THE NIST DIGITAL LIBRARY OF MATHEMATICAL FUNCTIONS PROJECT

DANIEL W. LOZIER

Abstract. The NIST DLMF Project, begun in 1997, is preparing a handbook and Web site to provide useful data about special functions to a wide audience. These data include mathematical formulas, graphs, references, methods of computation, and links to software. Special features of the Web site include 3D interactive graphics (scientific visualizations) and an equation search capability. More than 50 mathematicians, physicists and computer scientists from around the world are participating in the work. The handbook and Web site are scheduled for completion in 2003. The tools that are being used are, of necessity, ones that are widely available now, even though better tools are in active development. For example, LaTeX computer files are the common source for both the handbook and the Web site. This is the technology of choice for presentation of mathematics in print but it is not suited to equation search, for example, or for input to computer algebra systems. These and other problems, and some partially successful work-arounds, will be discussed in this paper.

1. Introduction

Scientific and engineering professionals who are are conducting research or developing applications frequently need ready access to specific technical data. Traditionally, applicable technical data has been gathered into handbooks, and often these publications become indispensable aids that persist through many editions. Accuracy—freedom from errors—is critical for a handbook to be successful, and also relevance to a significant community of users. Successful mathematics handbooks that come to mind are certain tables of integrals—and the ubiquitous *NBS Handbook of Mathematical Functions* [1].

The *NBS Handbook* is the motivation and model for the *NIST Digital Library of Mathematical Functions*. The *Handbook* was published in 1964, and except for correction of errata through the 10th printing in 1972 it has never been revised. The *DLMF* is currently under development in a large project headed by NIST. It will be published as traditional handbook of about 1000 pages, but there will also be a hypertext version that is to be made available on CD-ROM and from a public Web site at NIST.

The *Handbook* and *DLMF* have many features in common, of which two will be mentioned here. First, subject matter. The general goal of each reference work is to provide carefully selected and validated technical data about the elementary

Date: August 30, 2001.
Research was supported in part by NSF Grant 9980036.
This is a preliminary version of this article, prepared for the First International Workshop on Mathematical Knowledge Management, RISC, A-4232 Schloss Hagenberg, Austria, September 24–26, 2001.

and higher mathematical functions[1]. The main criterion for inclusion of data is its proven or potential use in disciplines outside mathematics. Information of interest only within pure mathematics is excluded. The majority of the chapters deal with individual groups of functions that share a common characteristic such as the property of being a solution of a given differential equation. The remaining chapters are methodological, providing concentrated accounts of mathematical techniques that are indispensable for working with special functions.

The second feature in common is intended audience. The level and presentation of the technical data is directed toward experienced professionals. Thus extended pedagogical descriptions are excluded so that the space saved can be used for additional technical data. It might be thought that space is not an issue for the DLMF hypertext version. However, there is benefit in requiring the authors of the chapters to consider carefully which data are paramount and to include only the most important data. Also, at least for the first edition, the project aims for maximum overlap between the typeset and hypertext versions. Offsetting this requirement to some extent, DLMF authors are allowed considerable freedom to point DLMF users toward references where additional data can be found.

Despite these similarities, the hypertext version of the *DLMF* is fundamentally different from the old *Handbook* because it comes with hyperlinks, interactive graphics, and tools for downloading and searching. An online record of feedback from users is planned, and tools for generating tables and graphs on demand may be added. Prototypes for all these capabilities exist now.

The print and hypertext versions of the *DLMF* issue from the same source, which is a *mathematical database* consisting (at present) of LaTeX computer files. Together with a special LaTeX class and software tools developed at NIST, the database preserves limited mathematical semantics of the objects it contains; see the paper presented at this conference by Miller and Youssef. Managing the mathematical knowledge in this database so that it will remain accessible and usable into the indefinite future is a serious concern for the DLMF project. The data is being collected at great financial cost. Though it will be preserved in conventional form as a published handbook, it would be very unfortunate if the database were lost due to future software and hardware incompatibilities.

2. 1964 NBS Handbook

The handbook [1] quickly became the reference of choice for applications of special functions, and it remains so today. This success was due to (i) its concise style that provided users with concentrated and relevant information, (ii) its orientation toward the large community of users in disciplines, such as the physical sciences and engineering, in which mathematics is an essential tool, and (iii) the care that was taken to engage the best available authors and editors.

It was conceived as early as 1952, active work was begun in 1956, and the technical parts were largely complete by 1960. The delay in publication to 1964 was largely due to the death in 1958 of its chief architect, Milton Abramowitz. It supplies the most important data relating to special functions (formulas, tables for computing by interpolation, and graphs) as determined by the needs of its time. Included are whole chapters on Elementary transcendental functions; Exponential integral

[1]The higher mathematical functions are also known as special functions. Examples are Bessel, Legendre and elliptic functions. The term higher transcendental functions has also been used.

and related functions; Gamma function and related functions; Error function and Fresnel integrals; Legendre functions; Bessel functions of integer order; Bessel functions of fractional order; Integrals of Bessel functions; Struve functions and related functions; Confluent hypergeometric functions; Coulomb wave functions; Hypergeometric functions; Jacobian elliptic functions and theta functions; Elliptic integrals; Weierstrass elliptic and related functions; Parabolic cylinder functions; Mathieu functions; Spheroidal wave functions; Orthogonal polynomials; Bernoulli and Euler polynomials and Riemann zeta function; Combinatorial functions; Probability functions; and Miscellaneous functions. Two additional chapters of a more methodological nature are Elementary analytical methods; and Numerical interpolation, differentiation and integration.

The success of the handbook is exhibited by its sales and citation record. The official government edition has been in print continuously and has sold more than 150,000 copies. Identical low-price printings in paperback have been marketed since 1965 by Dover Publications; we estimate these sales are 4 to 5 times greater than the government sales. A study of *Science Citation Index* shows the handbook was cited over 7000 times between 1992 and 1996, or approximately once every 1.5 hours of every working day. Furthermore, the citation rate was increasing even more rapidly than the *SCI* as a whole. No primarily mathematical journal ranked in the top 50 in frequency of citations, a group that includes journals in chemistry, earth sciences, electrical engineering, environmental sciences, optics, physics and statistics.

3. DLMF

3.1. **Motivation and Goals.** The continued appeal of the old handbook is due to the permanence of its collection of formulas, which take up less than half of its 1045 pages. Mathematical formulas that are found to be useful in applications hold their value for a long time, even forever. However, this collection is showing its age in some respects. For example, since 1960

- new properties have been found for many of the functions, such as integral representations, integrals, addition formulas and generating functions;
- additional functions have gained practical importance, such as Carlson's elliptic integrals, discrete orthogonal polynomials, new statistical distribution functions, Painlevé transcendents and basic hypergeometric functions;
- new fields of application of special functions have appeared, such as soliton theory and nonlinear dynamics;
- developments in asymptotic analysis have led to improved analytical approximation of special functions, for example through use of uniform asymptotics, asymptotics via distribution theory, and reexpansion of remainder terms.

Because of these developments there is an increasingly critical need for a major expansion of the old handbook.

The other half of the old handbook is irrelevant today. This part is devoted largely to massive tables of values and descriptions of how to calculate values of special functions, often with the aid of interpolation in the tables. The old method of interpolation in tables has been superseded by newer, more effective methods that depend upon the speed with which computers can perform arithmetic. In effect, a data-intensive and operation-conserving method is replaced by a data-conserving and operation-intensive one. Developments since 1960 that have contributed to this obsolescence include

- significant improvements in methods for accurate and efficient computation of special functions that have been obtained through advances in stability analysis, Padé approximation theory, boundary-value techniques for differential and difference equations, and application of the new developments in asymptotics;
- the rapid development of computing technology, both in hardware and in software;
- the emergence of powerful scientific software, such as subroutine libraries and computer algebra systems, which enable users to generate sophisticated symbolic and numerical solutions for a wide range of problems involving special functions.

Accordingly, a new reference work should eliminate tables and must pay attention to software.

The main goal of the DLMF project is to meet the need for an up-to-date replacement for the old handbook. The second goal is to develop foundations for constructing mathematical databases in general, together with tools for their convenient and effective use in computers and over computer networks. The combination of a database and associated tools is what we mean by a *digital library.* The experience gained in the construction of a nontrivial digital library in an important area of mathematics will be of great value in demonstrating feasibility, comparing alternative technologies, exposing weaknesses of current technologies, and suggesting avenues for future research. Also, because of the expected long life and wide community of users, the DLMF will advance the penetration of digital library technology into the customary practice of scientific researchers and technical professionals in all science-based fields of work.

3.2. **Beginnings.** The first discussions took place within NIST after October 1996, leading to a conference presentation [2, 3] and a workshop at NIST with invited external participants in summer 1997. The workshop resulted in the organizational structure for the project and an initial list of chapters with a brief synopsis for each chapter.

A difficult question at this early stage was how such a large project could be financed. NIST was making a large commitment but not enough to complete the project without obtaining additional funds. In 1998 a sample chapter was written [6], a prototype Web site[2] was constructed, a proposal to the National Science Foundation was submitted, and a contingency plan involving royalties on sales was prepared. The proposal failed but by that time NIST had named a board of associate editors and obtained tentative agreement from well-known researchers to write about 1/2 the chapters, if funding arrangements could be worked out. Two conference papers describing the project at this stage in more detail are [4] and [5].

In 1999 a second proposal to the National Science Foundation received funding at the level requested for three years starting in September of that year.

3.3. **Organizational Structure.** The structure identified in 1997 is in use today. There are five classes of project contributors: principal editors, associate editors,

[2] http://dlmf.nist.gov

authors, validators, and NIST staff. The principal editors and areas of responsibility are

- Dr. D. W. Lozier, NIST, General Editor;
- Prof. F. W. J. Olver, University of Maryland and NIST, Mathematics Editor;
- Dr. C. W. Clark, NIST, Physical Sciences editor;
- Dr. R. F. Boisvert, NIST, Information Technology Editor.

They form the executive committee, directing and bearing responsibility for the whole project.

The (unpaid) associate editors and areas of responsibility are

- Prof. R. A. Askey, University of Wisconsin, Special Functions;
- Prof. Sir M. V. Berry, University of Bristol, Physics;
- Prof. W. Gautschi, Purdue University, Special Functions;
- Prof. L. C. Maximon, George Washington University, Physics;
- Prof. M. Newman, University of California, Santa Barbara, Combinatorics and Number Theory;
- Prof. I. Olkin, Stanford University, Statistics;
- Prof. Dr. P. Paule, Johannes Kepler University, Symbolic Computing;
- Prof. W. P. Reinhardt, University of Washington, Chemistry;
- Dr. N. M. Temme, CWI Amsterdam, Special Functions.

They assist the principal editors in choice of technical content, selection of authors and validators, resolution of technical questions, and review of chapter drafts.

The authors and validators are drawn from around the world based on their research accomplishments. They are compensated by contract with NIST. The author's contract has two parts, the first requiring delivery of an initial draft that meets the strict editorial guidelines of the project and the second requiring coordination with all the project contributors in completing the DLMF Web site. The validator's contracts, one for each chapter, require verification of all formulas and other mathematical material.

Finally, the NIST staff consists of approximately 10 individuals, including

- Dr. B. R. Miller, Information Architecture;
- Dr. B. V. Saunders, Graphics;
- Prof. A. Youssef, Search Engine.

Prof. Youssef is on the faculty at George Washington University and is a guest researcher at NIST.

3.4. **Time Frame and Current Status.** The DLMF book and Web site are being developed simultaneously. The schedule calls for the book to be in print in 2003. The public announcement of the DLMF Web site will come soon after. Preliminary releases with controlled access will begin in 2002. NIST is constructing the Web site, which will be maintained indefinitely, and will establish a contract for publishing the book. The book will be priced low enough for individual purchasers, and the Web site will be free.

There will be almost 40 chapters in the DLMF. Currently, contracts are in effect for all of them. Initial drafts have been received for about 2/3. Of these about half have received thorough editorial review and their authors notified that the first part of the contract has been met. Some initial drafts have serious defects, which the editors are working with the authors to resolve.

4. DLMF Chapters

In this section we describe the two types of chapters, function chapters and methodology chapters, illustrating typical content with the title pages in the printed handbook format. The special features of the hypertext version are discussed in sections 5–7.

4.1. **Function Chapters.** See Fig. 1.

```
* Front matter
  - title and author
  - gallery
  - contents
  - acknowledgements
  - copyright
* Parts
  - notation
    . establish notation for the chapter
    . supply exact relation to other notations
    . no definitions in this part
    . notations common to all chapters gathered elsewhere
  - math properties
    . formulas
    . graphs
  - applications
  - computations
  - references
```

4.2. **Methodology Chapters.** See Fig. 2.

5. Metadata

In this section metadata and electronic-only material are discussed and illustrateds by figures. See Figs. 3–6. (The figures will be made into a collage, if feasible).

```
* Clickable metadata (author's notes)
 - can be associated with every chapter, section,
subsection and equation, but always at the highest level and
only rarely at the equation level
 - clickable means a web link is visible on the web site for
the purpose of producing the metadata on the screen
* Unclickable metadata
 - used to construct indexes, cross-reference tables and search engine
* Author's Notes
  - reference for proof of every property
  - acknowledge source of copyrighted material
  - can be supplied for every numbered entity
  - should be supplied at every section and subsection
  - should only rarely be supplied at equations
```

6. Graphs

See Fig. 7.

7. Search Engine

In this sections figures illustrating use of the search engine to find equations in the DLMF will be discussed.

8. Conclusions

References

1. M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55, U. S. Government Printing Office, Washington, D. C., 1964.
2. Barry A. Cipra, *A new testament for special functions?*, SIAM News **31** (1998), no. 2, http://www.siam.org/siamnews/03-98/function.htm.
3. D. W. Lozier, *Toward a Revised NBS Handbook of Mathematical Functions*, NISTIR 6072, National Institute of Standards and Technology, 1997, download from http://dlmf.nist.gov/about/publications/.
4. D. W. Lozier, B. R. Miller, and B. V. Saunders, *Design of a digital mathematical library for science, technology and education*, Proceedings of the IEEE Forum on Research and Technology Advances in Digital Libraries, May 19–21, 1999, Baltimore, Maryland, IEEE Computer Society, Los Alamitos, California, 1999, pp. 118–128.
5. Daniel W. Lozier, *The DLMF project: A new initiative in classical special functions*, Special Functions (Charles Dunkl, Mourad Ismail, and Roderick Wong, eds.), World Scientific, Singapore, 2000, pp. 207–220.
6. F. W. J. Olver, *Airy Functions, Chapter AI, Digital Library of Mathematical Functions Project*, National Institute of Standards and Technology, Gaithersburg, MD 20899-8910, U. S. A., 1998, http://dlmf.nist.gov.

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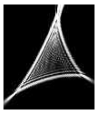
Chapter AI:

Airy and Related Functions

Frank W. J. Olver[1]

August 30, 2001 16:52

Gallery



Contents

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Acknowledgments: This chapter is based in part on Abramowitz and Stegun (1964, Chapter 10) by H. A. Antosiewicz. The graphs and visualizations in §AI.3 were constructed by R. R. Lipman, D. W. Lozier, B. V. Saunders, Q. Wang with the aid of the software package of Amos (1986). In §AI.9, B. R. Fabijonas supplied higher terms in some of the asymptotic expansions and computed the numerical tables. D. W. Lozier assisted with the compilation of §AI.21. The Web site was prepared by R. F. Boisvert and B. R. Miller.

FIGURE 1. Title page of DLMF chapter on Airy and Related Functions, hardcopy version. ©National Institute of Standards and Technology.

Chapter AS:

Asymptotic Approximations

Frank W. J. Olver and Roderick Wong

August 30, 2001 17:04

Contents

FIGURE 2. Title page of DLMF chapter on Asymptotic Approximations, hardcopy version. ©National Institute of Standards and Technology.

AI Airy and Related Functions
by Frank W. J. Olver
Mathematical Properties
§AI.2 Differential equation
§AI.2(i) Airy's Equation

Search Index Contents

National Institute of Standards and Technology

§AI.2 Differential equation §AI.2(ii) Initial Values

AI.2(i) Airy's Equation *About §AI.2(i)*

AI.2.1
$$\frac{d^2w}{dz^2} = zw.$$

Standard solutions are

AI.2.2
$$w = \mathrm{Ai}(z), \quad \mathrm{Bi}(z), \quad \mathrm{Ai}(ze^{-2\pi i/3}), \quad \mathrm{Ai}(ze^{2\pi i/3}).$$

All solutions are entire functions of z.

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FIGURE 3. Subsection of DLMF Chapter AI dealing withe Airy's differential equation and its standard solutions, hypertext version. ©National Institute of Standards and Technology.

Search Index Contents

AI Airy and Related Functions
by Frank W. J. Olver
Mathematical Properties
§AI.2 Differential equation
§AI.2(i) Airy's Equation
About §AI.2(i) Airy's Equation

About §AI.2(i) Airy's Equation

Also see *About Mathematical Properties* and *About AI*.

Notes

- See Miller (1946) and Olver (1974, pp. 392-393, 413-414).

Internal Metadata

Label

- `sec:AI.DE.AE`

Indexing Data

- Airy functions > differential equation
- Differential equation > for Airy functions

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Figure 4. Metadata for subsection AI.2(i). ©National Institute of Standards and Technology.

Search Index Contents

AI Airy and Related Functions
by Frank W. J. Olver
Mathematical Properties
§AI.6 Relations to Other Functions
§AI.6(iii) Airy Functions as Confluent Hypergeometric Functions

National Institute of Standards and Technology

§AI.6(ii) Bessel Functions as §AI.7 Asymptotic Expansions

AI.6(iii) Airy Functions as Confluent Hypergeometric Functions

About §AI.6(iii)

AI.6.21 $$\zeta = \tfrac{2}{3} z^{3/2}.$$

AI.6.22 $$\mathrm{Ai}(z) = \tfrac{1}{2}\pi^{-1/2} z^{-1/4} W_{0,1/3}(2\zeta) = 3^{-1/6}\pi^{-1/2}\zeta^{2/3}e^{-\zeta}U(\tfrac{5}{6},\tfrac{5}{3},2\zeta),$$

AI.6.23 $$\mathrm{Ai}'(z) = -\tfrac{1}{2}\pi^{-1/2} z^{1/4} W_{0,2/3}(2\zeta) = -3^{1/6}\pi^{-1/2}\zeta^{4/3}e^{-\zeta}U(\tfrac{7}{6},\tfrac{7}{3},2\zeta),$$

AI.6.24 $$\mathrm{Bi}(z) = \frac{1}{2^{1/3}\Gamma(\frac{2}{3})} z^{-1/4} M_{0,-1/3}(2\zeta) + \frac{3}{2^{5/3}\Gamma(\frac{1}{3})} z^{-1/4} M_{0,1/3}(2\zeta),$$

AI.6.25 $$\mathrm{Bi}'(z) = \frac{2^{1/3}}{\Gamma(\frac{1}{3})} z^{1/4} M_{0,-2/3}(2\zeta) + \frac{3}{2^{10/3}\Gamma(\frac{2}{3})} z^{1/4} M_{0,2/3}(2\zeta),$$

AI.6.26 $$\mathrm{Bi}(z) = \frac{1}{3^{1/6}\Gamma(\frac{2}{3})} e^{-\zeta}{}_1F_1(\tfrac{1}{6};\tfrac{1}{3};2\zeta) + \frac{3^{5/6}}{2^{2/3}\Gamma(\frac{1}{3})}\zeta^{2/3}e^{-\zeta}{}_1F_1(\tfrac{5}{6};\tfrac{5}{3};2\zeta),$$

AI.6.27 $$\mathrm{Bi}'(z) = \frac{3^{1/6}}{\Gamma(\frac{1}{3})} e^{-\zeta}{}_1F_1(-\tfrac{1}{6};-\tfrac{1}{3};2\zeta) + \frac{3^{7/6}}{2^{7/3}\Gamma(\frac{2}{3})}\zeta^{4/3}e^{-\zeta}{}_1F_1(\tfrac{7}{6};\tfrac{7}{3};\zeta).$$

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Figure 5. Subsection of DLMF Chapter AI supplying Airy functions in terms of confluent hypergeometric functions. ©National Institute of Standards and Technology.

Search Index Contents

AI Airy and Related Functions
by Frank W. J. Olver
Mathematical Properties
§AI.6 Relations to Other Functions
§AI.6(iii) Airy Functions as Confluent Hypergeometric Functions
About §AI.6(iii) Airy Functions as Confluent Hypergeometric Functions

About §AI.6(iii) Airy Functions as Confluent Hypergeometric Functions

Also see *About Mathematical Properties* and *About AI*.

Notes

- See (AI.6.21) for definition of ζ.
- See Chapter CH (Confluent Hypergeometric Functions) for Hypergeometric functions $W_{\kappa,\mu}$, U, $M_{\kappa,\mu}$ and ${}_pF_q$.
- These formulas are derivable from (AI.6.1)-(AI.6.5) and those given in CH (Confluent Hypergeometric Functions).

Internal Metadata

Label

- `sec:AI.RL.HY`

Indexing Data

- Airy functions > relation to > confluent hypergeometric functions
- Confluent hypergeometric functions > relation to > Airy functions

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Figure 6. Metadata for subsection AI.6(iii). ©National Institute of Standards and Technology.

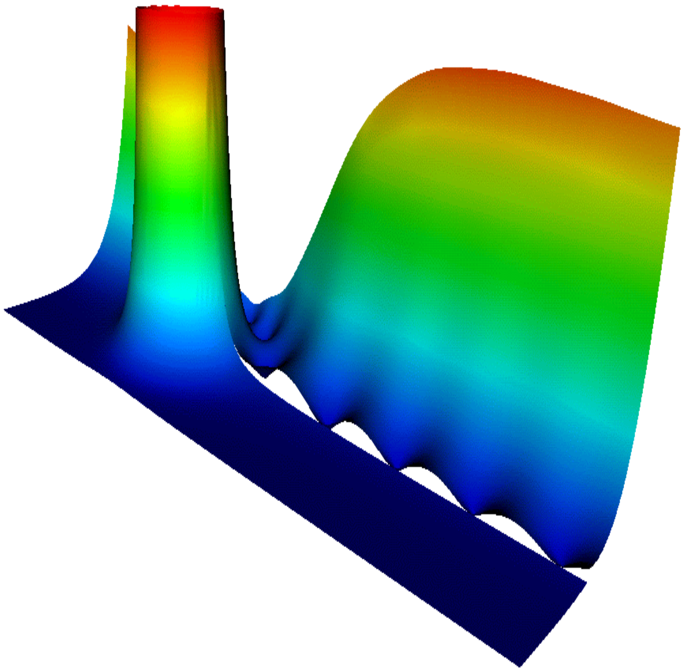

FIGURE 7. View of principal branch of the Hankel function $|H_5^{(1)}(x+iy)|$ showing pole at the origin, branch cut, location of zeroes near the cut, and exponential growth and decay in different parts of the complex plane. Five zeros around the pole are not fully visible in this view. In the DLMF, this view may be rotated and seen from any angle. The same technology can be used to generate views of additional branches. ©National Institute of Standards and Technology.