ is a unitary map. Then we have the following diagram.

$$\begin{tikzcd} A\dot{\otimes}_A X_1 \arrow[r, "\alpha_A"] & A\dot{\otimes}_A X_1 \arrow[d, "\chi_{X_2}"'] \\ X_1 \arrow[u, "\kappa_{X_1}"] \arrow[r, "h"] & X_2 \end{tikzcd}$$

Here $\chi_{X_2}(a\dot{\otimes}_A x) = a \bullet x$, $\kappa_{X_1} = \chi_{X_1}^*$ and $h = \chi_{X_2} \circ \alpha_A \circ \kappa_{X_1}$. Left multiplication in A are adjointable maps, so a natural transformation α_A has to preserve them. From [11] we know that κ_{X_1} and χ_{X_2} are unitary maps and bimodule homomorphisms. Hence h is unitary and bimodule homomorphism. The other direction is evident. □

The following theorem is an involutive quantale version of Morita's fundamental theorem.

Theorem 2.4. *Let A and B be m-regular involutive quantales. Then A and B are Morita-equivalent if and only if there are an essential right Hilbert*

$A-B$ bimodule X and an essential right Hilbert $B-A$ bimodule Y such that $X\dot{\otimes}_B Y \cong A$ and $Y\dot{\otimes}_A X \cong B$ as bimodules.

Proof. We have $F(-) \cong -\dot{\otimes}_A X$ and $G(-) \cong -\dot{\otimes}_B Y$. Composing these two we obtain

$$A \cong GF(A) \cong A\dot{\otimes}_A X\dot{\otimes}_B Y \cong X\dot{\otimes}_B Y.$$

Similarly, $Y\dot{\otimes}_A X \cong B$, and this identification and the last are unitarily, and as bimodules, the latter exactly as in pure algebra [3].

Conversely, let $\lambda : X\dot{\otimes}_B Y \to A$ be the unitary isomorphism, $F(-) \cong -\dot{\otimes}_A X$ and $G(-) \cong -\dot{\otimes}_B Y$. Then

$$(GF)(M) \cong M\dot{\otimes}_A X\dot{\otimes}_B Y.$$

Let us define

$$\tau_M = \pi_M \circ (\mathrm{id}_M \dot{\otimes}_A \lambda),$$

where $\pi_M : M\dot{\otimes}_A A \to M$ is the canonical unitary isomorphism defined by $m\dot{\otimes}_A a \mapsto m \diamond a$. We get a natural isomorphism $\tau : GF \to \mathrm{Id}_{\mathrm{mreg}-Hilb_A}$. By symmetry, A is Morita equivalent to B. □

Corollary 2.5. *Let A and B be unital involutive quantales. Then A and B are Morita-equivalent if and only if their categories of weakly projective m-regular right Hilbert modules are equivalent.*

Proof. It follows from the fact that weak projectivity is a categorical property. Hence weakly projective objects are mapped on weakly projective objects. Conversely, any equivalence between categories of weakly projective m-regular right Hilbert modules can be easily extended to an equivalence of m-regular right Hilbert modules. □

We recall that if A and B are m-regular involutive quantales then an imprimitivity Hilbert $A-B$ bimodule is an $A-B$ bimodule X, which is a full m-regular right Hilbert B-module, and also a full m-regular left Hilbert A-module, such that ${}_A\langle x, y\rangle z = x\langle y, z\rangle_B$ whenever $x, y, z \in X$. The existence of such an X is the definition of A and B being *strongly Morita equivalent.* Our theorem gives a functorial characterization of such an X, and of strong Morita equivalence of A and B.

Theorem 2.6. *Let A and B be m-regular involutive quantales. Suppose that $F\colon \mathrm{mreg}-Hilb_A \to \mathrm{mreg}-Hilb_B$, $G\colon \mathrm{mreg}-Hilb_B \to \mathrm{mreg}-Hilb_A$ are equivalence *-functors, with $FG \cong Id$ and $GF \cong Id$ via unitary natural isomorphisms. Then A and B are strongly Morita equivalent. Moreover, the $A-B$ bimodule $X = F(A)$ from the Eilenberg-Watts theorem is an imprimitivity bimodule implementing the strong Morita equivalence. As in that theorem, $F(-) \cong -\dot{\otimes}_A X$ naturally and unitarily. Conversely, any $A-B$ imprimitivity bimodule X implements such a functorial isomorphism.*

Proof. It follows from the fact that X is an imprimitivity $A-B$ bimodule if and only if there is an essential right Hilbert $B-A$ bimodule Y such that $X\dot{\otimes}_B Y \cong A$ and $Y\dot{\otimes}_A X \cong B$ as right-Hilbert bimodules (see [11]) and Theorem 2.4. $\square$

Proposition 2.7. *Let A, B and C be m-regular involutive quantales. Then the unitary isomorphism classes of equivalence functors* $\mathrm{mreg} - Hilb_A \to \mathrm{mreg} - Hilb_B$ *are in a 1-1 correspondence with the unitary equivalence classes of imprimitivity $A-B$ bimodules. Composition of such functors corresponds to the interior tensor product of the bimodules.*

Proof. Every imprimitivity $A - B$ bimodule X gives rise to an equivalence $(-)\dot{\otimes}_A X : \mathrm{mreg} - Hilb_A \to \mathrm{mreg} - Hilb_B$, the isomorphism type depends only on the isomorphism type of X (as we saw in Lemma 2.3). Conversely, if $F : \mathrm{mreg}-Hilb_A \to \mathrm{mreg}-Hilb_B$ is an equivalence, $F(A)$ is an imprimitivity $A-B$ bimodule and its isomorphism type depends only on that of F. If Y is an imprimitivity $B-C$ bimodule of an equivalence functor, the composition of the equivalences $\mathrm{mreg} - Hilb_A \to \mathrm{mreg} - Hilb_B \to \mathrm{mreg} - Hilb_C$ is given by $(-)\dot{\otimes}_A(X\dot{\otimes}_B Y)$. In particular, $X\dot{\otimes}_B Y$ is an imprimitivity $A - C$ bimodule. $\square$

Corollary 2.8. *Let A be an m-regular involutive quantale. Then the unitary isomorphism classes of self-equivalence functors* $\mathrm{mreg} - Hilb_A \to \mathrm{mreg} - Hilb_A$ *under a composition form a group isomorphic to the group of the unitary equivalence classes of imprimitivity $A - A$ bimodules.*

Corollary 2.9. *Let A and B be m-regular involutive quantales. Then A and B are Morita-equivalent if and only if A^d and B^d are Morita-equivalent.*

Proof. Let A and B are Morita-equivalent. Then we have an imprimitivity $A - B$ bimodule X. Then Y with the same order and inner products as X and equipped with operations $a \bullet_d x = a^* \bullet x$ and $x \diamond_d b = x \diamond x^*$ is evidently an imprimitivity $A^d - B^d$ bimodule. $\square$

Proposition 2.10. *Let A be an m-regular involutive quantale. Then A is Morita equivalent to the matrix quantale $\mathcal{M}^J(A)$.*

Proof. Note that $\mathcal{M}^J(A) \cong \mathcal{K}_A(A^J, A^J)$. $\square$

Proposition 2.11. *Let S be a sup-semilatticce with a duality. Then the involutive subquantale $\mathcal{Q}_0(S)$ of $\mathcal{Q}(S)$ that is generated by right-sided elements of $\mathcal{Q}(S)$ is Morita equivalent to the 2-element Boolean algebra $\mathbf{2}$. In particular, $\mathcal{M}^J(\mathbf{2})$ is Morita equivalent to $\mathbf{2}$.*

Proof. Recall that $\mathcal{Q}_0(S) \cong \mathcal{K}_{\mathbf{2}}(S, S)$. $\square$

Lemma 2.12. *Let E be an imprimitivity Hilbert B-A bimodule with $A = \mathcal{K}_{\mathbf{2}}(S_A)$, S_A being a sup-lattice with a duality. Then there is a sup-lattice with a duality S_B such that $B \cong \mathcal{K}_{\mathbf{2}}(S_B)$ and $E \cong \mathcal{K}_{\mathbf{2}}(S_A, S_B)$.*

Proof. We write ι for the identity representation of $\mathcal{K}_{\mathbf{2}}(S_A)$ on S_A and put $S_B = E\dot{\otimes}_A S_A$. Left multiplication in the first component of S_B then defines a faithful representation of $B \simeq \mathcal{K}_A(E)$ onto S_B i.e. $B \simeq \mathcal{K}_{\mathbf{2}}(S_B)$ (see [11, Corollary 1.13]).

Now we shall define a mapping

$$\Theta\colon E \to \mathcal{A}_{\mathbf{2}}(S_A, S_B),\ x \mapsto (s \mapsto x\dot{\otimes}_A s).$$

Namely,

$$\left\langle \Theta(x)(s), y\dot{\otimes}_A t\right\rangle = \left\langle x\dot{\otimes}_A s, y\dot{\otimes}_A t\right\rangle = \langle s, \langle x, y\rangle_A \bullet t\rangle$$

i.e.

$$\Theta(x)^*(y\dot{\otimes}_A t) = \langle x, y\rangle_A \bullet t.$$

We also observe that

$$\Theta(x)^*\Theta(y) = \langle x, y\rangle_A\,.$$

If $a \in A$, then

$$\Theta(x \diamond a)(s) = (x \diamond a)\dot{\otimes}_A s = x\dot{\otimes}_A a \bullet s,$$

so that

$$\Theta(E) = \Theta(E \diamond A) \subseteq \Theta(E) \circ \mathcal{K}_{\mathbf{2}}(S_A) \subseteq \mathcal{K}_{\mathbf{2}}(S_A, S_B).$$

Similarly,

$$\Theta_{y\dot{\otimes}_A t, s} = y\dot{\otimes}_A t \diamond \langle s, -\rangle = \Theta(y) \circ \Theta_{t,s}$$

i.e.

$$\mathcal{K}_{\mathbf{2}}(S_A, S_B) \subseteq \Theta(E) \circ \mathcal{K}_{\mathbf{2}}(S_A) \subseteq \Theta(E).$$

So we conclude that

$$\Theta(E) = \mathcal{K}_{\mathbf{2}}(S_A, S_B).$$

It is a straightforward computation that Θ is an isomorphism of Hilbert bimodules. □

Lemma 2.13. *For any m-regular involutive quantale A,*

$$\mathcal{A}_A(A) \cap {}_A\mathcal{A}(A) = HMOD_A(A) \cap {}_AHMOD(A).$$

Proof. It is enough to show that any bimodule endomorphism $f : A \to A$ is adjointable. Note that

$$\begin{array}{rcl} \langle m, f(n)\rangle & = & m^* \cdot f(n) \\ & = & f(m^* \cdot n) \\ & = & f(m^*) \cdot n \\ & = & (f(m^*)^*)^* \cdot n \\ & = & \langle f(m^*)^*, n\rangle\,\mu \text{ for all } m,\, n \in A \end{array}$$

i.e. f has an adjoint. □

Definition 2.14. *Let A be an m-regular involutive quantale. The set of adjointable natural transformations from the identity functor* $\mathrm{Id}_{\mathrm{mreg}-Hilb_A}$ *to itself is called* $\mathrm{ANat}(A)$*. It is a unital commutative involutive quantale* *with the composition of natural transformations as multiplication, with the adjoint of a natural transformation as involution*

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "\eta_N"] & N \end{tikzcd} \longmapsto \begin{tikzcd} M \arrow[r, "\eta_M^*"] & M \\ N \arrow[r, "\eta_N^*"] \arrow[u, "f^*"] & N \arrow[u, "f^*"'] \end{tikzcd}$$

and with the join given by

$$\begin{tikzcd} M \arrow[r, "\eta_{iM}"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "\eta_{iN}"] & N \end{tikzcd} , i \in I \quad \longmapsto \quad \begin{tikzcd} M \arrow[r, "\bigvee_i \eta_{iM}"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "\bigvee_i \eta_{iN}"] & N \end{tikzcd} .$$

The centre *of A is the unital commutative involutive quantale* $\mathrm{Cen}(A) = \mathcal{A}_A(A) \cap {}_A\mathcal{A}(A)$.

The elements of $\mathrm{Cen}(A)$ are by Lemma 2.13 A-bimodule endomorphisms on A. Note that evidently $\mathrm{Cen}(A)$ is commutative since, for all $a \in A$, $a = \bigvee_j a_j \cdot b_j$, f, $g \in \mathrm{Cen}(A)$, we have

$$\begin{aligned} (f \circ g)(a) &= (f \circ g)(\textstyle\bigvee_j a_j \cdot b_j) \\ &= \textstyle\bigvee_j f(g(a_j \cdot b_j)) \\ &= \textstyle\bigvee_j f(g(a_j) \cdot b_j) \\ &= \textstyle\bigvee_j g(a_j) \cdot f(b_j) \\ &= \textstyle\bigvee_j g(f(a_j \cdot b_j)) \\ &= (g \circ f)(\textstyle\bigvee_j a_j \cdot b_j) \\ &= (g \circ f)(a). \end{aligned}$$

Whenever A is commutative we can see A as a complete *-ideal of $\mathrm{Cen}(A)$ by identifying an element $a \in A$ with the bimodule endomorphism on A induced by multiplication by a.

The following theorem is based on the theorem 4.2 from [2] for involutive rings.

Theorem 2.15. *Let A, B be Morita equivalent m-regular involutive quantales. Then*

1. *$Cen(A)$ and $Cen(B)$ are isomorphic as involutive quantales.*
2. *If A and B are commutative then A and B are isomorphic as involutive quantales.*

Proof. (1). By Theorem 2.6 there is an $A-B$ imprimitivity bimodule X implementing the Morita equivalence. We shall now define a map $\gamma : \mathrm{Cen}(A) \to$

$\mathrm{Cen}(B)$ by the prescription

$$\gamma(f)(\bigvee_i \langle x_i, r_i \bullet y_i\rangle_B) = \bigvee_i \langle x_i, f(r_i) \bullet y_i\rangle_B$$

for all $x_i, y_i \in X$ and $r_i \in A$. Let us check that γ is well defined. Assume that

$$\bigvee_i \langle x_i, r_i \bullet y_i\rangle_B = \bigvee_j \langle u_j, p_j \bullet v_j\rangle_B .$$

Let $z, t \in X$ and $r' \in A$. Then

$$\begin{array}{rcl}\langle z, r' \bullet t\rangle_B \cdot (\bigvee_i \langle x_i, f(r_i) \bullet y_i\rangle_B) & = & \bigvee_i \langle z, (r' \cdot \langle t, x_i\rangle_A \cdot f(r_i)) \bullet y_i\rangle_B \\ & = & \bigvee_i \langle z, (f(r') \cdot \langle t, x_i\rangle_A \cdot r_i) \bullet y_i\rangle_B \\ & = & \langle z, f(r') \bullet t\rangle_B \cdot (\bigvee_i \langle x_i, r_i \bullet y_i\rangle_B) \\ & = & \langle z, f(r') \bullet t\rangle_B \cdot (\bigvee_j \langle u_j, p_j \bullet v_j\rangle_B) \\ & = & \langle z, r' \bullet t\rangle_B \cdot (\bigvee_j \langle u_j, f(p_j) \bullet v_j\rangle_B).\end{array}$$

We show that $\gamma(f) \in \mathrm{Cen}(B)$. Let $b_j \in B$,

$$b_j = \bigvee_{i \in I_j} \left\langle x_i^j, r_i^j \bullet y_i^j \right\rangle_B .$$

Then

$$\begin{array}{rcl}\gamma(f)(\bigvee_j b_j) & = & \gamma(f)(\bigvee_j \bigvee_{i\in I_j} \left\langle x_i^j, r_i^j \bullet y_i^j \right\rangle_B) \\ & = & \bigvee_j \bigvee_{i\in I_j} \left\langle x_i^j, f(r_i^j) \bullet y_i^j \right\rangle_B \\ & = & \bigvee_j \gamma(f)(\bigvee_{i\in I_j} \left\langle x_i^j, j_i \bullet y_i^j \right\rangle_B) \\ & = & \bigvee_j \gamma(f)(b_j)\end{array}$$

i.e. $\gamma(f)$ is a sup-lattice homomorphism. Now, let b, $c \in B$,

$$b = \bigvee_i \langle x_i, r_i \bullet y_i\rangle_B .$$

Then

$$\begin{array}{rcl}\gamma(f)(b \cdot c) & = & \gamma(f)(\bigvee_i \langle x_i, r_i \bullet y_i \diamond c\rangle_B) \\ & = & \bigvee_i \langle x_i, f(r_i) \bullet y_i \diamond c\rangle_B \\ & = & \bigvee_i \langle x_i, f(r_i) \bullet y_i\rangle_B \cdot c \\ & = & \gamma(f)(b) \cdot c\end{array}$$

and

$$\begin{array}{rcl}\gamma(f)(c \cdot b) & = & \gamma(f)(\bigvee_i \langle x_i \diamond c^*, r_i \bullet y_i\rangle_B) \\ & = & \bigvee_i \langle x_i \diamond c^*, f(r_i) \bullet y_i\rangle_B \\ & = & c \cdot \bigvee_i \langle x_i, f(r_i) \bullet y_i\rangle_B \\ & = & c \cdot \gamma(f)(b)\end{array}$$

i.e. $\gamma(f)$ is an bimodule endomorphism.

Let us check that γ is an involutive quantale homomorphism from $\mathrm{Cen}(A)$ to $\mathrm{Cen}(B)$. Let $f, g, f_j \in \mathrm{Cen}(A)$, $j \in J$. Then

$$\begin{aligned}
\gamma(\bigvee_j f_j)(\bigvee_i \langle x_i, r_i \bullet y_i\rangle_B) &= \bigvee_i \left\langle x_i, \bigvee_j f_j(r_i) \bullet y_i \right\rangle_B \\
&= \bigvee_j \bigvee_i \langle x_i, f_j(r_i) \bullet y_i\rangle_B \\
&= \bigvee_j \gamma(f_j)(\bigvee_i \langle x_i, r_i \bullet y_i\rangle_B),
\end{aligned}$$

$$\begin{aligned}
\gamma(f^*)(\bigvee_i \langle x_i, r_i \bullet y_i\rangle_B) &= \bigvee_i \langle x_i, f^*(r_i) \bullet y_i\rangle_B \\
&= \bigvee_i \langle x_i, f(r_i^*)^* \bullet y_i\rangle_B \\
&= \bigvee_i \langle f(r_i^*) \bullet x_i, y_i\rangle_B \\
&= (\bigvee_i \langle y_i, f(r_i^*) \bullet x_i\rangle_B)^* \\
&= (\gamma(f)(\bigvee_i \langle y_i, r_i^* \bullet x_i\rangle_B))^* \\
&= \gamma(f)^*(\bigvee_i \langle y_i, r_i^* \bullet x_i\rangle_B^*) \\
&= \gamma(f)^*(\bigvee_i \langle r_i^* \bullet x_i, y_i\rangle_B) \\
&= \gamma(f)^*(\bigvee_i \langle x_i, r_i \bullet y_i\rangle_B)
\end{aligned}$$

and

$$\begin{aligned}
\gamma(f \circ g)(\bigvee_i \langle x_i, r_i \bullet y_i\rangle_B) &= \bigvee_i \langle x_i, (f \circ g)(r_i) \bullet y_i\rangle_B \\
&= \bigvee_i \langle x_i, f(g(r_i) \bullet y_i\rangle_B \\
&= \gamma(f)(\bigvee_i \langle x_i, g(r_i) \bullet y_i\rangle_B) \\
&= (\gamma(f) \circ \gamma(g))(\bigvee_i \left\langle x_i, \bigvee_j r_i \bullet y_i \right\rangle_B).
\end{aligned}$$

Similarly, we have an involutive quantale homomorphism δ from $\mathrm{Cen}(B)$ to $\mathrm{Cen}(A)$ defined by

$$\delta(g)(\bigvee_i {}_A\langle x_i \diamond s_i, y_i\rangle) = \bigvee_i {}_A\langle x_i \diamond g(s_i), y_i\rangle$$

for $g \in \mathrm{Cen}(B)$ and $x_i, y_i \in X$, $s_i \in B$. One can easily check that δ is the inverse of γ i.e. γ provides an involutive quantale isomorphism from $\mathrm{Cen}(A)$ to $\mathrm{Cen}(B)$.

(2). Let us assume that A and B are commutative i.e. each of these involutive quantales can be viewed as a complete *-ideal of its centre. It is enough to show that $\gamma(A) = B$ and $\delta(B) = A$. Assume that $r \in A$ and $x, y \in X$. Then $\gamma(r)(\langle x, y\rangle_B) = \langle x, r \bullet y\rangle_B$. Since $r = \bigvee_j {}_A\langle z_j \diamond s_j, t_j\rangle$ for suitable $z_j, t_j \in X$ and $s_j \in B$ and B is commutative we have

$$\begin{aligned}
\gamma(r)(\langle x, y\rangle_B) &= \bigvee_j \langle x, {}_A\langle z_j \diamond s_j, t_j\rangle \bullet y\rangle_B \\
&= \bigvee_j \langle x, (z_j \diamond s_j) \langle t_j, y\rangle_B\rangle_B \\
&= \bigvee_j \langle x, z_j \diamond (s_j \cdot \langle t_j, y\rangle_B)\rangle_B \\
&= \bigvee_j \langle x, z_j \diamond (\langle t_j, y\rangle_B \cdot s_j)\rangle_B \\
&= \bigvee_j \langle x, {}_A\langle z_j, t_j\rangle \bullet y\rangle_B \cdot s_j \\
&= \left(\bigvee_j s_j \cdot \gamma({}_A\langle z_j, t_j\rangle)\right) \bullet \langle x, y\rangle_B\,.
\end{aligned}$$

Since B is an ideal in $\mathrm{Cen}(B)$ we have that

$$q = \bigvee_j s_j \cdot \gamma({}_A\langle z_j, t_j\rangle) \in B.$$

Hence $\gamma(r) = q$ i.e. $\gamma(A) \subseteq B$ and therefore also $A \subseteq \delta(B)$. Similarly $\delta(B) \subseteq A$ and $B \subseteq \gamma(A)$. $\square$

Lemma 2.16. *Let A be an m-regular involutive quantale. Then* $\mathrm{Cen}(A)$ *and $ANat(A)$ are isomorphic as involutive quantales.*

Proof. Let $\psi \in \mathrm{Cen}(A)$ and let $M \in \text{m-reg}-Hilb_A$. Then, since $A \cong \mathcal{K}_A(A)$ as involutive quantales M is a right Hilbert $\mathcal{K}_A(A)$-module. In particular, we have a right action $\diamond_{\mathcal{K}_A(A)} : M \times \mathcal{K}_A(A) \to M$ that gives rise to a non-degenerate involutive quantale homomorphism $f : \mathcal{K}_A(A) \to \mathcal{A}_A(M)$. Then we can find its unique extension $g : \mathcal{A}_A(A) \to \mathcal{A}_A(M)$ by Corollary 1.5 in [10] i.e. $M \in MOD_{\mathcal{A}_A(A)}$ with the right action $\diamond_{\mathcal{A}_A(A)} : M \times \mathcal{A}_A(A) \to M$. Moreover the map $\sigma_M : M \to M$ given by $m \mapsto m \diamond_{\mathcal{A}_A(A)} \psi$ is adjointable. Namely,

$$\begin{aligned}
b \cdot \langle n, \sigma_M(m)\rangle \cdot a &= b \cdot \langle n, m \diamond \psi\rangle \cdot a \\
&= b \cdot \langle n, (m \diamond \psi) \bullet a\rangle \\
&= b \cdot \langle n, m \diamond \psi(a)\rangle \\
&= b \cdot \langle n, m\rangle \cdot \psi(a) \\
&= b \cdot \psi(\langle n, m\rangle \cdot a) \\
&= \psi(b \cdot \langle n, m\rangle) \cdot a \\
&= \psi(b) \cdot \langle n, m\rangle \cdot a \\
&= \langle n \diamond \psi(b)^*, m\rangle \cdot a \\
&= \langle n \diamond \psi^*(b^*), m\rangle \cdot a \\
&= \langle n \diamond \psi^* \circ b^*, m\rangle \cdot a \\
&= b \cdot \langle n \diamond \psi^*, m\rangle \cdot a.
\end{aligned}$$

Therefore $\langle n, \sigma_M(m)\rangle = \langle n \diamond \psi^*, m\rangle$ i.e. σ_M is adjointable. Hence the map $\Phi : \mathrm{Cen}(A) \to \mathrm{ANat}(A)$ given by $\psi \longmapsto (\sigma_M : M \to M,\ m \mapsto m \diamond_{\mathcal{A}_A(A)} \psi)$ defines an adjointable natural transformation since any adjointable map preserves the right action. This is easily checked since

$$\begin{aligned}
(h \circ \sigma_M)(m) \diamond a &= h(\sigma_M(m) \diamond a) \\
&= h(m \diamond \psi \diamond a) \\
&= h(m \diamond \psi(a)) \\
&= h(m) \diamond \psi(a) \\
&= h(m) \diamond \psi \diamond a \\
&= \sigma_M(h(m)) \diamond a \\
&= (\sigma_M \circ h)(m) \diamond a
\end{aligned}$$

i.e. $h \circ \sigma_M = \sigma_M \circ h$. Conversely, if $\sigma : \mathrm{Id}_{\mathrm{mreg}-Hilb_A} \to \mathrm{Id}_{\mathrm{mreg}-Hilb_A}$ is an adjointable natural transformation then $\sigma_A \in \mathrm{Cen}(A)$. $\square$

The following theorem provides a fully categorical proof of the first part of the Theorem 2.15.

Theorem 2.17. *Let A, B be Morita equivalent m-regular involutive quantales. Then $ANat(A)$ and $ANat(B)$ are isomorphic as involutive quantales.*

Proof. Let $F : \mathrm{mreg} - Hilb_A \to \mathrm{mreg} - Hilb_B$ be the unitary equivalence functor. Then there is a functor $G : \mathrm{mreg} - Hilb_B \to \mathrm{mreg} - Hilb_A$ such that we have natural unitary isomorphisms $\eta : GF \to \mathrm{Id}_{\mathrm{mreg}-Hilb_A}$ and $\zeta : FG \to \mathrm{Id}_{\mathrm{mreg}-Hilb_B}$. Similarly as in [6] let us define maps $T : \mathrm{ANat}(A) \to \mathrm{ANat}(B)$ and $S : \mathrm{ANat}(B) \to \mathrm{ANat}(A)$ by the prescription

$$\sigma \mapsto (\zeta_N \circ F(\sigma_{G(N)}) \circ \zeta_N^{-1})_N \text{ and } \rho \mapsto (\eta_M \circ G(\rho_{F(M)}) \circ \eta_M^{-1})_M.$$

Note that since η_M is a unitary isomorphism we have that

$$\sigma_{GF(M)} = GF(\sigma_M) \text{ and } \sigma_{GFG(N)} = GF(\sigma_{G(N)}).$$

So we have the following pair of commuting diagrams, the second diagram is an application of the functor F to the first diagram.

$$\begin{tikzcd}
GFGF(M) \arrow[r, "GF(\sigma_{GF(M)})"] \arrow[d, "G(\zeta_{F(M)})"] & GFGF(M) \arrow[d, "G(\zeta_{F(M)})"'] \\
GF(M) \arrow[r, "GF(\sigma_M)", "\sigma_{GF(M)}"'] \arrow[d, "\eta_M"] & GF(M) \arrow[d, "\eta_{FM}"'] \\
M \arrow[r, "\sigma_M"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
FGFGF(M) \arrow[r, "FGF(\sigma_{FGF(M)})"] \arrow[d, "FG(\zeta_{F(M)})"] & FGFGF(M) \arrow[d, "FG(\zeta_{F(M)})"'] \\
FGF(M) \arrow[r, "FGF(\sigma_M)", "F(\sigma_{GF(M)})"'] \arrow[d, "F(\eta_M)"] & FGF(M) \arrow[d, "F(\eta_{FM})"'] \\
F(M) \arrow[r, "F(\sigma_M)"] & F(M)
\end{tikzcd}$$

Hence, from the second diagram, we have

$$F(\sigma_M) \circ F(\eta_M) \circ FG(\zeta_{F(M)}) = F(\eta_M) \circ FG(\zeta_{F(M)}) \circ FGF(\sigma_{FGF(M)}).$$

In particular,

$$F(\sigma_M) = F(\eta_M) \circ FG(\zeta_{F(M)}) \circ FGF(\sigma_{FGF(M)}) \circ FG(\zeta_{F(M)}^{-1}) \circ F(\eta_M^{-1})$$

i.e.

$$F(\sigma_M) = F(\eta_M) \circ FG(T(\sigma)_{F(M)}) \circ F(\eta_M^{-1}).$$

Hence $\sigma_M = ST(\sigma)_M$. Similarly, $\rho_N = TS(\rho)_N$ i.e. $ST = \mathrm{id}_{\mathrm{ANat}(A)}$ and $TS = \mathrm{id}_{\mathrm{ANat}(B)}$. Note that we have the following commuting diagram

$$\begin{tikzcd}
GFG(N) \arrow[r, "GF(\sigma_{G(N)})", "\sigma_{GFG(N)}"'] \arrow[d, "G(\zeta_N)"'] & GFG(N) \arrow[d, "G(\zeta_N)"'] \\
G(N) \arrow[r, "\sigma_{G(N)}", "G(\zeta_N) \circ GF(\sigma_{G(N)}) \circ G(\zeta_N^{-1})"'] & G(N)
\end{tikzcd}$$

This gives us that

$$\sigma_{G(N)} = G(\zeta_N) \circ GF(\sigma_{G(N)}) \circ G(\zeta_N^{-1})$$

i.e. the diagram

$$\begin{tikzcd}
G(N_1) \arrow[r, "\sigma_{G(N_1)}", "G(\zeta_{N_1}) \circ GF(\sigma_{G(N_1)}) \circ G(\zeta_{N_1}^{-1})"'] \arrow[d, "G(f)"'] & G(N_1) \arrow[d, "G(f)"'] \\
G(N_2) \arrow[r, "\sigma_{G(N_2)}", "G(\zeta_{N_2}) \circ GF(\sigma_{G(N_2)}) \circ G(\zeta_{N_2}^{-1})"'] & G(N_2)
\end{tikzcd}$$

commutes. Hence also the following diagram commutes

$$\begin{tikzcd}
N_1 \arrow[r, "\zeta_{N_1} \circ F(\sigma_{G(N_1)}) \circ \zeta_{N_1}^{-1}"'] \arrow[d, "f"'] & N_1 \arrow[d, "f"'] \\
N_2 \arrow[r, "\zeta_{N_2} \circ F(\sigma_{G(N_2)}) \circ \zeta_{N_2}^{-1}"'] & N_2
\end{tikzcd}$$

i.e. $T(\sigma)$ is a natural transformation and it is evidently adjointable. Similarly for $S(\rho)$. Evidently, T and S are involutive quantale isomorphisms. $\square$

Corollary 2.18. *A unital involutive quantale is Morita equivalent to a commutative m-regular involutive quantale C if and only if it is equivalent to its own centre.*

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