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ABOUT BAND LIMITED UP TO CONGRUENCES WAVELETS

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Dedicated to the memory of Chicho

Abstract. If ψ is a BLc wavelet, a Chui-Shi family of redundant tight frames $\{\Psi(k); k \in \mathbb{Z}\}$ can be built. Its redundancy is studied through the subspaces of coefficients in $l^2(\mathbb{Z}^2)$. It will be proved that these subspaces converge to zero when ψ is a uniformly continuous wavelet. Also a Whittaker-Shannon-Kotel'nikov type sampling theorem will be given for discrete vector functions associated to $\Psi(k)$. Even more, it will be proved that does not exist a Shannon type sampling result for modifications in the scaling parameter.

1. Introduction

In many situations it is much more useful to manage a frame than an orthonormal basis, mainly when the application needs to reconstruct signals avoiding noise brought by perturbation in coefficients of the signal in an analysing system.

The redundancy of frames gets to reduce this noise, as can be seen in [M]; but the overcompleteness also brings the growing of computations needed for both analysis and synthesis. For these reasons it turns to be really interesting to construct redundant systems from a complete exact one, such as a Riesz basis, through a perturbation.

Among the different perturbations of a frame system in a Hilbert space that we can find (see [Ch], [CaCh] and [ChChp]) we will deal on the Charles K. Chui and Xianliang Shi one (see [ChuS1]), written as:

$$\Psi(k) = \{\psi^k_{j,n}(t) = 2^{\frac{j}{2}}\psi(2^j t - \tfrac{n}{k}); \, j, n \in \mathbb{Z}\}$$

for any $0 \neq k \in \mathbb{Z}$.

In [ChuS1] and [ChuS2], the authors prove that

Teorema 1.1. *Let ψ be a wavelet such that $\Psi(1)$ is an A-B-frame. Then $\Psi(k)$ is also a frame with admisible bounds kA and kB for any odd k.*

This result is not true in general if k is even: take ψ the Haar wavelet. Clearly $\Psi(2)$ is not tight although $\Psi(1)$ is an orthonormal basis.

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In [C] is given a characterization (extension of a result of Wang [W] and Gripenberg [G] as appears in [HWe]) of the wavelets that make $\Psi(k)$ to be a k-tight frame, and then a characterization of the wavelets that maintain the relation between the frame bounds (tightness for an orthonormal basis) of the system through a Chui-Shi perturbation.

Teorema 1.2. *It will be said that a function* $f \in L^2(\mathbb{R})$ *is band limited up to congruences, with band width* B *(we will write* $f \in B$*-BLc) if there exist bounded intervals* $\{I_k, k \in \mathbb{Z}\}$ *such that* $J_k = I_k - kB$ *are disjoint,* $J_k \subset [-B, B]$ *and* $\operatorname{supp} \hat{f} \subset \cup_{k \in K} I_k$.

Denoting with $\mathcal{F}_k$ the set of wavelets $\psi \in L^2(\mathbb{R})$ such that $\Psi(k)$ is a k-tight frame in [C] is got the following characterization result:

Teorema 1.3. *Let* $\psi \in \mathcal{F}_1$*. The following are equivalent:*

(a) $\psi \in \bigcap_{n \in \mathbb{N}} \mathcal{F}_{2n}$,

(b) $\psi \in \bigcap_{r \in \mathbb{N}} \mathcal{F}_{2^r}$,

(c) ψ *is* π*-BLc.*

And so we have that

Corollary 1.4. *A wavelet* $\psi \in \mathcal{F}_1$ *is in* $\mathcal{F}_n$ *for any* $n \in \mathbb{N}$ *if and only if it is* π*-BLc.*

From now on we will only consider ψ a BLc wavelet that generates an orthonormal basis by the action of the diadic affine group, and so the family $\Psi(k)$ will be a tight frame in $L^2(\mathbb{R})$ for any $0 \neq k \in \mathbb{Z}$.

In Section 2 we will study the redundancy of the frames of the families $\{\Psi(k); k \in \mathbb{N}\}$ through the subspaces of sequences of $l^2(\mathbb{Z}^2)$ that are the coefficients in the frame of the space of functions $L^2(\mathbb{R})$, namely

$$S(\Psi(k)) = S_k = \left\{ \left(\langle f, \psi^k_{j,n} \rangle \right)_{j,n \in \mathbb{Z}}; \; f \in L^2(\mathbb{R}) \right\}.$$

Clearly $S_1 = l^2(\mathbb{Z}^2)$, and for any other $k \neq 1$ we just know, when ψ is band-limited up to congruences, that $S_k \subset l^2(\mathbb{Z}^2)$. The following section will prove that $\cap_{k \in \mathbb{N}} S_k = \{0\}$ when ψ is in $L^1(\mathbb{R})$, even more, $\lim_{k \to \infty} S_k = \{0\}$.

In Section 3 we will study another important feature of redundant systems: the relation between the coefficients in such a system. It will be proved that they verify a Shannon's sampling type theorem. In Section 4 we will introduce a different perturbation and will see that is also maintains the relation between the frame bounds, but the redundancy got does not provide a Shannon type formula for the new frame coefficients.

2. REDUNDANCY

Lemma 2.1. *Let* $\psi \in L^1(\mathbb{R})$ *be a uniformly continuous wavelet. Then, for any* $j, p, l, h \in \mathbb{Z}$,

$$\langle \psi(2^j \cdot - h), \psi(2^p \cdot - \tfrac{l}{k}) \rangle \xrightarrow{k \to \infty} \langle \psi(2^j \cdot - h), \psi(2^p \cdot) \rangle = 2^{-p} \delta_h \delta_{j,p}.$$

Proof. The Theorem of continuity of functions defined by means of a parametric integral will be applied to

$$f(x,y) = \psi(2^j x - h)\,\overline{\psi(2^p x - y)}$$

for $x \in \mathbb{R}$, $y \in [0,\infty)$. It is a continuous function as ψ does.

As ψ is a uniformly continuous wavelet, given $\varepsilon = 1$ there exists $\delta > 0$ such that for any $x \in \mathbb{R}$ and $|z| < \delta$, $|\psi(x+z) - \psi(x)| < 1$.

Take $y_0 \in [0,\infty)$ and $V = (y_0 - \gamma, y_0 + \gamma)$ with $\gamma \leq \min\{\delta, y_0\}$ if $y_0 \neq 0$, and $V = [0,\gamma)$ with $\gamma \leq \delta$ if $y_0 = 0$.

Let's see that there exists an integrable function g_{y_0} such that $|f(x,y)| \leq g_{y_0}(x)$ for any $x \in \mathbb{R}$, $y \in V$:

$$\begin{aligned}
|f(x,y)| &= |\psi(2^j x - h)\,\psi(2^p x - y)| \\
&= |\psi(2^j x - h)|\,|\psi(2^p x - y) - \psi(2^p x - y_0) + \psi(2^p x - y_0)| \\
&\leq |\psi(2^j x - h)|\,\{|\psi(2^p x - y_0 + (y_0 - y)) - \psi(2^p x - y_0)| + |\psi(2^p x - y_0)|\} \\
&\leq |\psi(2^j x - h)| + |\psi(2^j x - h)|\,|\psi(2^p x - y_0)|.
\end{aligned}$$

Take $g_{y_0}(x) = |\psi(2^j x - h)| + |\psi(2^j x - h)||\psi(2^p x - y_0)|$, which is integrable on $\mathbb{R}$ as $\psi \in L^1(\mathbb{R})$, and verifies that $|f(x,y)| \leq g_{y_0}(x)$ for any $x \in \mathbb{R}$, $y \in V$. So

$$F(y) = \int_{\mathbb{R}} f_y(x)\,dx = \int_{\mathbb{R}} \psi(2^j x - h)\,\overline{\psi(2^p x - y)}\,dx$$

is continuous in $[0,\infty)$; in particular, taking $y = \frac{l}{k}$ and making $k \to \infty$ we have the lemma. □

Definition 2.2. *A sequence of sets* $\{\Gamma_n\}_{n\in\mathbb{N}}$ *is convergent to* Γ *if* $\overline{\lim}\,\Gamma_n = \underline{\lim}\,\Gamma_n = \Gamma$*, where*

$$\overline{\lim}\,\Gamma_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \Gamma_n \qquad \textit{and} \qquad \underline{\lim}\,\Gamma_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \Gamma_n.$$

For more details see [Mu] for e.g.

Let us consider the subspaces

$$S_k = \{\{\langle f, \psi^k_{j,n\in\mathbb{Z}}\rangle\}_{j,n\in\mathbb{Z}};\ f \in L^2(\mathbb{R})\}$$

in $l^2(\mathbb{Z}^2)$.

Teorema 2.3. *Let* $\psi \in L^1(\mathbb{R})$ *be a BLc uniformly continuous wavelet. Then*

$$\bigcap_{k\in\mathbb{N}} S_k = \{0\}.$$

Proof. Let $\mathbf{x} = (x_{j,n}) \in \bigcap_{k\in\mathbb{N}} S_k$. As $\mathbf{x} \in S_k$ there exists $f_k \in L^2(\mathbb{R})$ such that

$$\mathbf{x} = \Big(\langle f_k, 2^{\frac{j}{2}}\psi(2^j \cdot - \tfrac{n}{k})\rangle\Big)_{j,n\in\mathbb{Z}}. \tag{1}$$

Consider the subsequence $x_{j,kh} = \langle f_k, 2^{\frac{j}{2}}\psi(2^j\cdot -h)\rangle$. As $\Psi(1)$ is an orthonormal basis in $L^2(\mathbb{R})$, f_k can be expressed as

$$f_k = \sum_{j,h\in\mathbb{Z}} \langle f_k, \psi_{j,h}\rangle \psi_{j,h} = \sum_{j,h\in\mathbb{Z}} x_{j,kh}\, 2^{\frac{j}{2}}\,\psi(2^j\cdot -h) \tag{2}$$

and so, taking (2) to (1) we have that for any $p, l \in \mathbb{Z}$

$$\begin{aligned} x_{p,l} &= \langle \sum_{j,h\in\mathbb{Z}} x_{j,kh}\, 2^{\frac{j}{2}}\,\psi(2^j\cdot -h), 2^{\frac{p}{2}}\,\psi(2^p\cdot -\tfrac{l}{k})\rangle \\ &= \sum_{j,h\in\mathbb{Z}} x_{j,kh}\, 2^{\frac{j+p}{2}} \langle \psi(2^j\cdot -h), \psi(2^p\cdot -\tfrac{l}{k})\rangle \\ &= \sum_{j\in\mathbb{Z}} 2^{\frac{j+p}{2}} \sum_{h\in\mathbb{Z}} x_{j,kh}\langle \psi(2^j\cdot -h), \psi(2^p\cdot -\tfrac{l}{k})\rangle. \end{aligned}$$

Grouping terms accurately we have

$$\begin{aligned} x_{p,l} = &\sum_{\substack{j\in\mathbb{Z}\\ j\neq p}} \sum_{h\in\mathbb{Z}} x_{j,kh}\langle 2^{\frac{j}{2}}\psi(2^j\cdot -h), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})\rangle \\ &+ \sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}} x_{p,kh}\langle 2^{\frac{p}{2}}\psi(2^p\cdot -h), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})\rangle \\ &+ x_{p,0}\langle 2^{\frac{p}{2}}\psi(2^p\cdot), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})\rangle. \end{aligned}$$

Call A, B and C these three terms respectively. Taking modulus and by triangular inequality we have that

$$\begin{aligned} |x_{p,l}| \leq &\sum_{\substack{j\in\mathbb{Z}\\ j\neq p}} \sum_{h\in\mathbb{Z}} |x_{j,kh}| \left|\langle 2^{\frac{j}{2}}\psi(2^j\cdot -h), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})\rangle\right| \\ &+ \sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}} |x_{p,kh}| \left|\langle 2^{\frac{p}{2}}\psi(2^p\cdot -h), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})\rangle\right| \\ &+ |x_{p,0}| \left|\langle 2^{\frac{p}{2}}\psi(2^p\cdot), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})\rangle\right|. \end{aligned} \tag{3}$$

As ψ is uniformly continuous, given $\varepsilon > 0$ there exists $k_1 > 0$ such that for any $k \in \mathbb{Z}$ with $|k| > k_1$, $\|2^{\frac{p}{2}}\psi(2^p\cdot) - 2^{\frac{p}{2}}\psi(2^p\cdot -\frac{l}{k})\| < \varepsilon$.

We also have that, using Lemma 2.1, given $\varepsilon > 0$ there exists $k_2 > 0$ such that for any $k \in \mathbb{Z}$ with $|k| > k_2$, $|\langle 2^{\frac{p}{2}}\psi(2^p\cdot), 2^{\frac{p}{2}}\psi(2^p\cdot -\frac{l}{k})\rangle| < 1 + \varepsilon$.

Taking $k_0 \geq \max\{k_1, k_2\}$, for any $|k| > k_0$ they two both hold.

Let's take the first term A of $x_{p,l}$:

$$\begin{aligned}|A| &\leq \sum_{\substack{j\in\mathbb{Z}\\ j\neq p}}\sum_{h\in\mathbb{Z}} |x_{j,kh}| \left|\langle 2^{\frac{j}{2}}\psi(2^j\cdot -h), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})\rangle\right| \\ &\leq \left(\sum_{\substack{j\in\mathbb{Z}\\ j\neq p}}\sum_{h\in\mathbb{Z}} |x_{j,kh}|^2\right)^{1/2} \left(\sum_{\substack{j\in\mathbb{Z}\\ j\neq p}}\sum_{h\in\mathbb{Z}} \left|\langle 2^{\frac{j}{2}}\psi(2^j\cdot -h), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})\rangle\right|^2\right)^{1/2} \\ &\leq \|\mathbf{x}\|_{l^2(\mathbb{Z}^2)} \left(\sum_{\substack{j\in\mathbb{Z}\\ j\neq p}} \left|\left|P_j[2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})]\right|\right|^2\right)^{1/2}\end{aligned}$$

where we have applied the Cauchy-Schwarz inequality and that P_j is the projection on W_j. As the summation index does not equal p and $2^{\frac{p}{2}}\psi(2^p\cdot) \in W_p$, we have that:

$$\begin{aligned}&\|\mathbf{x}\|_{l^2(\mathbb{Z}^2)} \left(\sum_{\substack{j\in\mathbb{Z}\\ j\neq p}} \left|\left|P_j[2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})]\right|\right|^2\right)^{1/2} \\ &= \|\mathbf{x}\|_{l^2(\mathbb{Z}^2)} \left(\sum_{\substack{j\in\mathbb{Z}\\ j\neq p}} \left|\left|P_j[2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k}) - 2^{\frac{p}{2}}\psi(2^p\cdot)]\right|\right|^2\right)^{1/2} \\ &\leq \|\mathbf{x}\|_{l^2(\mathbb{Z}^2)} \left|\left|2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k}) - 2^{\frac{p}{2}}\psi(2^p\cdot)\right|\right| \\ &\leq \|\mathbf{x}\|_{l^2(\mathbb{Z}^2)}\ \varepsilon\end{aligned}$$

for $|k| > k_0$.

A similar process can be followed with the second term of $x_{p,l}$:

$$\begin{aligned}|B| &\leq \sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}} |x_{p,kh}| \left|\langle 2^{\frac{p}{2}}\psi(2^p\cdot -h), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})\rangle\right| \\ &\leq \left(\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}} |x_{p,kh}|^2\right)^{1/2} \left(\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}} \left|\langle 2^{\frac{p}{2}}\psi(2^p\cdot -h), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k})\rangle\right|^2\right)^{1/2} \\ &\leq \|\mathbf{x}_p\|_{l^2(\mathbb{Z})} \left(\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}} \left|\langle 2^{\frac{p}{2}}\psi(2^p\cdot -h), 2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k}) - 2^{\frac{p}{2}}\psi(2^p\cdot)\rangle\right|^2\right)^{1/2} \\ &\leq \|\mathbf{x}_p\|_{l^2(\mathbb{Z})} \left|\left|P_p[2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k}) - 2^{\frac{p}{2}}\psi(2^p\cdot)]\right|\right| \\ &\leq \|\mathbf{x}_p\|_{l^2(\mathbb{Z})} \left|\left|2^{\frac{p}{2}}\psi(2^p\cdot -\tfrac{l}{k}) - 2^{\frac{p}{2}}\psi(2^p\cdot)\right|\right| \\ &\leq \|\mathbf{x}_p\|_{l^2(\mathbb{Z})}\ \varepsilon.\end{aligned}$$

In this case, orthogonality between $\psi(2^p \cdot -h)$ and $\psi(2^p \cdot)$ for any $h \neq 0$ is used.

These bounds for the first two terms and the application of the Lemma 2.1 at the third one, take (3) to:

$$\begin{aligned}|x_{p,l}| &\leq \|\mathbf{x}\|_{l^2(\mathbb{Z}^2)}\,\varepsilon + \|\mathbf{x}_p\|_{l^2(\mathbb{Z})}\,\varepsilon + |x_{p,0}|\,(1+\varepsilon)\\ &= \varepsilon\left(\|\mathbf{x}\|_{l^2(\mathbb{Z}^2)} + \|\mathbf{x}_p\|_{l^2(\mathbb{Z})} + |x_{p,0}|\right) + |x_{p,0}|\end{aligned}$$

for any $\varepsilon > 0$. So, for any $l \in \mathbb{Z}$

$$|x_{p,l}| \leq |x_{p,0}|. \tag{4}$$

If $x_{p,0} = 0$, $\mathbf{x}_p = 0$ for all p and so $\mathbf{x} \equiv 0$. In other case, by inverse triangular inequality, we have that $|x_{p,l}| \geq \frac{1}{3}|x_{p,0}|$. As $\mathbf{x} \in l^2(\mathbb{Z}^2)$, $\mathbf{x}_p \in l^2(\mathbb{Z})$ and so does the constant sequence equal to $x_{p,0}$. So $x_{p,0} = 0$ and then by (4) $\mathbf{x}_p \equiv 0$ for all $p \in \mathbb{Z}$, and so $\mathbf{x} \equiv 0$. ☐

Corollary 2.4. *Let ψ be a wavelet as in Theorem 2.3. Then there exists the limit of S_k and equals $\{0\}$.*

Proof. As $\underline{\lim}\, S_k \subset \overline{\lim}\, S_k$ we just have to see that $\overline{\lim}\, S_k = \{0\}$.

Take $\mathbf{x} \in \overline{\lim}\, S_k$, then $\mathbf{x}$ belongs to S_k for an infinite number of values of k, and so there exists a subsequence of natural numbers $\{k_n\}$ such that $\mathbf{x} \in \bigcap_{n\in\mathbb{N}} S_{k_n}$.

The same proof of Theorem 2.3 can be followed to obtain that $\bigcap_{n\in\mathbb{N}} S_{k_n} = \{0\}$, and so the announced result. ☐

3. DISCRETE VECTOR SAMPLING THEOREM

Let f be a function in $L^2(\mathbb{R})$. Consider the biinfinite matrix

$$\mathbf{x} = (x_{j,n})_{j,n\in\mathbb{Z}} = (\langle f, \psi^k_{j,n}\rangle)_{j,n\in\mathbb{Z}}.$$

As $\Psi(1)$ is an orthonormal basis in $L^2(\mathbb{R})$, $f = \sum\limits_{j,n\in\mathbb{Z}} \langle f, \psi_{j,n}\rangle \psi_{j,n}$ and so

$$x_{l,p} = \sum_{j,n\in\mathbb{Z}} x_{j,nk}\,\langle \psi_{j,n}, \psi^k_{l,p}\rangle.$$

Call $\mathbf{x}^n$ the n-th column of $\mathbf{x}$. It is clear that the matrix $\mathbf{x}$ can be got just from the columns $\{\mathbf{x}^{hk};\, h \in \mathbb{Z}\}$.

This is a situation that reminds a reconstruction from samples. We will see in this paper (in what will be called Discrete Vector Sampling Theorem, DVST) that it is a Shannon type one. With "Shannon type" we mean that the reconstruction will have the same structure of Shannon's Sampling Theorem for band-limited functions:

Teorema 3.1 (Whittaker-Shannon-Kotel'nikov)**.** *Let $f \in L^2(\mathbb{R})$ with* $\operatorname{supp} \hat{f} \subset [-B, B]$. *Then*

$$f(\cdot) = \sum_{n\in\mathbb{Z}} f(n\tau)\,\operatorname{sinc}\left(\frac{\cdot}{\tau} - n\right) \tag{5}$$

for any $\tau \leq \tau_0 = \frac{1}{2B}$, called Nyquist frequency. This equality holds in $L^2(\mathbb{R})$.

See, by e.g., [Z], pg. 15 and ff., for more details.

We will begin defining the space of discrete functions where we are going to work from now on. Given $F : \mathbb{Z} \longrightarrow l^2(\mathbb{Z})$, the j-th element of the sequence $F(n)$ will be denoted by $F_j(n) = [F(n)]_j$. We will call $\mathcal{L}^2(\mathbb{Z})$ the space $l_2(l_2(\mathbb{Z}))$.

The subspace of $\mathcal{L}^2(\mathbb{Z})$ whose functions will verify the DVST, and that will play the analogous role of the band limited functions in Shannon's Sampling Theorem, is the following:

Definition 3.2. *We will call*

$$\mathcal{B}_k = \{F : \mathbb{Z} \longrightarrow l^2(\mathbb{Z});\ \exists f \in L^2(\mathbb{R})\,,\ F_j(n) = \langle f, \psi^k_{j,n}\rangle\}. \tag{6}$$

As $\Psi(k)$ is a frame $\mathcal{B}_k \subset \mathcal{L}^2(\mathbb{Z})$.

Proposition 3.3. $(\mathcal{B}_k, \|\cdot\|_{\mathcal{L}^2(\mathbb{Z})})$ *is a Hilbert space.*

Proof. Let $(F^m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}_k$. Then, given $m \in \mathbb{N}$, there exists $f_m \in L^2(\mathbb{R})$ such that $F^m_j(n) = \langle f_m, \psi^k_{j,n}\rangle$ for all $j, n \in \mathbb{Z}$.

And, by other side, we have that for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ so that for any $p, h > N_0$

$$\sum_{j,n\in\mathbb{Z}} \left|F^p_j(n) - F^h_j(n)\right|^2 < \varepsilon.$$

As

$$\sum_{j,n\in\mathbb{Z}} \left|F^p_j(n) - F^h_j(n)\right|^2 = \sum_{j,n\in\mathbb{Z}} \left|\langle f_p - f_h, \psi^k_{j,n}\rangle\right|^2$$

and $\Psi(k)$ is a frame in $L^2(\mathbb{R})$ (let $A(k)$ and $B(k)$ be its bounds) we have that

$$A(k)\,\|f_p - f_h\|^2 \le \sum_{j,n\in\mathbb{Z}} \left|F^p_j(n) - F^h_j(n)\right|^2 \le B(k)\,\|f_p - f_h\|^2.$$

This makes the family $\{f_m; m \in \mathbb{N}\}$ to be a Cauchy sequence in $L^2(\mathbb{R})$ and so convergent to a function $f \in L^2(\mathbb{R})$.

Take $F \in \mathcal{L}^2(\mathbb{Z})$ associated to f by (6). Clearly $F \in \mathcal{B}_k$ and

$$\begin{aligned}\|F^m - F\|^2_{\mathcal{L}^2(\mathbb{Z})} &= \sum_{j,n\in\mathbb{Z}} \left|F^m_j(n) - F_j(n)\right|^2 = \sum_{j,n\in\mathbb{Z}} \left|\langle f_m - f, \psi^k_{j,n}\rangle\right|^2 \\ &\le B(k)\,\|f_m - f\|^2.\end{aligned}$$

This proves that, for all $k \in \mathbb{Z}$, $\mathcal{B}_k$ is complete and so closed in $\mathcal{L}^2(\mathbb{Z})$, what concludes the result of the proposition. □

We will define now the basic function that will take the role of the function sinc.

Definition 3.4. *Let us define* $S : \mathbb{Q} \times \mathbb{Z} \longrightarrow l^2(\mathbb{Z})$ *as:*

$$[S(l, m)]_j = \langle \psi, \psi_{j,l-2^j m}\rangle = \langle \psi, 2^{\frac{j}{2}}\psi(2^j \cdot -l + 2^j m)\rangle.$$

Note that it is well defined; just take $l = \frac{p}{q} \in \mathbb{Q}$ and verify that

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |\langle \psi, \psi_{j,l-2^j m}\rangle|^2 = \sum_{j\in\mathbb{Z}} |\langle \psi, \psi^q_{j,p-2^j mq}\rangle|^2 &= \sum_{j\in\mathbb{Z}} |\langle \psi_{0,m}, \psi^q_{j,p}\rangle|^2 \\ &\leq B(q)\|\psi\|^2.\end{aligned}$$

We can give now the DVST:

Teorema 3.5. *Let $k \in \mathbb{N}$ and $F \in \mathcal{B}_k$. Let S be the function defined upwards. Then*

$$F(\cdot) = \sum_{n\in\mathbb{Z}} F(nk) * S(\tfrac{\cdot}{k}, n) \tag{7}$$

where $*$ is the convolution product in $l^2(\mathbb{Z})$. This equality holds in $\mathcal{L}^2(\mathbb{Z})$.

Proof. First of all we will see the equality holds pointwise, it is

$$[F(l)]_p = \sum_{n\in\mathbb{Z}} [F(nk) * S(\tfrac{l}{k}, n)]_p$$

for all $p, l \in \mathbb{Z}$.

Take $F \in \mathcal{B}_k$ and let $f \in L^2(\mathbb{R})$ be the function associated to it by (6). As $\Psi(1)$ is an orthonormal basis in $L^2(\mathbb{R})$, $f = \sum_{j,n} \langle f, \psi_{j,n}\rangle \psi_{j,n}$. This gives:

$$\begin{aligned}F_p(l) &= \sum_{j,n\in\mathbb{Z}} \langle f, \psi_{j,n}\rangle\langle \psi_{j,n}, \psi^k_{p,l}\rangle = \sum_{j,n\in\mathbb{Z}} \langle f, \psi^k_{j,nk}\rangle\langle \psi_{j,n}, \psi^k_{p,l}\rangle \\ &= \sum_{j,n\in\mathbb{Z}} F_j(nk)\langle \psi_{j,n}, \psi^k_{p,l}\rangle = \sum_{j,n\in\mathbb{Z}} F_j(nk)\langle \psi, \psi_{p-j,\frac{l}{k}-2^{p-j}n}\rangle \\ &= \sum_{j,n\in\mathbb{Z}} [F(nk)]_j [S(\tfrac{l}{k}, n)]_{p-j} = \sum_{n\in\mathbb{Z}} [F(nk) * S(\tfrac{l}{k}, n)]_p \\ &= [\sum_{n\in\mathbb{Z}} F(nk) * S(\tfrac{l}{k}, n)]_p\end{aligned}$$

for any $p \in \mathbb{Z}$.

For getting the equality in $\mathcal{L}^2(\mathbb{Z})$ it is not too difficult to see that what we have to prove is that

$$\sum_{l,p\in\mathbb{Z}} \Big| \sum_{|n|>N} [a^n(l)]_p \Big|^2 \longrightarrow 0,$$

where

$$\begin{aligned}[a^n(l)]_p = [F(nk) * S(\tfrac{l}{k}, n)]_p &= \sum_{m\in\mathbb{Z}} [F(nk)]_m [S(\tfrac{l}{k}, n)]_{p-m} \\ &= \sum_{m\in\mathbb{Z}} \langle f, \psi^k_{m,nk}\rangle\langle \psi, \psi_{p-m,\frac{l}{k}-2^{p-m}n}\rangle \\ &= \langle \sum_{m\in\mathbb{Z}} \langle f, \psi_{m,n}\rangle \psi_{m,n}, \psi^k_{p,l}\rangle.\end{aligned}$$

From here, and using that $\Psi(k)$ is a frame we have that there exists $B(k) > 0$ such that

$$\begin{aligned}\sum_{l,p\in\mathbb{Z}}\Big|\sum_{|n|>N}[a^n(l)]_p\Big|^2 &= \sum_{l,p\in\mathbb{Z}}\Big|\langle\sum_{|n|>N}\sum_{m\in\mathbb{Z}}\langle f,\psi_{m,n}\rangle\psi_{m,n},\psi^k_{p,l}\rangle\Big|^2\\ &\le B(k)\Big\|\sum_{|n|>N}\sum_{m\in\mathbb{Z}}\langle f,\psi_{m,n}\rangle\psi_{m,n}\Big\|^2_{L^2(\mathbb{R})}\\ &= B(k)\sum_{|n|>N}\sum_{m\in\mathbb{Z}}|\langle f,\psi_{m,n}\rangle|^2.\end{aligned}$$

As $\Psi(1)$ is an orthonormal basis and $f \in L^2(\mathbb{R})$ we have the result of the theorem. □

Note that Theorems 3.5 and 3.1 are completely analogous, not only because of the type of convergence got in both of them, but also by the similar expressions (7) and (5) respectively. For this reason Theorem 3.5 can be said to be a Whittaker-Shannon-Kotel'nikov type theorem.

4. COLUMNS BY ROWS: THE FAMILY $\Psi^{[k]}$

By now only modifications in the translation parameter of an orthonormal wavelet basis have been considered, but the same type of modification can be made in the scaling parameter. We will call $\Psi^{[k]}$ the family

$$\Psi^{[k]} = \{\,{}^k\psi_{j,n}(\cdot) = 2^{\frac{j}{2k}}\psi(2^{\frac{j}{k}}\cdot - n);\ j,n\in\mathbb{Z}\,\},$$

where $k \in \mathbb{N}$ and ψ is a wavelet such that $\Psi(1)$ is an orthonormal wavelet basis.

Proposition 4.1. *The family $\Psi^{[k]}$ is a tight frame in $L^2(\mathbb{R})$ for any $k \in \mathbb{N}$ and the frame bound is k.*

Proof. Take $f \in L^2(\mathbb{R})$ and consider the k different classes in $\mathbb{Z}/k\mathbb{Z}$.

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}}\big|\langle f,2^{\frac{m}{2k}}\psi(2^{\frac{m}{k}}\cdot-n)\rangle\big|^2 &= \sum_{m,n\in\mathbb{Z}}\sum_{h=0}^{k-1}\big|\langle f(t)\,{}^k\psi_{mk+h,n}\rangle\big|^2\\ = \sum_{h=0}^{k-1}\sum_{m,n\in\mathbb{Z}}\Big|\langle f_{-\frac{h}{k},0},\psi_{m,n}\rangle\Big|^2 &= \sum_{h=0}^{k-1}\|f_{-\frac{h}{k},0}\|^2_{L^2(\mathbb{R})} = k\,\|f\|_{L^2(\mathbb{R})}.\end{aligned}$$

□

Note that this result is got for any wavelet ψ such that $\Psi(1)$ is an orthonormal basis in $L^2(\mathbb{R})$, and finite band-width is not required.

Associated to the family $\Psi^{[k]}$ we can also try to obtain a discrete vector sampling theorem as before. We will see that a Shannon type theorem cannot be got.

Definition 4.2. *Let ψ be a wavelet such that $\Psi(1)$ is an orthonormal basis in $L^2(\mathbb{R})$ and $k \in \mathbb{N}$. We will call $\mathcal{A}_k$ the space*

$$\mathcal{A}_k = \{G:\mathbb{Z}\longrightarrow l^2(\mathbb{Z});\ \exists g\in L^2(\mathbb{R}),\ G_j(n) = \langle g,^k\psi_{j,n}\rangle\}. \tag{8}$$

By Proposition 4.1 we have that $\mathcal{A}_k \subset \mathcal{L}^2(\mathbb{Z})$.

Proposition 4.3. *There is no Shannon type sampling theorem for the functions in* $\mathcal{A}_k$.

Proof. Take $G \in \mathcal{A}_k$, and let $g \in L^2(\mathbb{R})$ be the function associated to G by (8). As $\Psi(1)$ is an orthonormal basis in $L^2(\mathbb{R})$

$$\begin{aligned}[G(l)]_j &= \sum_{n,q\in\mathbb{Z}} \langle g, \psi_{n,q}\rangle \langle \psi_{n,q}, {}^k\psi_{l,j}\rangle \\ &= \sum_{n,q\in\mathbb{Z}} [G(nk)]_q \, \langle \psi_{n,q}, \psi_{\frac{l}{k},j}\rangle.\end{aligned}$$

Suppose there exists $T : \mathbb{Q} \times \mathbb{Z} \longrightarrow l^2(\mathbb{Z})$ such that

$$G(l) = \sum_{n\in\mathbb{Z}} G(nk) * T(\tfrac{l}{k}, n)$$

for all $l \in \mathbb{Z}$. Then

$$[G(l)]_j = \sum_{n,q\in\mathbb{Z}} [G(nk)]_q \; [T(\tfrac{l}{k}, n)]_{j-q}.$$

From these two expressions of $[G(l)]_j$ we obtain that

$$[T(\tfrac{l}{k}, n)]_{j-q} = \langle \psi_{n,q}, \psi_{\frac{l}{k},j}\rangle,$$

what is the same, given $\lambda \in \mathbb{Q}$ and $n \in \mathbb{Z}$

$$[T(\lambda, n)]_{j-q} = \langle \psi_{n,q}, \psi_{\lambda,j}\rangle.$$

This would make $\langle \psi_{n,q}, \psi_{\lambda,j}\rangle$ to depend on j and q just by its difference, and so

$$\langle \psi_{n,q}, \psi_{\lambda,j}\rangle = \langle \psi_{n,0}, \psi_{\lambda,j-q}\rangle.$$

But it is not true, as $\langle \psi_{n,q}, \psi_{\lambda,j}\rangle = \langle \psi_{n,0}, \psi_{\lambda,2^{\lambda-n}q-j}\rangle$. ☐

Let us observe that the only, but essential, difference between the redundancy introduced in the family $\Psi(k)$ and in $\Psi^{[k]}$ is that $\langle \psi_{n,q}, \psi_{j,\frac{l}{k}}\rangle$ does depend on j and q just by its difference, and so we can build the function S so that

$$[F(l)]_j = \sum_{n,q\in\mathbb{Z}} [F(qk)]_n \, \langle \psi_{n,q}, \psi_{j,\frac{l}{k}}\rangle = \sum_{n,q\in\mathbb{Z}} [F(qk)]_n \, [S(\tfrac{l}{k}, n)]_{j-q}.$$

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