Proceedings of the Ninth Prague Topological Symposium
Contributed papers from the symposium held in
Prague, Czech Republic, August 19–25, 2001

ON TYCHONOFF-TYPE HYPERTOPOLOGIES

GEORGI DIMOV, FRANCO OBERSNEL, AND GINO TIRONI

Abstract. In 1975, M. M. Choban [5] introduced a new topology on the set of all closed subsets of a topological space, similar to the *Tychonoff topology* but weaker than it. In 1998, G. Dimov and D. Vakarelov [8] used a generalized version of this new topology, calling it *Tychonoff-type topology*. The present paper is devoted to a detailed study of Tychonoff-type topologies on an arbitrary family $\mathcal{M}$ of subsets of a set X. When $\mathcal{M}$ contains all singletons, a description of all Tychonoff-type topologies $\mathcal{O}$ on $\mathcal{M}$ is given. The continuous maps of a special form between spaces of the type $(\mathcal{M}, \mathcal{O})$ are described in an isomorphism theorem. The problem of *commutability between hyperspaces and subspaces with respect to a Tychonoff-type topology* is investigated as well. Some topological properties of the hyperspaces $(\mathcal{M}, \mathcal{O})$ with Tychonoff-type topologies $\mathcal{O}$ are briefly discussed.

1. Introduction

In 1975, M. M. Choban [5] introduced a new topology on the set of all closed subsets of a topological space for obtaining a generalization of the famous Kolmogoroff Theorem on operations on sets. This new topology is similar to the *Tychonoff topology* (known also as *upper Vietoris topology*, or *upper semi-finite topology* ([13]), or *kappa-topology*) but is weaker than it. In 1998, G. Dimov and D. Vakarelov [8] used a generalized version of this new topology for proving an isomorphism theorem for the category of all Tarski

2000 *Mathematics Subject Classification.* Primary 54B20, 54B05; Secondary 54B30, 54D10, 54G99.

Key words and phrases. Tychonoff topology, Tychonoff-type topology, T-space, commutative space, $\mathcal{O}$-commutative space, $\mathcal{M}$-cover, $\mathcal{M}$-closed family, P_∞-space.

The first author was partially supported by a Fellowship for Mathematics of the NATO-CNR Outreach Fellowships Programme 1999, Bando 219.32/16.07.1999.

The second and the third authors were supported by the National Group “Analisi reale” of the Italian Ministry of the University and Scientific Research at the University of Trieste.

Georgi Dimov, Franco Obersnel and Gino Tironi, *On Tychonoff-type hypertopologies*, Proceedings of the Ninth Prague Topological Symposium, (Prague, 2001), pp. 51–70, Topology Atlas, Toronto, 2002; arXiv:math.GN/0204121.

consequence systems. This generalized version was called *Tychonoff-type topology*.

The present paper is devoted to a detailed study of Tychonoff-type topologies on an arbitrary family $\mathcal{M}$ of subsets of a set X. When $\mathcal{M}$ is a *natural family*, i.e. it contains all singletons, a description of all Tychonoff-type topologies $\mathcal{O}$ on $\mathcal{M}$ is given (see Proposition 2.32). For doing this, the notion of *T-space* is introduced. The natural morphisms for T-spaces are not enough to describe all continuous maps between spaces of the type $(\mathcal{M}, \mathcal{O})$, where $\mathcal{M}$ is a natural family and $\mathcal{O}$ is a Tychonoff-type topology on it; we obtain a characterization of those continuous maps which correspond to the morphisms between T-spaces. This is done by defining suitable categories and by proving that these categories are isomorphic (see Theorem 2.37). In such a way we extend to any natural family $\mathcal{M}$ on X the corresponding result obtained in [8] for the family $\mathcal{F}in(X)$ of all finite subsets of X. We investigate also the problem of *commutability between hyperspaces and subspaces with respect to a Tychonoff-type topology*, i.e. when the hyperspace of any subspace A of a topological space Y is canonically representable as a subspace of the hyperspace of Y. Such investigations were done previously by H.-J. Schmidt [14] for the lower Vietoris topology, by G. Dimov [7, 6] for the Tychonoff topology and for the Vietoris topology, and by B. Karaivanov [12] for other hypertopologies. We study also such a problem for a fixed subspace A of Y. Some results of [7, 6, 15] are generalized. Finally, we study briefly some topological properties (separation axioms, compactness, weight, density, isolated points, P_∞) of the hyperspaces $(\mathcal{M}, \mathcal{O})$ with Tychonoff-type topologies $\mathcal{O}$. Some results of [10, 8] are generalized.

Let us fix the notations.

Notations 1.1. We denote by ω the set of all *positive* natural numbers, by $\mathbb{R}$ — the real line, and by $\mathbb{Z}$ — the set of all integers. We put $\mathbb{N} = \omega \cup \{0\}$.

Let X be a set. We denote by $\mathcal{P}(X)$ the set of all subsets of X. Let $\mathcal{M}, \mathcal{A} \subseteq \mathcal{P}(X)$ and $A \subseteq X$. We will use the following notations:

- $A^+_{\mathcal{M}} := \{M \in \mathcal{M} : M \subseteq A\}$;
- $\mathcal{A}^+_{\mathcal{M}} := \{A^+_{\mathcal{M}} : A \in \mathcal{A}\}$;
- $\mathcal{F}in(X) := \{M \subseteq X : |M| < \aleph_0\}$;
- $\mathcal{F}in_n(X) := \{M \subseteq X : |M| \leq n\}$, where $n \in \omega$.

We will denote by $\mathcal{A}^\cap$ (respectively by $\mathcal{A}^\cup$) the closure under finite intersections (unions) of the family $\mathcal{A}$. In other words,

- $\mathcal{A}^\cap := \{\bigcap_{i=1}^k A_i : k \in \omega, A_i \in \mathcal{A}\}$ and
- $\mathcal{A}^\cup := \{\bigcup_{i=1}^k A_i : k \in \omega, A_i \in \mathcal{A}\}$.

Let $(X, \mathcal{T})$ be a topological space. We put

- $\mathcal{CL}(X) := \{M \subseteq X : M \text{ is closed in } X,\ M \neq \emptyset\}$ and
- $\mathcal{C}omp(X) := \{M \subseteq X : M \text{ is compact}\}$.

The closure of a subset A of X in $(X, \mathcal{T})$ will be denoted by $cl_X A$ or $\overline{A}^X$; as usual, for $U \subseteq A \subseteq X$, we put

- $Ex_{A,X}U := X \setminus cl_X(A \setminus U)$.

By a *base* of $(X, \mathcal{T})$ we will always mean an open base. The weight (resp., the density) of $(X, \mathcal{T})$ will be denoted by $w(X, \mathcal{T})$ (resp., $d(X, \mathcal{T})$).

If $\mathcal{C}$ denotes a category, we write $X \in |\mathcal{C}|$ if X is an object of the category $\mathcal{C}$.

For all undefined here notions and notations, see [9] and [11].

2. Hypertopologies of Tychonoff-type

Fact 2.1. Let X be a set and $\mathcal{M}, \mathcal{A} \subseteq \mathcal{P}(X)$. Then:

(a) $\bigcap \mathcal{A}^+_{\mathcal{M}} = (\bigcap \mathcal{A})^+_{\mathcal{M}}$;
(b) $A \subseteq B$ implies that $A^+_{\mathcal{M}} \subseteq B^+_{\mathcal{M}}$ for all A, $B \subseteq X$.

Definition 2.2. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{M} \subseteq \mathcal{P}(X)$. The topology $\mathcal{O}_{\mathcal{T}}$ on $\mathcal{M}$, having as a base the family $\mathcal{T}^+_{\mathcal{M}}$, will be called *Tychonoff topology on $\mathcal{M}$ generated by* $(X, \mathcal{T})$. When $\mathcal{M} = \mathcal{CL}(X)$, then $\mathcal{O}_{\mathcal{T}}$ is just the classical Tychonoff topology on $\mathcal{CL}(X)$.

Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. A topology $\mathcal{O}$ on $\mathcal{M}$ is called a *Tychonoff topology on* $\mathcal{M}$ if there exists a topology $\mathcal{T}$ on X such that $\mathcal{T}^+_{\mathcal{M}}$ is a base of $\mathcal{O}$.

Definition 2.3. Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. A topology $\mathcal{O}$ on the set $\mathcal{M}$ is called a *topology of Tychonoff-type on* $\mathcal{M}$ if the family $\mathcal{O} \cap \mathcal{P}(X)^+_{\mathcal{M}}$ is a base for $\mathcal{O}$.

Clearly, a Tychonoff topology on $\mathcal{M}$ is always a topology of Tychonoff-type on $\mathcal{M}$, but not viceversa (see Example 2.42).

Fact 2.4. Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then the family $\mathcal{B}_{\mathcal{O}} := \{A \subseteq X : A^+_{\mathcal{M}} \in \mathcal{O}\}$ is closed under finite intersections, $X \in \mathcal{B}_{\mathcal{O}}$, and, hence, $\mathcal{B}_{\mathcal{O}}$ is a base for a topology $\mathcal{T}_{\mathcal{O}}$ on X. The family $(\mathcal{B}_{\mathcal{O}})^+_{\mathcal{M}}$ is a base of $\mathcal{O}$.

Definition 2.5. Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. We will say that the topology $\mathcal{T}_{\mathcal{O}}$ on X, introduced in Fact 2.4, is *induced by the topological space* $(\mathcal{M}, \mathcal{O})$.

Proposition 2.6. *Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. A topology $\mathcal{O}$ on $\mathcal{M}$ is a topology of Tychonoff-type if and only if there exists a topology $\mathcal{T}$ on X and a base $\mathcal{B}$ for $\mathcal{T}$ (which contains X and is closed under finite intersections) such that $\mathcal{B}^+_{\mathcal{M}}$ is a base for $\mathcal{O}$.*

Proof. Suppose $\mathcal{O}$ is a topology of Tychonoff-type on $\mathcal{M}$. Then the topology $\mathcal{T}_{\mathcal{O}}$ induced by the topological space $(\mathcal{M}, \mathcal{O})$ (see Fact 2.4 and Definition 2.5) and the base $\mathcal{B}_{\mathcal{O}}$ have the required property.

Conversely, suppose $\mathcal{T}$ and $\mathcal{B}$ are given as in the statement. Then $\mathcal{B}^+_{\mathcal{M}}$ is a base for $\mathcal{O}$, and therefore also $\mathcal{O} \cap \mathcal{P}(X)^+_{\mathcal{M}}$ is a base for $\mathcal{O}$. □

Definition 2.7. Let X be a set and $\mathcal{M},\mathcal{B} \subseteq \mathcal{P}(X)$. When $\mathcal{B}^+_{\mathcal{M}}$ is a base for a topology $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{M}$, we will say that $\mathcal{B}$ *generates a topology* on $\mathcal{M}$. (Obviously, the topology $\mathcal{O}_{\mathcal{B}}$ is of Tychonoff-type.

Proposition 2.8. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(X)$. The family $\mathcal{B}$ generates a topology $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{M}$ if and only if the family $\mathcal{B}$ satisfies the following conditions:*

(MB1) *For any $M \in \mathcal{M}$ there exists a $U \in \mathcal{B}$ such that $M \subseteq U$;*
(MB2) *For any U_1, $U_2 \in \mathcal{B}$ and any $M \in \mathcal{M}$ with $M \subseteq U_1 \cap U_2$ there exists a $U_3 \in \mathcal{B}$ such that $M \subseteq U_3 \subseteq U_1 \cap U_2$.*

Proof. It follows from Proposition 1.2.1 [9]. □

Corollary 2.9. *Let X be a set and $\mathcal{M},\mathcal{B} \subseteq \mathcal{P}(X)$. If $\mathcal{B} = \mathcal{B}^\cap$ and $X \in \mathcal{B}$, then $\mathcal{B}$ generates a Tychonoff-type topology on $\mathcal{M}$.*

Definition 2.10. Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. We say that $\mathcal{M}$ is a *natural family in X* if $\{x\} \in \mathcal{M}$ for all $x \in X$.

Corollary 2.11. *Let X be a set and $\mathcal{M}$ be a natural family in X. If $\mathcal{B} \subseteq \mathcal{P}(X)$ generates a topology on $\mathcal{M}$ (see Definition 2.7), then $\mathcal{B}$ is a base for a topology on X.*

Proof. By Proposition 2.8, $\mathcal{B}$ satisfies the conditions (MB1) and (MB2). Since $\mathcal{M}$ is natural, this clearly implies that $\mathcal{B}$ satisfies the hypotesis of Proposition 1.2.1 [9]. So $\mathcal{B}$ is a base for a topology on X. □

Remark 2.12. Trivial examples show that there exist sets X and (non-natural) families $\mathcal{M}$,$\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\mathcal{B}^+_{\mathcal{M}}$ is a base for a topology on $\mathcal{M}$ but

(a) $\bigcup \mathcal{B} \neq X$, so that $\mathcal{B}$ cannot serve even as subbase of a topology on X (take $X = \{0, 1\}$, $\mathcal{M} = \mathcal{B} = \{\{0\}\}$);
(b) $\mathcal{B}$ is not a base of a topology of X, although $\bigcup \mathcal{B} = X$ (take $X = \{0, 1, 2\}$, $\mathcal{M} = \mathcal{B} = \{\{0, 1\}, \{0, 2\}\}$).

The example of (b) shows also that if we substitute in 2.11 naturality of $\mathcal{M}$ with the condition "$\bigcup \mathcal{M} = X$" then we cannot prove that $\mathcal{B}$ is a base of a topology on X; however, it is easy to show that the condition "$\bigcup \mathcal{M} = X$" implies that $\bigcup \mathcal{B} = X$, i.e. $\mathcal{B}$ can serve as a subbase of a topology on X.

Of course, as it follows from Fact 2.1, if $\mathcal{B}^+_{\mathcal{M}}$ is a base of a topology $\mathcal{O}$ on $\mathcal{M}$, then $\tilde{\mathcal{B}} = \mathcal{B}^\cap \cup \{X\}$ is a base for a topology on X and $\tilde{\mathcal{B}}^+_{\mathcal{M}}$ is a base of $\mathcal{O}$.

Corollary 2.13. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} \subseteq \mathcal{T}$ be a base of $(X, \mathcal{T})$, closed under finite unions. Then $\mathcal{B}$ generates a topology of Tychonoff-type on $\mathcal{F}in(X)$ and $\mathcal{C}omp(X)$.*

Proof. It follows easily from Proposition 2.8. □

Proposition 2.14. *Let X be a set, $\mathcal{M}$,$\mathcal{B}_1$,$\mathcal{B}_2 \subseteq \mathcal{P}(X)$, and suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ generate, respectively, some topologies (of Tychonoff-type) $\mathcal{O}_{\mathcal{B}_1}$ and $\mathcal{O}_{\mathcal{B}_2}$ on $\mathcal{M}$. Then $\mathcal{O}_{\mathcal{B}_1} = \mathcal{O}_{\mathcal{B}_2}$ if and only if the following conditions are satisfied:*

(CO1) *For any $M \in \mathcal{M}$ and any $U_1 \in \mathcal{B}_1$ such that $M \subseteq U_1$ there exists $U_2 \in \mathcal{B}_2$ with $M \subseteq U_2 \subseteq U_1$;*

(CO2) *For any $M \in \mathcal{M}$ and any $U_2 \in \mathcal{B}_2$ such that $M \subseteq U_2$ there exists $U_1 \in \mathcal{B}_1$ with $M \subseteq U_1 \subseteq U_2$.*

Proof. It follows from 1.2.B [9]. □

Corollary 2.15. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}_1$, $\mathcal{B}_2$ be bases of $(X, \mathcal{T})$, closed under finite unions. Then they generate (see Corollary 2.13) equal topologies on $\mathcal{F}in(X)$ and $\mathcal{C}omp(X)$. In particular, every topology of Tychonoff-type on $\mathcal{F}in(X)$ or on $\mathcal{C}omp(X)$, generated by a base of $(X, \mathcal{T})$ which is closed under finite unions, coincides with the Tychonoff topology generated by $(X, \mathcal{T})$ on the corresponding set.*

Proof. Check that conditions (CO1) and (CO2) of Proposition 2.14 are satisfied. □

Corollary 2.16. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq \mathcal{P}(X)$, $\mathcal{B} \subseteq \mathcal{T}$, and suppose that $\mathcal{B}$ generates a topology of Tychonoff-type $\mathcal{O}$ on $\mathcal{M}$. Then $\mathcal{O}$ is the Tychonoff topology on $\mathcal{M}$ generated by $(X, \mathcal{T})$ if and only if for all $M \in \mathcal{M}$ and for all $V \in \mathcal{T}$ such that $M \subseteq V$, there exists $U \in \mathcal{B}$ with $M \subseteq U \subseteq V$. In this case we will say that $\mathcal{B}$ is an $\mathcal{M}$-*base *for $(X, \mathcal{T})$. Clearly, if $\mathcal{M}$ is a natural family, then every $\mathcal{M}$-base of $(X, \mathcal{T})$ is also a base of $(X, \mathcal{T})$.*

Proof. Put $\mathcal{B}_1 := \mathcal{T}$ and $\mathcal{B}_2 := \mathcal{B}$. Then condition (CO2) of Proposition 2.14 is trivially satisfied. The condition required in the statement is exactly condition (CO1). □

Definition 2.17. Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $A \subseteq X$. A family $\mathcal{U} \subseteq \mathcal{P}(X)$ will be called an $\mathcal{M}$-*cover* of A if $A = \bigcup \mathcal{U}$ and for all $M \in \mathcal{M}$ with $M \subseteq A$ there exists some $U \in \mathcal{U}$ such that $M \subseteq U$.

Proposition 2.18. *Let X be a set and $\mathcal{M}, \mathcal{A} \subseteq \mathcal{P}(X)$. Then the following conditions are equivalent:*

(U1) *For all $U \in \mathcal{A}$ and for all $x \in U$, there exists an $M \in \mathcal{M}$ with $x \in M \subseteq U$.*

(U2) *For any $U \in \mathcal{A} \cup \mathcal{M}$ and for any subfamily $\{U_\delta : \delta \in \Delta\}$ of $\mathcal{A} \cup \mathcal{M}$, the equality $U^+_{\mathcal{M}} = \bigcup_{\delta \in \Delta} (U_\delta)^+_{\mathcal{M}}$ holds if and only if the family $\{U_\delta\}_{\delta \in \Delta}$ is an $\mathcal{M}$-cover of U.*

Proof. Observe that, trivially, in condition (U1) we can replace the requirement 'for all $U \in \mathcal{A}$" with "for all $U \in \mathcal{A} \cup \mathcal{M}$".

(U1)⇒(U2). Let $U^+_{\mathcal{M}} = \bigcup_{\delta \in \Delta} (U_\delta)^+_{\mathcal{M}}$, with U,$U_\delta \in \mathcal{A} \cup \mathcal{M}$ for all $\delta \in \Delta$. We will prove first that $\bigcup_{\delta \in \Delta} U_\delta = U$.

Let $x \in \bigcup_{\delta\in\Delta} U_\delta$. Then there exists a $\delta \in \Delta$ such that $x \in U_\delta$. By assumption, there exists an $M \in \mathcal{M}$ with $x \in M \subseteq U_\delta$. Hence $M \in (U_\delta)^+_{\mathcal{M}}$. Since $U^+_{\mathcal{M}} = \bigcup_{\delta\in\Delta} (U_\delta)^+_{\mathcal{M}}$, we obtain that $M \subseteq U$. Thus $x \in U$. Therefore, $\bigcup_{\delta\in\Delta} U_\delta \subseteq U$.

Conversely, let $x \in U$. By assumption, there exists an $M \in \mathcal{M}$ such that $x \in M \subseteq U$. Hence $M \in U^+_{\mathcal{M}} = \bigcup_{\delta\in\Delta} (U_\delta)^+_{\mathcal{M}}$. Therefore there exists a $\delta \in \Delta$ such that $M \in (U_\delta)^+_{\mathcal{M}}$, i.e. $M \subseteq U_\delta$ and $x \in U_\delta \subseteq \bigcup_{\delta\in\Delta} U_\delta$.

We have verified that $\bigcup_{\delta\in\Delta} U_\delta = U$.

Suppose $M \in \mathcal{M}$ and $M \subseteq U$. Then $M \in U^+_{\mathcal{M}}$ and therefore there exists some $\gamma \in \Delta$ with $M \in (U_\gamma)^+_{\mathcal{M}}$. Hence $M \subseteq U_\gamma$.

This shows that the family $\{U_\delta\}_{\delta\in\Delta}$ is an $\mathcal{M}$-cover of U.

The other implication can be easily proved. (Let's remark that condition (U1) is not used in the proof of this last implication.)

(U2)$\Rightarrow$(U1). Suppose $U \in \mathcal{A}$ and $x \in U$. Clearly, we have

$$U^+_{\mathcal{M}} = \bigcup\{M^+_{\mathcal{M}} : M \in \mathcal{M},\ M \subseteq U\}.$$

Then, by assumption, the family $\{M \in \mathcal{M} : M \subseteq U\}$ is an $\mathcal{M}$-cover of U. Therefore $U = \bigcup\{M : M \in \mathcal{M},\ M \subseteq U\}$. Hence there exists an $M \in \mathcal{M}$ with $x \in M \subseteq U$. $\square$

Proposition 2.19. *Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. Then the following conditions are equivalent:*

(a) *$\mathcal{M}$ is a natural family;*

(b) *For any $U \subseteq X$ and for any subfamily $\{U_\delta : \delta \in \Delta\}$ of $\mathcal{P}(X)$, the equality $U^+_{\mathcal{M}} = \bigcup_{\delta\in\Delta} (U_\delta)^+_{\mathcal{M}}$ holds if and only if the family $\{U_\delta\}_{\delta\in\Delta}$ is an $\mathcal{M}$-cover of U.*

Proof. Put $\mathcal{A} = \mathcal{P}(X)$ in Proposition 2.18. $\square$

Proposition 2.20. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$, $\mathcal{O}$ be a Tychonoff topology on $\mathcal{M}$ generated by a topology $\mathcal{T}$ on X and $\mathcal{M}$ be a network in the sense of Arhangel′skiĭ for $\mathcal{T}_{\mathcal{O}}$. Then $\mathcal{T} = \mathcal{B}_{\mathcal{O}}$ and $\mathcal{O}$ is generated by a unique topology on X, namely by $\mathcal{T}_{\mathcal{O}}$ (see Fact 2.4 for the notation $\mathcal{B}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$).*

Proof. We only need to show that $\mathcal{B}_{\mathcal{O}} \subseteq \mathcal{T}$. Assume $A \in \mathcal{B}_O$. Then $A^+_{\mathcal{M}} \in \mathcal{O}$. Since $\mathcal{T}$ generates $\mathcal{O}$, we have $A^+_{\mathcal{M}} = \bigcup_{\delta\in\Delta} (U_\delta)^+_{\mathcal{M}}$, where $U_\delta \in \mathcal{T}$ for all $\delta \in \Delta$. Clearly $U_\delta \in \mathcal{B}_O$ for all $\delta \in \Delta$. By Proposition 2.18, we obtain that $A = \bigcup_{\delta\in\Delta} U_\delta$ and therefore $A \in \mathcal{T}$. $\square$

Remark 2.21. Trivial examples show that there exist sets X, families $\mathcal{M} \subseteq \mathcal{P}(X)$ and Tychonoff topologies on $\mathcal{M}$ which are generated by more than one topology on X.

Corollary 2.22. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$, $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$ and $\mathcal{M}$ be a network in the sense of Arhangel′skiĭ for $\mathcal{T}_{\mathcal{O}}$. Then $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$ if and only if $\mathcal{B}_{\mathcal{O}} = \mathcal{T}_{\mathcal{O}}$ (see Fact 2.4 for the notations $\mathcal{B}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$).*

Proof. Suppose $\mathcal{B}_\mathcal{O} = \mathcal{T}_\mathcal{O}$. Then the topology $\mathcal{O}$ is generated by the topology $\mathcal{T}_\mathcal{O}$ on X and hence, by definition, $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$.

Suppose $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$. Then $\mathcal{O}$ is generated by some topology $\mathcal{T}$ on X. By Proposition 2.20, we get $\mathcal{T} = \mathcal{B}_\mathcal{O}$. Hence $\mathcal{T}_\mathcal{O} = \mathcal{B}_\mathcal{O}$. $\square$

Corollary 2.23. *Let X be a set, $\mathcal{M}$ be a natural family in X and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$ if and only if $\mathcal{B}_\mathcal{O} = \mathcal{T}_\mathcal{O}$.*

Proof. A natural family $\mathcal{M}$ satisfies the hypothesis of Corollary 2.22. $\square$

Proposition 2.24. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M}$ be a natural family in X, $\mathcal{B} \subseteq \mathcal{T}$ and suppose that $\mathcal{B}$ generates a topology $\mathcal{O}_\mathcal{B}$ on $\mathcal{M}$. Then*

$$\mathcal{B}_{\mathcal{O}_\mathcal{B}} = \{A \subseteq X : A \quad \textit{is } \mathcal{M}\textit{-covered} \quad \textit{by a} \quad \textit{subfamily} \quad \textit{of } \mathcal{B}\},$$

$\mathcal{B}_{\mathcal{O}_\mathcal{B}} \subseteq \mathcal{T}$ and $\mathcal{B}^\cap \subseteq \mathcal{B}_{\mathcal{O}_\mathcal{B}}$ (see Fact 2.4 for the notation $\mathcal{B}_{\mathcal{O}_\mathcal{B}}$).

Proof. It follows from Proposition 2.19 and Fact 2.4. $\square$

Proposition 2.25. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M}$ be a natural family in X, $\mathcal{B}$ be a base for $(X, \mathcal{T})$ and suppose that $\mathcal{B}$ generates a topology $\mathcal{O}_\mathcal{B}$ on $\mathcal{M}$. Let $\mathcal{T}_{\mathcal{O}_\mathcal{B}}$ be the topology on X induced by $(\mathcal{M}, \mathcal{O}_\mathcal{B})$. Then $\mathcal{T}_{\mathcal{O}_\mathcal{B}} = \mathcal{T}$.*

Proof. We have, by Proposition 2.24, that $\mathcal{B} \subseteq \mathcal{B}_{\mathcal{O}_\mathcal{B}} \subseteq \mathcal{T}_{\mathcal{O}_\mathcal{B}}$. Thus $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{O}_\mathcal{B}}$. As it is shown in Proposition 2.24, $\mathcal{B}_{\mathcal{O}_\mathcal{B}} \subseteq \mathcal{T}$ and hence $\mathcal{T}_{\mathcal{O}_\mathcal{B}} \subseteq \mathcal{T}$. So, $\mathcal{T} = \mathcal{T}_{\mathcal{O}_\mathcal{B}}$. $\square$

Example 2.26. Let us show that in Proposition 2.25 the requirement "$\mathcal{M}$ is a natural family" is essential.

Let $X = (0,1) \subset \mathbb{R}$ be the open unit interval with the usual topology, $\mathcal{M} = \{[a,b] : 0 < a < b < 1\}$, $\mathcal{B} = \{(a,b) : 0 < a < b < 1\}$. Then the family $\mathcal{B}$ satisfies conditions (MB1) and (MB2). Consider the set $A = (\frac{1}{2}, \frac{3}{4}) \cup \{\frac{1}{4}\}$. We have $A^+_\mathcal{M} = (\frac{1}{2}, \frac{3}{4})^+_\mathcal{M} \in \mathcal{B}^+_\mathcal{M}$ and therefore $A \in \mathcal{B}_{\mathcal{O}_\mathcal{B}}$ even though A is not open in X.

Definition 2.27. Let X be a set and $\mathcal{M}, \mathcal{U} \subseteq \mathcal{P}(X)$. We will say that $\mathcal{U}$ is an *$\mathcal{M}$-closed family* if for all $A \subseteq X$ such that A is $\mathcal{M}$-covered by some subfamily of $\mathcal{U}$, we have that $A \in \mathcal{U}$.

Proposition 2.28. *Let X be a set and $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{U} \subseteq \mathcal{P}(X)$, $\mathcal{M} \subseteq \mathcal{M}'$. Suppose that $\mathcal{U}$ is an $\mathcal{M}$-closed family. Then $\mathcal{U}$ is an $\mathcal{M}'$-closed family too.*

Proof. Suppose $A \subseteq X$ is $\mathcal{M}'$-covered by some subfamily $\mathcal{U}'$ of $\mathcal{U}$. Since $\mathcal{M} \subseteq \mathcal{M}'$, the set A is also $\mathcal{M}$-covered by $\mathcal{U}'$. By the hypothesis, $\mathcal{U}$ is an $\mathcal{M}$-closed family. Hence $A \in \mathcal{U}$. $\square$

Proposition 2.29. *Let X be a set, $\mathcal{M}$ be a natural family in X and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then $\mathcal{B}_\mathcal{O}$ is an $\mathcal{M}$-closed family (see Fact 2.4 for the notation $\mathcal{B}_\mathcal{O}$).*

Proof. It follows from Proposition 2.19. $\square$

Proposition 2.30. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M}$ be a natural family in X and $\mathcal{B} \subseteq \mathcal{T}$ be an $\mathcal{M}$-closed base of $(X, \mathcal{T})$. Suppose that $\mathcal{B}$ generates a topology $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{M}$. Then $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} = \mathcal{B}$, $X \in \mathcal{B}$ and $\mathcal{B}^{\cap} = \mathcal{B}$ (see Fact 2.4 for the notation $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}}$).*

Proof. Obviously, $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} \supseteq \mathcal{B}$. Let us show that $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} \subseteq \mathcal{B}$.

Let $A \in \mathcal{B}_{\mathcal{O}_{\mathcal{B}}}$. Then, by Proposition 2.24, A is $\mathcal{M}$-covered by some subfamily of $\mathcal{B}$ and, since $\mathcal{B}$ is $\mathcal{M}$-closed, we conclude that $A \in \mathcal{B}$. So, $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} = \mathcal{B}$. Now Fact 2.4 implies that $X \in \mathcal{B}$ and $\mathcal{B}^{\cap} = \mathcal{B}$. □

Definition 2.31. Let X be a set and $\mathcal{M}, \mathcal{B} \subseteq \mathcal{P}(X)$. The ordered triple $(X, \mathcal{B}, \mathcal{M})$ will be called a *T-space* if $\mathcal{B}$ is an $\mathcal{M}$-closed family, $X \in \mathcal{B}$ and $\mathcal{B}^{\cap} = \mathcal{B}$.

Note that if $(X, \mathcal{B}, \mathcal{M})$ is a T-space, then $\mathcal{B}$ is a base for a topology on X.

Proposition 2.32. *Let X be a set and $\mathcal{M}$ be a natural family in X. Let $TTT(X, \mathcal{M})$ be the set of all topologies of Tychonoff-type on $\mathcal{M}$. Denote by $T\text{-}Sp(X, \mathcal{M})$ the set of all T-spaces of the form $(X, \mathcal{B}, \mathcal{M})$. Then there is a bijective correspondence between the sets $TTT(X, \mathcal{M})$ and $T\text{-}Sp(X, \mathcal{M})$. Namely, consider the function $\alpha : TTT(X, \mathcal{M}) \to T\text{-}Sp(X, \mathcal{M})$, defined by $\alpha(\mathcal{O}) = (X, \mathcal{B}_{\mathcal{O}}, \mathcal{M})$ (see Fact 2.4 for the notation $\mathcal{B}_{\mathcal{O}}$), and the function $\beta : T\text{-}Sp(X, \mathcal{M}) \to TTT(X, \mathcal{M})$, defined by $\beta((X, \mathcal{B}, \mathcal{M})) = \mathcal{O}_{\mathcal{B}}$ (see Corollary 2.9 and Definition 2.7); then α and β are bijections and each one is the inverse of the other one.*

Proof. Let us show that the function α is well-defined. Let $\mathcal{O} \in TTT(X, \mathcal{M})$. By Fact 2.4, the family $\mathcal{B}_{\mathcal{O}}$ is closed under finite intersections and $X \in \mathcal{B}_{\mathcal{O}}$. By Proposition 2.29, $\mathcal{B}_{\mathcal{O}}$ is $\mathcal{M}$-closed. Hence $(X, \mathcal{B}_{\mathcal{O}}, \mathcal{M}) \in T\text{-}Sp(X, \mathcal{M})$.

We will prove now that the function β is well defined. Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be an $\mathcal{M}$-closed family, closed under finite intersections and such that $X \in \mathcal{B}$. Then, by Corollary 2.9, $\mathcal{O}_{\mathcal{B}}$ is a topology of Tychonoff-type on $\mathcal{M}$.

Proposition 2.30 gives that $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} = \mathcal{B}$, i.e. $\alpha \circ \beta = \mathrm{id}_{T\text{-}Sp(X,\mathcal{M})}$.

To show that $\beta \circ \alpha = \mathrm{id}_{TTT(X,\mathcal{M})}$, let $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then $\beta(\alpha(\mathcal{O})) = \mathcal{O}_{\mathcal{B}_{\mathcal{O}}}$. Since $\mathcal{O}$ is a topology of Tychonoff-type, $(\mathcal{B}_{\mathcal{O}})^{+}_{\mathcal{M}}$ is a base of $\mathcal{O}$. On the other hand, $(\mathcal{B}_{\mathcal{O}})^{+}_{\mathcal{M}}$ is, by definition, a base of $\mathcal{O}_{\mathcal{B}_{\mathcal{O}}}$. Hence $\mathcal{O} = \mathcal{O}_{\mathcal{B}_{\mathcal{O}}}$. □

Definition 2.33. We denote by $\mathcal{HT}$ (Hypertopologies of Tychonoff-type) the category defined as follows: its objects are all ordered triples $(X, \mathcal{M}, \mathcal{O})$ where X is a set, $\mathcal{M}$ is a natural family in X and $\mathcal{O}$ is a topology of Tychonoff-type on $\mathcal{M}$. To define the morphisms of $\mathcal{HT}$, let $(X, \mathcal{M}, \mathcal{O})$, $(X', \mathcal{M}', \mathcal{O}')$ be objects of $\mathcal{HT}$ and $f : X \to X'$ be a function between the sets X and X'. We will say that f *generates a morphism f_H of $\mathcal{HT}$* between $(X, \mathcal{M}, \mathcal{O})$ and $(X', \mathcal{M}', \mathcal{O}')$ if $f(\mathcal{M}) \subseteq \mathcal{M}'$ and the induced function on $\mathcal{M}$, $f_m : (\mathcal{M}, \mathcal{O}) \to (\mathcal{M}', \mathcal{O}')$, defined by $f_m(M) := f(M)$ (where the M on the left-handside is regarded as an element of $\mathcal{M}$ and the M on the right-handside is regarded as a subset of X) is continuous. The morphisms of $\mathcal{HT}$ are defined to be all f_H generated in this way.

Remark 2.34. It is easy to see that not any continuous map between spaces of the type $(\mathcal{M}, \mathcal{O})$ appears as some f_H (see Definition 2.33 for the notations). Indeed, let $(X, \mathcal{T})$ be a discrete space having more than one point; then the constant function $c : (\mathcal{CL}(X), \mathcal{O}_{\mathcal{T}}) \to (\mathcal{CL}(X), \mathcal{O}_{\mathcal{T}})$, defined by $c(F) = X$ for all $F \in \mathcal{CL}(X)$, is continuous but is not of the type f_H (here $\mathcal{O}_{\mathcal{T}}$ is the classical Tychonoff topology on $\mathcal{CL}(X)$ (see Definition 2.2)).

Definition 2.35. We denote by $\mathcal{TH}$ the category defined as follows: its objects are all T-spaces $(X, \mathcal{B}, \mathcal{M})$ (see Definition 2.31). To define the morphisms of $\mathcal{TH}$, let $(X, \mathcal{B}, \mathcal{M})$, $(X', \mathcal{B}', \mathcal{M}')$ be objects of $\mathcal{TH}$ and $f : X \to X'$ be a function between the sets X and X'. We will say that f *generates a morphism* f^T *of* $\mathcal{TH}$ between $(X, \mathcal{B}, \mathcal{M})$ and $(X', \mathcal{B}', \mathcal{M}')$ if $f(\mathcal{M}) \subseteq \mathcal{M}'$ and $f^{-1}(\mathcal{B}') \subseteq \mathcal{B}$. The morphisms of $\mathcal{TH}$ are defined to be all f^T generated in this way.

Lemma 2.36. *Let X, X' be sets, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{M}' \subseteq \mathcal{P}(X')$. Let $f : X \to X'$ be a function such that $f(\mathcal{M}) \subseteq \mathcal{M}'$ and let $f_m : \mathcal{M} \to \mathcal{M}'$ be defined by $f_m(M) := f(M)$. Then, for all $A' \subseteq X'$, we have*

$$f_m^{-1}\left((A')^+_{\mathcal{M}'}\right) = \left(f^{-1}(A')\right)^+_{\mathcal{M}}.$$

Proof. Let $M \in f_m^{-1}\left((A')^+_{\mathcal{M}'}\right)$. Then $f_m(M) \in (A')^+_{\mathcal{M}'}$, i.e. $f(M) \subseteq A'$. Since $M \subseteq f^{-1}(f(M)) \subseteq f^{-1}(A')$, we obtain $M \in \left(f^{-1}(A')\right)^+_{\mathcal{M}}$.

Let $M \in \left(f^{-1}(A')\right)^+_{\mathcal{M}}$. Then $M \subseteq f^{-1}(A')$ and hence $f(M) \subseteq A'$. Therefore $f_m(M) \in (A')^+_{\mathcal{M}'}$, i.e. $M \in f_m^{-1}\left((A')^+_{\mathcal{M}'}\right)$. □

Theorem 2.37. *The categories $\mathcal{HT}$ and $\mathcal{TH}$ are isomorphic.*

Proof. We define a functor $F : \mathcal{HT} \to \mathcal{TH}$ as follows: for all $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$, we put $F((X, \mathcal{M}, \mathcal{O})) := (X, \mathcal{B}_{\mathcal{O}}, \mathcal{M})$ (see Fact 2.4 for the notation $\mathcal{B}_{\mathcal{O}}$); for all morphisms $f_H : (X, \mathcal{M}, \mathcal{O}) \to (X', \mathcal{M}', \mathcal{O}')$, we put $F(f_H) := f^T$ (see Definitions 2.33 and 2.35 for the notations f_H and f^T).

We define a functor $G : \mathcal{TH} \to \mathcal{HT}$ as follows: for all $(X, \mathcal{B}, \mathcal{M}) \in |\mathcal{TH}|$, we put $G((X, \mathcal{B}, \mathcal{M})) := (X, \mathcal{M}, \mathcal{O}_{\mathcal{B}})$ (see Definition 2.7 for the notation $\mathcal{O}_{\mathcal{B}}$); for all morphisms $f^T : (X, \mathcal{B}, \mathcal{M}) \to (X', \mathcal{B}', \mathcal{M}')$, we put $G(f^T) := f_H$.

Let us check that F and G are well-defined.

Let $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. By Proposition 2.32, the triple $(X, \mathcal{B}_O, \mathcal{M})$ is a T-space, so that $F(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{TH}|$.

Let $f_H : (X, \mathcal{M}, \mathcal{O}) \to (X', \mathcal{M}', \mathcal{O}')$ be a morphism in $\mathcal{HT}$. Then $f(\mathcal{M}) \subseteq \mathcal{M}'$ and the induced function on $\mathcal{M}$, $f_m : (\mathcal{M}, \mathcal{O}) \to (\mathcal{M}', \mathcal{O}')$, defined by $f_m(M) = f(M)$, is continuous. To check that

$$f^T : F(X, \mathcal{M}, \mathcal{O}) \to F(X', \mathcal{M}', \mathcal{O}')$$

(i.e. $f^T : (X, \mathcal{B}_{\mathcal{O}}, \mathcal{M}) \to (X', \mathcal{B}_{\mathcal{O}'}, \mathcal{M}'))$ is a morphism in $\mathcal{TH}$, we need to show that $f^{-1}(\mathcal{B}_{\mathcal{O}'}) \subseteq \mathcal{B}_{\mathcal{O}}$. Let $B' \in \mathcal{B}_{\mathcal{O}'}$. Then $(B')^+_{\mathcal{M}'} \in \mathcal{O}'$. By the continuity of f_m, we have that $f_m^{-1}\left((B')^+_{\mathcal{M}'}\right) \in \mathcal{O}$. By Lemma 2.36, $f_m^{-1}\left((B')^+_{\mathcal{M}'}\right) = \left(f^{-1}(B')\right)^+_{\mathcal{M}}$. Hence $\left(f^{-1}(B')\right)^+_{\mathcal{M}} \in \mathcal{O}$, i.e. $f^{-1}(B') \in \mathcal{B}_{\mathcal{O}}$. So, we have proved that F is well-defined. Clearly, F is a functor.

Let now $(X, \mathcal{B}, \mathcal{M}) \in |\mathcal{TH}|$. Then, by Proposition 2.32, the topology $\mathcal{O}_{\mathcal{B}}$ is a Tychonoff-type topology on $\mathcal{M}$. Hence, $G((X, \mathcal{B}, \mathcal{M})) \in |\mathcal{HT}|$.

Let $f^T : (X, \mathcal{B}, \mathcal{M}) \to (X', \mathcal{B}', \mathcal{M}')$ be a morphism in $\mathcal{TH}$. Then $f(\mathcal{M}) \subseteq \mathcal{M}'$ and $f^{-1}(\mathcal{B}') \subseteq \mathcal{B}$. To check that

$$f_H : (X, \mathcal{M}, \mathcal{O}_{\mathcal{B}}) \to (X', \mathcal{M}', \mathcal{O}_{\mathcal{B}'})$$

is a morphism in $\mathcal{HT}$, we need to check that the function $f_m : (\mathcal{M}, \mathcal{O}_{\mathcal{B}}) \to (\mathcal{M}', \mathcal{O}_{\mathcal{B}'})$, defined by $f_m(M) := f(M)$, is continuous. By definition (see 2.7), $(\mathcal{B}')^+_{\mathcal{M}'}$ is a base of the topology $\mathcal{O}_{\mathcal{B}'}$. Let $B' \in \mathcal{B}'$. Then $(B')^+_{\mathcal{M}'} \in (\mathcal{B}')^+_{\mathcal{M}'}$. By assumption, $f^{-1}(B') \in \mathcal{B}$. Hence $\left(f^{-1}(B')\right)^+_{\mathcal{M}} \in \mathcal{O}_{\mathcal{B}}$. Since, by Lemma 2.36, $\left(f^{-1}(B')\right)^+_{\mathcal{M}} = f_m^{-1}\left((B')^+_{\mathcal{M}'}\right)$, we obtain that $f_m^{-1}\left((B')^+_{\mathcal{M}'}\right) \in \mathcal{O}_{\mathcal{B}}$. Therefore, the function f_m is continuous. So, G is well-defined. Obviously, G is a functor.

By Proposition 2.32, we have $F \circ G = \mathrm{id}_{\mathcal{TH}}$ and $G \circ F = \mathrm{id}_{\mathcal{HT}}$ on the objects. The equalities are clearly true for the morphisms. Hence F and G are isomorphisms. □

We recall that a topological space $(X, \mathcal{T})$ is called a *P_∞-space* (see [1, 8]) if $\mathcal{T}$ is closed under arbitrary intersections.

Lemma 2.38. *A space $(X, \mathcal{T})$ is a P_∞-space if and only it it has a base $\mathcal{B}$ closed under arbitrary intersections.*

Proof. Assume $\mathcal{T}$ has a base $\mathcal{B}$ closed under arbitrary intersections. Let $\mathcal{U} \subseteq \mathcal{T}$. Since $\emptyset$ is an open set, we can assume that $\bigcap \mathcal{U} \neq \emptyset$. Let $x \in \bigcap \mathcal{U}$. For any $U \in \mathcal{U}$, let $B_U \in \mathcal{B}$ be such that $x \in B_U \subseteq U$. Then

$$x \in \bigcap \{B_U : U \in \mathcal{U}\} \subseteq \bigcap \mathcal{U}$$

and, by assumption, $\bigcap \{B_U : U \in \mathcal{U}\} \in \mathcal{B}$. Hence, $\bigcap \mathcal{U} \in \mathcal{T}$. □

Proposition 2.39. *Let $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$. The topological space $(\mathcal{M}, \mathcal{O})$ is a P_∞-space if and only if the family $\mathcal{B}_{\mathcal{O}}$ is closed under arbitrary intersections. If $(\mathcal{M}, \mathcal{O})$ is a P_∞-space then $(X, \mathcal{T}_{\mathcal{O}})$ is a P_∞-space. If $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$, then $(\mathcal{M}, \mathcal{O})$ is a P_∞-space if and only if $(X, \mathcal{T}_{\mathcal{O}})$ is a P_∞-space (see Fact 2.4 for the notations $\mathcal{B}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$).*

Proof. The first two assertions follow from Lemma 2.38, Fact 2.1(a) and the definitions of $\mathcal{B}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$. The last assertion follows now from Corollary 2.22. □

Corollary 2.40. *Let $\mathcal{HT}_\infty$ be the full subcategory of $\mathcal{HT}$ having as objects all triples $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$ for which the space $(\mathcal{M}, \mathcal{O})$ is a P_∞-space. Let*

$\mathcal{TH}_\infty$ be the full subcategory of $\mathcal{TH}$ whose objects are all $(X, \mathcal{B}, \mathcal{M}) \in |\mathcal{TH}|$ such that the family $\mathcal{B}$ is closed under arbitrary intersections. Then $\mathcal{HT}_\infty$ and $\mathcal{TH}_\infty$ are isomorphic.

Proof. It follows from (the proof of) Theorem 2.37 and Proposition 2.39. □

Example 2.41. We will show that there exists $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$ such that $(X, \mathcal{T}_\mathcal{O})$ is a P_∞-space but $(\mathcal{M}, \mathcal{O})$ is not a P_∞-space.

Let $X = \omega$,

$$\mathcal{B} = \{\{n\} : n \in \omega\} \cup \{A \subseteq \omega : |\omega \setminus A| < \aleph_0\} \cup \{\emptyset\},$$

and $\mathcal{P}(\omega) \supseteq \mathcal{M} \supseteq \mathcal{F}in_2(\omega) \setminus \{\emptyset\}$. The family $\mathcal{B}$ is a base of the discrete topology $\mathcal{T}$ on ω, it is closed under finite intersections (but not under infinite intersections) and $X \in \mathcal{B}$. Let us show that $\mathcal{B}$ is $\mathcal{M}$-closed.

Let $\{B_\delta\}_{\delta \in \Delta}$ be a subfamily of $\mathcal{B} \setminus \{\emptyset\}$ which is an $\mathcal{M}$-cover of a subset B of X. Without loss of generality, we can assume that there exist at least two indices δ_1 and δ_2 such that $B_{\delta_1} \neq B_{\delta_2}$. Then there is at least one $\delta \in \Delta$ such that $|\omega \setminus B_\delta| < \aleph_0$; otherwise we would have $B_{\delta_1} = \{n_{\delta_1}\}$, $B_{\delta_2} = \{n_{\delta_2}\}$ and the set $F = \{n_{\delta_1}, n_{d_2}\}$, which belongs to $\mathcal{M}$, would be contained in B without being contained in B_δ for any $\delta \in \Delta$. Therefore $|\omega \setminus B| < \aleph_0$ and, hence, $B \in \mathcal{M}$.

Put $\mathcal{O} := \mathcal{O}_\mathcal{B}$. Then, by Proposition 2.30, $\mathcal{B} = \mathcal{B}_\mathcal{O}$ and hence $\mathcal{T} = \mathcal{T}_\mathcal{O}$. Since $\mathcal{B}$ is not closed under arbitrary intersections, we obtain, by Proposition 2.39, that $(\mathcal{M}, \mathcal{O})$ is not a P_∞-space. Clearly, the space $(X, \mathcal{T})$ is a P_∞-space because it is discrete. Observe that $\mathcal{O}$ is not a Tychonoff topology since $\mathcal{B}_\mathcal{O} \neq \mathcal{T}_\mathcal{O}$.

Example 2.42. Two more examples of Tychonoff-type, non Tychonoff topologies on some families $\mathcal{M} \subseteq \mathcal{P}(X)$.

Let X be a set with more than two elements. Let $\mathcal{M} = \mathcal{F}in(X) \setminus \{\emptyset\}$ and let $\mathcal{O} = \{\{\{x\} : x \in A\} : A \subseteq X\} \cup \{\mathcal{M}\} \cup \{\emptyset\}$. Then $\mathcal{O}$ is a topology on $\mathcal{M}$.

$\mathcal{O}$ is a topology of Tychonoff-type since

$$\mathcal{O} \cap \mathcal{P}(X)^+_\mathcal{M} = \{\{x\} : x \in X\} \cup \{X^+_\mathcal{M}\} \cup \{\emptyset\}$$

is a base for $\mathcal{O}$. Clearly $(\mathcal{M}, \mathcal{O})$ is a P_∞-space.

We have $\mathcal{B}_\mathcal{O} = \{\{x\} : x \in X\} \cup \{X\} \cup \{\emptyset\}$, and therefore $\mathcal{T}_\mathcal{O}$ is the discrete topology. By Proposition 2.22, since $\mathcal{B}_\mathcal{O} \neq \mathcal{T}_\mathcal{O}$, $\mathcal{O}$ is not a Tychonoff topology on $\mathcal{M}$.

Observe that $(\mathcal{M}, \mathcal{O})$ is not a T_0-space. In fact, the only neighbourhood of any $F \in \mathcal{M}$ such that $|F| \geq 2$ is $\mathcal{M}$, and we are assuming $|X| > 2$.

We consider now the natural family $\mathcal{M}' = \mathcal{P}(X) \setminus \{\emptyset\}$ and we define $\mathcal{O}$ as above. $\mathcal{O}$ is a Tychonoff-type topology on $\mathcal{M}'$ but it is not a Tychonoff topology. Again $\mathcal{T}_\mathcal{O}$ is the discrete topology on X. Hence $\mathcal{M}' = \mathcal{CL}((X, \mathcal{T}_\mathcal{O}))$. We observe as before that $(\mathcal{M}', \mathcal{O})$ is not a T_0-space. Note also that the family $\mathcal{B}_\mathcal{O}$ is $\mathcal{M}''$-closed for every natural family $\mathcal{M}''$ in X which contains all two-points subsets of X and $\emptyset \notin \mathcal{M}''$.

We will briefly discuss now some topological properties of the hyperspaces $(\mathcal{M},\mathcal{O})$ with Tychonoff-type topologies $\mathcal{O}$.

Fact 2.43. Let X be a set, $\mathcal{M}\subseteq\mathcal{P}(X)$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$ generated by a subfamily $\mathcal{B}$ of $\mathcal{P}(X)$. Then:

(a) the topological space $(\mathcal{M},\mathcal{O})$ is a T_0-space (resp., T_1-space) if and only if for any $F,G\in\mathcal{M}$ with $F\neq G$, there exists a $B\in\mathcal{B}$ such that either $F\subseteq B$ and $G\not\subseteq B$, or $G\subseteq B$ and $F\not\subseteq B$ (resp., $F\subseteq B$ and $G\not\subseteq B$).

(b) if for any $x\in X$ and for any $F\in\mathcal{M}$ with $x\notin F$, there exists a $B\in\mathcal{B}$ such that $F\subseteq B$ and $x\notin B$, then $(\mathcal{M},\mathcal{O})$ is a T_0-space.

Remark 2.44. Let us note that Fact 2.43(b) implies the following assertion, which was mentioned in [5], section 2 (after Lemma 3) (the requirement that $X\in\Omega$ has to be added there): *if $(X,\mathcal{T})$ is a regular T_1-space, $\mathcal{M}$ is a family consisting of closed subsets of $(X,\mathcal{T})$ and $\mathcal{B}$ is a base of $(X,\mathcal{T})$ such that $\mathcal{B}=\mathcal{B}^\cap$ and $U\in\mathcal{B}$ implies that $X\setminus\overline{U}\in\mathcal{B}$, then $(\mathcal{M},\mathcal{O}_\mathcal{B})$ is a T_0-space.*

Fact 2.45. Let $(X,\mathcal{M},\mathcal{O})\in|\mathcal{HT}|$. Then the correspondence $(X,\mathcal{T}_\mathcal{O})\to(\mathcal{M},\mathcal{O})$, $x\mapsto\{x\}$, is a homeomorphic embedding. Hence, we have, in particular, that:

(a) $w(X,\mathcal{T}_\mathcal{O})\leq w(\mathcal{M},\mathcal{O})$;

(b) if $(\mathcal{M},\mathcal{O})$ is a T_0-space then $(X,\mathcal{T}_\mathcal{O})$ is a T_0-space.

Fact 2.46.

(a) Let X be a set, $\mathcal{M}\subseteq\mathcal{P}(X)$ be a family such that there exist $F,G\in\mathcal{M}$ with $F\subset G$ and $F\neq G$, and let $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then $(\mathcal{M},\mathcal{O})$ is not a T_1-space.

(b) If $(X,\mathcal{M},\mathcal{O})\in|\mathcal{HT}|$ then $(\mathcal{M},\mathcal{O})$ is a T_1-space if and only if $(X,\mathcal{T}_\mathcal{O})$ is a T_1-space and $\mathcal{M}=\{\{x\}:x\in X\}$.

Fact 2.47. Let $(X,\mathcal{M},\mathcal{O})\in|\mathcal{HT}|$. Then $(\mathcal{M},\mathcal{O})$ is a compact space if and only if any $\mathcal{M}$-cover of X, consisting of elements of $\mathcal{B}_\mathcal{O}$, has a finite $\mathcal{M}$-subcover.

Proof. It follows from Proposition 2.19. □

Examples 2.48. There are many examples of "very nice" spaces X with non-T_0-hyperspaces $(\mathcal{M},\mathcal{O}_\mathcal{B})$ (see Examples 2.42 and 3.19). As an example of a non-T_0-space $(X,\mathcal{T})$ with a T_0-hyperspace $(\mathcal{M},\mathcal{O}_\mathcal{T})$, consider the two-points space $X=\{0,1\}$, with $\mathcal{T}=\mathcal{M}=\{\emptyset,X\}$.

There exist non-compact spaces X such that $(\mathcal{CL}(X),\mathcal{O}_\mathcal{B})$ is a compact non-T_0-space (e.g., the space $(\mathcal{CL}(\mathbb{R}),\mathcal{O}_\mathcal{B})$, described in Example 3.19).

To get an example of a non-compact space X and a natural family $\mathcal{M}$ such that $(\mathcal{M},\mathcal{O}_\mathcal{B})$ is a compact T_0-space, consider $X:=\mathbb{R}$ with its natural topology, $\mathcal{M}:=\mathcal{F}in_2(\mathbb{R})\cup\{\mathbb{R}\}$ and take $\mathcal{B}$ as in Example 3.19.

As an example of a compact space $(X,\mathcal{T})$ with a non-compact hyperspace $(\mathcal{M},\mathcal{O}_\mathcal{T})$, consider the unit interval $X=[0,1]$ with its natural topology and put $\mathcal{M}=\{\{x\}:x\in(0,1]\}$.

The next three propositions are generalizations of, respectively, Propositions 1, 2 and 3 of [10], and have proofs similar to those given in [10]. (Let us note that in Proposition 2 of [10] the requirement "$\emptyset \notin \mathcal{C}$" has to be added.)

Proposition 2.49. *Let $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$, $w(\mathcal{M}, \mathcal{O}) = \aleph_0$, $(X, \mathcal{T}_{\mathcal{O}})$ be a T_1-space, $\mathcal{B}_{\mathcal{O}}$ be closed under countable unions and $\mathcal{M}$ contain all infinite countable closed subsets of $(X, \mathcal{T}_{\mathcal{O}})$. Then $(X, \mathcal{T}_{\mathcal{O}})$ is a compact space.*

Proposition 2.50. *Let $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$ and $\emptyset \notin \mathcal{M}$. Then $d(\mathcal{M}, \mathcal{O}) = d(X, \mathcal{T}_{\mathcal{O}})$.*

Proposition 2.51. *Let $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$ and $\emptyset \notin \mathcal{M}$. Then $(\mathcal{M}, \mathcal{O})$ has isolated points if and only if $(X, \mathcal{T}_{\mathcal{O}})$ has isolated points.*

3. On $\mathcal{O}$-commutative spaces

3.1. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Recall that A is said to be *2-combinatorially embedded in X* (see [4]) if the closures in X of any two disjoint closed in A subsets of A are disjoint.

Definition 3.2. Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$ and $\mathcal{B} \subseteq \mathcal{P}(X)$. We will say that A is *$2_{\mathcal{B}}$-combinatorially embedded in X* if for any $F \in \mathcal{CL}(A)$ and for any $U \in \mathcal{B}$ with $F \subseteq U$, there exists a $V \in \mathcal{B}$ such that $\overline{F}^X \subseteq V$ and $V \cap A \subseteq U$.

Proposition 3.3. *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then A is 2-combinatorially embedded in X if and only if A is $2_{\mathcal{T}}$-combinatorially embedded in X.*

Proof. ($\Rightarrow$) Let $H \in \mathcal{CL}(A)$, $V \in \mathcal{T}$ and $H \subseteq V$. We put $U = V \cap A$ and $F = A \setminus U$. Then F and H are two disjoint closed subsets of A. Hence, by assumption, they have disjoint closures in X, i.e. $\overline{F}^X \cap \overline{H}^X = \emptyset$. Let $W = X \setminus \overline{F}^X$. Then W is open in X, $W \cap A = U = V \cap A$ and $\overline{H}^X \subseteq W$.

($\Leftarrow$) Let F and G be two disjoint closed subsets of A. Put $V = X \setminus \overline{G}^X$. Then V is open in X and $F \subseteq V$. Hence, by assumption, there exists an open set W such that $\overline{F}^X \subseteq W$ and $W \cap A \subseteq V \cap A$. Let $U = A \setminus G$. Then $Ex_{A,X} U = V$. Hence $V \cap A = U$ and $W \subseteq V$. We conclude that $\overline{F}^X \subseteq W \subseteq V = X \setminus \overline{G}^X$, i.e. $\overline{F}^X \cap \overline{G}^X = \emptyset$. □

Remark 3.4. In Example 3.22 below we will show that there exist spaces $(X, \mathcal{T})$, subspaces A of X and bases $\mathcal{B}$ of $\mathcal{T}$ such that A is $2_{\mathcal{B}}$-combinatorially embedded in X but A is not 2-combinatorially embedded in X.

Proposition 3.5. *Let $(X, \mathcal{T})$ be a T_1-space, $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$ and $A \subseteq X$. Put $\mathcal{B}_A = \{U \cap A : U \in \mathcal{B}_{\mathcal{O}}\}$ (see Fact 2.4 for the notation $\mathcal{B}_{\mathcal{O}}$). The family $\mathcal{B}_A$ generates a topology of Tychonoff-type $\mathcal{O}_A$ on $\mathcal{CL}(A)$. The function $i_{A,X} : (\mathcal{CL}(A), \mathcal{O}_A) \to (\mathcal{CL}(X), \mathcal{O})$, defined by $i_{A,X}(F) := \overline{F}^X$, is inversely continuous (i.e. it is injective and its inverse,* *defined on $i_{A,X}(\mathcal{CL}(A))$, is a continuous function) if and only if the set A is $2_{\mathcal{B}_{\mathcal{O}}}$-combinatorially embedded in X.*

Proof. The family $\mathcal{B} := \mathcal{B}_{\mathcal{O}}$ is closed under finite intersections and $X \in \mathcal{B}$ (see Fact 2.4). Hence the family $\mathcal{B}_A$ is closed under finite intersections and $A \in \mathcal{B}_A$. Therefore, by Corollary 2.9, $\mathcal{B}_A$ generates a topology of Tychonoff-type $\mathcal{O}_A$ on $\mathcal{CL}(A)$.

The function $i_{A,X} : (\mathcal{CL}(A), \mathcal{O}_A) \to (\mathcal{CL}(X), \mathcal{O})$ is clearly injective. Denote by g its inverse defined on $i_{A,X}(\mathcal{CL}(A))$, i.e. $g : i_{A,X}(\mathcal{CL}(A)) \to \mathcal{CL}(A)$.

($\Rightarrow$) Let $H \in \mathcal{CL}(A)$, $U \in \mathcal{B}$ and $H \subseteq U$. Then $H \in (U \cap A)^+_{\mathcal{CL}(A)} \in \mathcal{O}_A$. Since $g(\overline{H}^X) = H$, the continuity of g implies that there exists a $V \in \mathcal{B}$ such that

$$\overline{H}^X \subseteq V \text{ and } g(V^+_{\mathcal{CL}(X)} \cap i_{A,X}(\mathcal{CL}(A))) \subseteq (U \cap A)^+_{\mathcal{CL}(A)}.$$

Then $V \cap A \subseteq U \cap A$. Indeed, let $x \in V \cap A$. Since X is a T_1-space, we obtain that

$$\{x\} \in V^+_{\mathcal{CL}(X)} \cap i_{A,X}(\mathcal{CL}(A)) \text{ and } g(\{x\}) = \{x\}.$$

Hence $x \in U \cap A$. So, A is $2_{\mathcal{B}}$-combinatorially embedded in X.

($\Leftarrow$) Let $F \in i_{A,X}(\mathcal{CL}(A))$ and $g(F) = H$. Then $F = \overline{H}^X$ and $H \in \mathcal{CL}(A)$. Let $U \in \mathcal{B}_A$ be such that $H \subseteq U$. Then there exists a $V \in \mathcal{B}$ with $V \cap A = U$. Hence $H \subseteq V$. Since A is $2_{\mathcal{B}}$-combinatorially embedded in X, there exists a $W \in \mathcal{B}$ such that $F = \overline{H}^X \subseteq W$ and $W \cap A \subseteq V \cap A = U$. Then $F \in W^+_{\mathcal{CL}(X)} \in \mathcal{O}$. We will show that

$$g(W^+_{\mathcal{CL}(X)} \cap i_{A,X}(\mathcal{CL}(A))) \subseteq U^+_{\mathcal{CL}(A)}.$$

Indeed, let $K \in \mathcal{CL}(A)$, $G = \overline{K}^X$ and $G \subseteq W$. Then $g(G) = K$ and

$$K = G \cap A \subseteq W \cap A \subseteq V \cap A = U,$$

i.e. $K \in U^+_{\mathcal{CL}(A)}$, as we have to show. Hence, g is a continuous function. $\square$

Corollary 3.6 ([6], Theorem 2.1). *If in Proposition 3.5 we take $\mathcal{O}$ to be the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X, \mathcal{T})$ then the function $i_{A,X}$ is inversely continuous if and only if A is 2-combinatorially embedded in X.*

Proof. It follows from Propositions 3.5, 2.20 and 3.3. $\square$

Corollary 3.7. *Let $(X, \mathcal{T})$ be a T_2-space, $A \subseteq X$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$ generated by a subfamily of $\mathcal{T}$. Let $i_{A,X}$ be inversely continuous (see Proposition 3.5 for the notation $i_{A,X}$). Assume that the following condition is satisfied:*

(*) *For any $U \in \mathcal{T}$ and for all countable $F \in \mathcal{CL}(A)$ such that $|A \setminus F| \geq \aleph_0$ and $F \subseteq U$, there exists a $V \in \mathcal{B}_{\mathcal{O}}$ with $F \subseteq V \subseteq U$.*

Then the set A is sequentially closed.

Proof. Put $\mathcal{B} := \mathcal{B}_{\mathcal{O}}$. Then, by Proposition 2.24, $\mathcal{B} \subseteq \mathcal{T}$. Assume that the set A is not sequentially closed. Then there exists a sequence $(x_n)_{n\in\omega}$ in A and a point $x \in X \setminus A$ such that $\lim_{n\to\infty} x_n = x$. Without loss of generality we can assume $x_n \neq x_m$ for all $n \neq m$.

Let us consider the sets $F = \{x_{2n} : n \in \omega\}$ and $G = \{x_{2n-1} : n \in \omega\}$. Put $U = X \setminus \overline{G}^X$. Then F is a countable closed subset of A, $|A \setminus F| \geq \aleph_0$, $F \subseteq U$ and $U \in \mathcal{T}$. By (*), there exists a $V \in \mathcal{B}$ such that $F \subseteq V \subseteq U$. Since we are assuming that the function $i_{A,X}$ is inversely continuous, we obtain, by Proposition 3.5, that the set A is $2_{\mathcal{B}}$-combinatorially embedded in X. Hence there exists a $W \in \mathcal{B}$ such that $\overline{F}^X \subseteq W$ and $W \cap A \subseteq V \cap A$. Then $x \in W$, because $x \in \overline{F}^X$. Since $W \in \mathcal{T}$ and x is a limit point of G, we have $G \cap W \neq \emptyset$. However this is a contradiction because

$$W \cap A \subseteq V \cap A \subseteq U = X \setminus \overline{G}^X,$$

and hence $G \cap W = \emptyset$. Therefore, A is sequentially closed. □

Remark 3.8. In Example 3.22 below we will show that condition (*) of Corollary 3.7 is essential, i.e., if we omit it, then the set A could fail to be sequentially closed.

Corollary 3.9 ([6], Corollary 2.3)**.** *Let $(X, \mathcal{T})$ be a T_2-space, $A \subseteq X$, $\mathcal{O}$ be the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X, \mathcal{T})$ and $i_{A,X}$ be inversely continuous (see Proposition 3.5 for the notation $i_{A,X}$). Then the set A is sequentially closed.*

Proof. We have, by Proposition 2.20, that $\mathcal{B}_{\mathcal{O}} = \mathcal{T}$. Then condition (*) of Corollary 3.7 is trivially satisfied. Hence, by Corollary 3.7, A is sequentially closed. □

Corollary 3.10. *Let $(X, \mathcal{T})$ be a sequential T_2-space, $A \subseteq X$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$ generated by a subfamily of $\mathcal{T}$. Assume that condition (*) of Corollary 3.7 is satisfied. Then the following conditions are equivalent (see Proposition 3.5 for the notation $i_{A,X}$):*

(a) *$i_{A,X}$ is a homeomorphic embedding;*
(b) *$i_{A,X}$ is inversely continuous;*
(c) *A is closed in X.*

Proof. It is clear that (a) implies (b). The implication (c)⇒(a) is true for any X, because if A is a closed subset of X then $i_{A,X}$ is the inclusion map. Let us show that (b) implies (c). By Corollary 3.7, A is sequentially closed. Since X is a sequential space, we obtain that the set A is closed. □

Corollary 3.11 ([6], Corollary 2.4)**.** *Let $(X, \mathcal{T})$ be a sequential T_2-space, $A \subseteq X$ and $\mathcal{O}$ be the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X, \mathcal{T})$. Then the following conditions are equivalent (see Proposition 3.5 for the notation $i_{A,X}$):*

(a) *$i_{A,X}$ is a homeomorphic embedding;*

(b) $i_{A,X}$ *is inversely continuous;*
(c) A *is closed in* X.

Definition 3.12. Let $(X,\mathcal{T})$ be a topological space and let $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$. The space $(X,\mathcal{T})$ is called *$\mathcal{O}$-commutative* if for any $A \subseteq X$ the function $i_{A,X}$, defined in Proposition 3.5, is a homeomorphic embedding.

When $\mathcal{O}$ is the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X,\mathcal{T})$, the notion of "$\mathcal{O}$-commutative space" coincides with the notion of "*commutative space*", introduced in [7, 6].

Corollary 3.13. *Let $(X,\mathcal{T})$ be a sequential T_2-space, $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$ generated by a subfamily of $\mathcal{T}$ and let condition (*) of Corollary 3.7 be satisfied for every subspace A of X. Then X is $\mathcal{O}$-commutative if and only if X is discrete.*

Proof. It follows from Corollary 3.10. □

Corollary 3.14 ([6], Corollary 2.5)**.** *If X is a sequential T_2-space then X is commutative if and only if X is discrete.*

Example 3.15. Let us show that there exist spaces X and topologies $\mathcal{O}$ of Tychonoff-type on $\mathcal{CL}(X)$ that are not Tychonoff topologies and that satisfy all hypothesis of Corollary 3.13.

Let $X = D(\aleph_1)$ be the discrete space of cardinality $\aleph_1$. Let

$$\mathcal{B} = \{A \subset X : |A| \leq \aleph_0\} \cup \{X\}$$

and $\mathcal{M} = \mathcal{CL}(X)$. Then $\mathcal{M}$ is a natural family on X, $\mathcal{B}$ is $\mathcal{M}$-closed, $\mathcal{B}^{\cap} = \mathcal{B}$, $X \in \mathcal{B}$ and $\mathcal{B}$ is a base for the discrete topology on X. Let $\mathcal{O}_{\mathcal{B}}$ be the topology on $\mathcal{M}$ generated by $\mathcal{B}$. Then $\mathcal{O}_{\mathcal{B}}$ is a topology of Tychonoff-type on $\mathcal{M}$, however it is not a Tychonoff topology. In fact, by Proposition 2.30, $\mathcal{B} = \mathcal{B}_{\mathcal{O}_{\mathcal{B}}}$. Hence $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} \neq \mathcal{T}_{\mathcal{O}_{\mathcal{B}}}$ and, by Corollary 2.23, $\mathcal{O}$ cannot be a Tychonoff topology. Obviously, $\mathcal{B} = \mathcal{B}_{\mathcal{O}_{\mathcal{B}}}$ satisfies condition (*) of Corollary 3.7 for any subspace A of X.

Definition 3.16. Let $(X,\mathcal{T})$ be a topological space and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$. The space $(X,\mathcal{T})$ is called *$\mathcal{O}$-HS-space* if, for any $A \subseteq X$, the function $i_{A,X}$, defined in Proposition 3.5, is continuous.

When $\mathcal{O}$ is the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X,\mathcal{T})$, the notion of "$\mathcal{O}$-HS-space" coincides with the notion of "*HS-space*", introduced in [2, 3].

Corollary 3.17. *Let $(X,\mathcal{T})$ be a T_1-space and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$. Then X is an $\mathcal{O}$-commutative space if and only if X is an $\mathcal{O}$-HS-space and every subset A of X is $2_{\mathcal{B}_{\mathcal{O}}}$-combinatorially embedded in X.*

Proof. It follows from Proposition 3.5. □

Corollary 3.18 ([6],Corollary 2.2)**.** *A T_1-space X is commutative if and only if X is an HS-space and every subspace of X is 2-combinatorially embedded in X.*

Proof. It follows from Corollary 3.17 and Proposition 3.3. □

Example 3.19. We will describe two Tychonoff-type, non Tychonoff topologies on two different subfamilies of $\mathcal{P}(\mathbb{R})$ generated by the family $\mathcal{B}$ of all open intervals of $\mathbb{R}$. One of the resulting spaces will be T_0 and the other one will not.

Let $\mathcal{T}$ be the natural topology on $X := \mathbb{R}$. Then the family $\mathcal{B}$ of all open intervals in X is a base for $\mathcal{T}$, it is closed under finite intersections and $X \in \mathcal{B}$. Put $\mathcal{M} := \mathcal{CL}(X, \mathcal{T})$ and $\mathcal{M}' := \mathcal{F}in_2(X)$. They are natural families. The family $\mathcal{B}$ is both $\mathcal{M}'$-closed and $\mathcal{M}$-closed. Indeed, let $U \subseteq X$ be $\mathcal{M}'$-covered by a subfamily $\mathcal{B}_U$ of $\mathcal{B}$. Then $U \in \mathcal{T}$ and for every $x, y \in U$ there exists an open interval $(\alpha, \beta) \in \mathcal{B}_U$ containing the points x and y. Hence U is a connected open set in $\mathbb{R}$, i.e. $U \in \mathcal{B}$. Therefore, $\mathcal{B}$ is an $\mathcal{M}'$-closed family. Since $\mathcal{M}' \subset \mathcal{M}$, we obtain, by Proposition 2.28, that $\mathcal{B}$ is an $\mathcal{M}$-closed family as well.

By Corollary 2.9, $\mathcal{B}$ generates Tychonoff-type topologies $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{M}$ and $\mathcal{O}'_{\mathcal{B}}$ on $\mathcal{M}'$. As it follows from Proposition 2.30, $\mathcal{B}_{\mathcal{O}_{\mathcal{B}}} = \mathcal{B}_{\mathcal{O}'_{\mathcal{B}}} = \mathcal{B} \neq \mathcal{T}$. Hence, by Corollary 2.23, $\mathcal{O}_{\mathcal{B}}$ and $\mathcal{O}'_{\mathcal{B}}$ are not Tychonoff topologies on $\mathcal{M}$, respectively $\mathcal{M}'$.

It is easy to see that $(\mathcal{M}', \mathcal{O}'_{\mathcal{B}})$ is a T_0-space. Indeed, let $\{x, y\}$ and $\{u, v\}$ be two distinct elements in $\mathcal{M}'$. We can assume $x < x + \varepsilon < u \leq v$ for some $\varepsilon > 0$. Consider the interval $B = (x + \varepsilon, +\infty)$. Then $\{u, v\} \in B^+_{\mathcal{M}'}$ but $\{x, y\} \notin B^+_{\mathcal{M}'}$.

Let's prove that $(\mathcal{M}, \mathcal{O}_{\mathcal{B}})$ is not a T_0-space. Put $F = \{2k : k \in \mathbb{Z}\}$ and $G = \{2k + 1 : k \in \mathbb{Z}\}$. Then $F, G \in \mathcal{M}$ and $F \neq G$ but the only neighbourhood of both F and G in $\mathcal{M}$ is $X^+_{\mathcal{M}} = \mathcal{M}$.

Example 3.20. In the notations of Example 3.19, we will show that $(\mathbb{R}, \mathcal{T})$ *is an $\mathcal{O}_{\mathcal{B}}$-HS-space.*

We are working now with the space $(\mathcal{M}, \mathcal{O}_{\mathcal{B}})$ from Example 3.19. We will write simply $\mathcal{O}$ instead of $\mathcal{O}_{\mathcal{B}}$.

Let $A \subseteq X$. We have to show that the function

$$i_{A,X} : (\mathcal{CL}(A), \mathcal{O}_A) \to (\mathcal{CL}(X), \mathcal{O}),$$

where the topology $\mathcal{O}_A$ on $\mathcal{CL}(A)$ is generated by the family

$$\mathcal{B}_A = \{A \cap U : U \in \mathcal{B}\},$$

is continuous (see Proposition 3.5 for the notation $i_{A,X}$). Let $B \in \mathcal{B}$. We will show that $i^{-1}_{A,X}(B^+_{\mathcal{M}})$ is an open set. Take an $F \in i^{-1}_{A,X}(B^+_{\mathcal{M}})$. Then $F \in \mathcal{CL}(A)$ and $\overline{F}^X \subseteq B$. There exists an $E \in \mathcal{B}$ such that

$$\overline{F}^X \subseteq E \subseteq \overline{E}^X \subseteq B$$

(this is clear if F is bounded, since in this case $\overline{F}^X$ is compact; if F is unbounded below, but is bounded above, then $B = (-\infty, \beta)$, for some $\beta \in \mathbb{R}$, and we can pick $E = (-\infty, \gamma)$ with $\sup F < \gamma < \beta$; similarly if F is unbounded above but not below; if F is unbounded both above and below then we have $B = \mathbb{R}$ and we put $E := B$). Then

$$F \in (E \cap A)^{+}_{\mathcal{C}L(A)} \subseteq i^{-1}_{A,X}(B^{+}_{\mathcal{M}}).$$

Indeed, let $G \in (E \cap A)^{+}_{\mathcal{C}L(A)}$. Then

$$\overline{G}^{X} \subseteq \overline{E}^{X} \subseteq B,$$

i.e. $i_{A,X}(G) \in B^{+}_{\mathcal{M}}$.

Remark 3.21. Let us note that a similar proof shows that *every subspace Y of $(\mathbb{R}, \mathcal{T})$ is an $\mathcal{O}_{\mathcal{B}_Y}$-HS-space* (see Examples 3.19 and 3.20 for the notations).

More generally, let Y be a topological space and $\mathcal{D}$ be a base of Y. We will say that Y is *$\mathcal{D}$-normal* if for every $F \in \mathcal{C}L(Y)$ and for every $D \in \mathcal{D}$ such that $F \subseteq D$ there exists an $E \in \mathcal{D}$ with $F \subseteq E \subseteq \overline{E}^{Y} \subseteq D$. Now, arguing as in Example 3.20, we can prove that *if Y is a $\mathcal{D}$-normal space, $\mathcal{D} = \mathcal{D}^{\cap}$ and $Y \in \mathcal{D}$, then Y is an $\mathcal{O}$-HS-space, where $\mathcal{O}$ is the Tychonoff-type topology on $\mathcal{C}L(Y)$ generated by $\mathcal{D}$.* This generalizes the result of M. Sekanina [15] that any normal space is a HS-space.

Example 3.22. In the notations of Examples 3.19 and 3.20, we will show that *the function $i_{A,\mathbb{R}}$ is a homeomorphic embedding for any open interval A.*

We will argue for $A = (0, 1)$; the proof for any other open interval is similar. We know, by Example 3.20, that the function $i_{A,X}$ is continuous. Therefore we only need to prove that $i_{A,X}$ is inversely continuous. By Proposition 3.5, it is enough to show that the set A is $2_{\mathcal{B}}$-combinatorially embedded in X. So, let H be a closed subset of $(0, 1)$ and let $B = (\alpha, \beta) \in \mathcal{B}$ be such that $H \subseteq B$. We have to find a $D \in \mathcal{B}$ such that $\overline{H}^{X} \subset D$ and $D \cap (0, 1) \subseteq B$. Clearly, $\overline{H}^{X} \subseteq [0, 1]$. If $\overline{H}^{X} \subset (0, 1)$, we can take $D = B$. If $0 \in \overline{H}^{X}$ but $1 \notin \overline{H}^{X}$ then $\alpha \leq 0$ and we can put $D = (-1, \beta)$. If $1 \in \overline{H}^{X}$ but $0 \notin \overline{H}^{X}$ then $\beta \geq 1$ and we can put $D = (\alpha, 2)$. If $0, 1 \in \overline{H}^{X}$ then $\alpha \leq 0$ and $\beta \geq 1$ and we put $D = (-1, 2)$. Therefore, *A is $2_{\mathcal{B}}$-combinatorially embedded in $(\mathbb{R}, \mathcal{T})$.*

Note that *A is not 2-combinatorially embedded in $(\mathbb{R}, \mathcal{T})$.*

Observe that the triple $((\mathbb{R}, \mathcal{T}), A, \mathcal{O})$ satisfies all hypothesis of Corollary 3.7 except for condition (*), but A is not sequentially closed.

Example 3.23. Let $Y \subseteq \mathbb{R}$. We will say, as usual, that a point $x \in Y$ is *isolated from the right (left) (in Y)* if there exists an $\varepsilon > 0$ such that if we put $U = (x, x + \varepsilon)$ $(U = (x - \varepsilon, x))$ then $U \cap Y = \emptyset$. Now, in the notations of Examples 3.19 and 3.20, we have: *a subspace Y of $(\mathbb{R}, \mathcal{T})$ is $\mathcal{O}_{\mathcal{B}_Y}$-commutative if and only if every point of Y is either isolated from the right or from the left.*

We first show that a space Y that has a point y_0 which is non-isolated both from the left and from the right cannot be $\mathcal{O}_{\mathcal{B}_Y}$-commutative. Indeed, put

$A = Y \setminus \{y_0\}$. We will prove that A is not $2_{\mathcal{B}_Y}$-combinatorially embedded in Y. By Proposition 3.5, this will imply that the function $i_{A,Y}$ is not inversely continuous and hence the space Y will be not $\mathcal{O}_{\mathcal{B}_Y}$-commutative. Let $H = \{y_n : n \in \omega\}$ be a decreasing sequence in Y converging to y_0. Then H is a closed subset of A and $H \subset (y_0, +\infty) \cap Y$. Suppose that there exists $B \in \mathcal{B}$ such that $cl_Y H \subseteq B$ and $B \cap A \subseteq (y_0, +\infty)$. Since $y_0 \in cl_Y H \subseteq B$ and y_0 is not isolated from the left, we have that $(B \cap A) \setminus (y_0, +\infty) \neq \emptyset$, which is a contradiction. Hence, A is not $2_{\mathcal{B}_Y}$-combinatorially embedded in Y.

Now we will show that a space Y having only points which are isolated either from the left or from the right is $\mathcal{O}_{\mathcal{B}_Y}$-commutative. Let $A \subset Y$. We know, by Remark 3.21, that the function $i_{A,Y}$ is continuous. Hence it is enough to show that it is inversely continuous, i.e., according to Proposition 3.5, that A is $2_{\mathcal{B}_Y}$-combinatorially embedded in Y. So, let $H \in \mathcal{CL}(A)$ and let $H \subseteq B \cap Y$ for some $B = (\alpha, \beta)$. We have to find a $D \in \mathcal{B}$ such that $\overline{H}^Y \subset D \cap Y$ and $D \cap A \subseteq B$. We have $cl_Y H \subseteq \overline{B}^X = [\alpha, \beta]$. If $cl_Y H \subseteq B$, we can take $D = B$ and we are done. If $\alpha \in cl_Y H$ and $\beta \notin cl_Y H$ then $\alpha \in Y$ and α is not isolated from the right, being a limit point of H. Hence, by the assumption, α is isolated from the left. Thus there exists a $\gamma < \alpha$ such that $(\gamma, \alpha) \cap Y = \emptyset$. Then $D = (\gamma, \beta)$ is the required interval. The other two possible cases are treated analogously.

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Department of Mathematics, University of Sofia, Blvd. J. Bourchier 5, 1126 Sofia, Bulgaria

E-mail address: gdimov@fmi.uni-sofia.bg

Department of Mathematical Sciences and Computer Science, University of Trieste, Via A. Valerio 12/1, 34127 Trieste, Italy

E-mail address: obersnel@mathsun1.univ.trieste.it

E-mail address: tironi@univ.trieste.it