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MANIFOLDS WITH INDEFINITE METRICS WHOSE SKEW-SYMMETRIC CURVATURE OPERATOR HAS CONSTANT EIGENVALUES

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Abstract. In this expository note, we survey some recent results in the pseudo-Riemannian setting giving geometrical consequences when the skew-symmetric curvature operator is assumed to have constant eigenvalues.

§1 Introduction

Let (M, g_M) be a smooth connected pseudo-Riemannian manifold of signature (p, q). We shall suppose henceforth that $p \leq q$ since we can always replace g_M by $-g_M$ and reverse the roles of p and q.

Let ∇ be the Levi-Civita connection on TM and let $R(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}$ be the *Riemann curvature operator*. The *associated curvature tensor* $R(x, y, z, w) := g_M(R(x, y)z, w)$ has symmetries:

$$\begin{aligned} &R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z),\\ &R(x,y,z,w) = R(z,w,x,y), \quad \text{and} \\ &R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0. \end{aligned} \tag{1.1.a}$$

The curvature tensor carries crucial geometric information about the manifold. However, the full curvature tensor is quite complicated. One can use the curvature tensor to define natural endomorphisms of the tangent bundle; the Jacobi operator $J_R(x) : y \to R(y, x)x$, the Szabó operator $S_R(x) : y \to \nabla_x R(y, x)x$, and the skew-symmetric curvature operator $R(\cdot)$ are examples of such operators. The natural domain of J_R and S_R is the unit tangent bundle $S(TM)$; the natural domain of $R(\cdot)$ is the oriented Grassmannian of non-degenerate 2-planes.

If one assumes that the eigenvalues of such an operator are constant on the natural domain of definition, then the possible geometries are usually quite restricted. We work in the Riemannian setting for the moment. If the Szabó operator S_R has constant eigenvalues, then (M, g_M) is a local symmetric space [38]. If the Jacobi operator J_R has constant eigenvalues, then (M, g_M) is a rank 1 symmetric space if $m \not\equiv 0 \bmod 4$ [9], [10], [11]; for other related work concerning the Jacobi operator we refer to [2]–[7], [12]–[20], [22], [25], [26], [29]–[36]. The proof of these results uses techniques from both differential geometry and from algebraic topology; in particular the work of Adams [1], Borel [8], and Stong [37] plays a central role.

In this paper, we shall deal with the skew-symmetric curvature operator; this operator was first studied in this context by Ivanova and Stanilov [28]. In the Riemannian setting, this operator has been studied by Gilkey [21], by Gilkey, Leahy, and Sadofsky [23], and by Ivanov and Petrova [27]. It is convenient to pass to a purely algebraic context and work with the space of algebraic curvature tensors. Let g be a non-degenerate symmetric bilinear form of signature (p,q) on a finite dimensional real vector space V. A 4 tensor $R \in \otimes^4(V^*)$ is said to be an *algebraic curvature tensor* if the equations displayed in (1.1.a) are satisfied. We note that the Riemann curvature tensor R of a manifold (M, g_M) defines an algebraic curvature tensor on T_PM for every P in M; conversely, given a metric g_P on T_PM and an algebraic curvature tensor R_P, there exists the germ of a metric $\tilde{g}_M$ on M extending g_P so that R_P is the curvature tensor of $\tilde{g}_M$ at P. Thus the study of algebraic curvature tensors is important in differential geometry. We refer to [20], [29] for expository accounts of this field and for a more detailed bibliography than can be presented here.

Here is a brief outline of this note. In §2, we shall introduce some notational conventions. In §3, we shall review results of [21], [23], [27] in the Riemannian setting. In §4, we discuss the corresponding generalizations of these results to the pseudo-Riemannian setting. We conclude with a short bibliography.

§2 Notational conventions

Let $\mathbb{R}^{p,q}$ be the vector space of real $(p+q)$-tuples of the form

$$x = (x_1, \dots, x_p, x_{p+1}, \dots, x_{p+q})$$

with the non-degenerate symmetric bilinear form of signature (p,q)

$$g(x,y) := -\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{p+q} x_i y_i.$$

Let π be a 2-plane in $\mathbb{R}^{p,q}$; π is said to be *non-degenerate* if the restriction of g to π is non-degenerate. Let $\{x,y\}$ be a basis for π; π is non-degenerate if and only if $g(x,x)g(y,y) - g(x,y)^2 \neq 0$. Let $Gr_2^+(\mathbb{R}^{p,q})$ (resp. $Gr_2(\mathbb{R}^{p,q})$) be the manifold of all oriented (resp. unoriented) spacelike 2-planes in $\mathbb{R}^{p,q}$. Let $\{x,y\}$ be an oriented basis for $\pi \in Gr_2^+(\mathbb{R}^{p,q})$. We define the *skew-symmetric curvature operator* $R(\pi)$ by

$$R(\pi) := \{g(x,x)g(y,y) - g(x,y)^2\}^{-\frac{1}{2}} R(x,y);$$

$R(\pi)$ is independent of the particular oriented basis chosen for π.

An algebraic curvature tensor R is said to be *of rank* r if $\operatorname{rank} R(\pi) = r$ on all $\pi \in Gr_2^+(\mathbb{R}^{p,q})$. An algebraic curvature tensor R is said to be *IP* if $R(\pi)$ has constant eigenvalues on all $\pi \in Gr_2^+(\mathbb{R}^{p,q})$. A metric g_M on a manifold M is said to be *IP* if $R(\pi)$ is IP at every point $P \in M$; the eigenvalues are permitted to depend on $P \in M$.

IP algebraic curvature tensors and IP metrics were first studied by IVANOV and PETROVA [27] in the context of four dimensional Riemannian geometry. Subsequently GILKEY [21], and GILKEY, LEAHY and SADOFSKY [23] classified the IP algebraic curvature tensors and IP metrics in the Riemannian setting except in dimension 7; some partial results regarding dimension 7 can be found in GILKEY and SEMMELMAN [24].

We say that (C,ϕ) is an *admissible pair* if C is a nonzero constant and if ϕ is a linear map of $\mathbb{R}^{p,q}$ so that $\phi^2=\varepsilon\cdot\mathrm{id}$ and that $g(\phi(u),\phi(v))=\varepsilon\cdot g(u,v)$ where $\varepsilon=\pm1$. If $\varepsilon=1$, then ϕ is said to be an *involutive isometry*; if $\varepsilon=-1$, then ϕ is said to be a *skew-involutive skew-isometry.* If (C,ϕ) is an admissible pair, we define

$$R_{C,\phi}(x,y)z:=C\{g(\phi(y),z)\phi(x)-g(\phi(x),z)\phi(y)\}.$$

We remark that $\varepsilon=-1$ is only possible when $p=q$. We note that if ϕ is the identity map, then $R_{C,\phi}$ has constant sectional curvature C.

We say that an algebraic curvature tensor R is *spacelike* (resp. *timelike*) if $\mathrm{Range}\,(R(\pi))$ is spacelike (resp. timelike) for every spacelike 2-plane π. If R is a rank 2 IP algebraic curvature tensor, then R is said to be *mixed* if $\mathrm{Range}\,(R(\pi))$ is of type $(1,1)$ for every spacelike 2-plane π; R is said to be *null* if $\mathrm{Range}\,(R(\pi))$ is a degenerate 2-plane for every spacelike 2-plane π and $R(\pi)$ has only the zero eigenvalue.

§3 CLASSIFICATION OF IP ALGEBRAIC CURVATURE TENSORS AND IP METRICS IN THE RIEMANNIAN SETTING

In this section, we review previous work of [21], [23], [27] on the classification of IP algebraic curvature tensors and IP metrics in the Riemannian setting. The following theorem classifies IP algebraic curvature tensors in the Riemannian setting if $m=5,6$ or if $m\geq 8$:

Theorem 3.1 (GILKEY [21], GILKEY, LEAHY and SADOFSKY [23])**.** *Let R be an IP algebraic curvature tensor. Assume that $(p,q)=(0,m)$. Let $m\geq 5$.*

1. *If $m\neq 7$, then $\mathrm{rank}\,R\leq 2$.*
2. *If $\mathrm{rank}\,R=2$, then there exists an admissible pair (C,ϕ) with ϕ an involutive isometry of $\mathbb{R}^m$ so that $R=R_{C,\phi}$.*

The four dimensional case is exceptional. We have:

Theorem 3.2 (IVANOV and PETROVA [27])**.** *Let R be an IP algebraic curvature tensor. Let $(p,q)=(0,4)$.*

1. *If $\mathrm{rank}\,R=2$, then there exists an admissible pair (C,ϕ) with ϕ an involutive isometry of $\mathbb{R}^4$ so that $R=R_{C,\phi}$.*
2. *If $\mathrm{rank}\,R=4$, then R is equivalent to a nonzero multiple of the "exotic" rank 4 tensor:*

$$\begin{aligned} R_{1212} &= 2,\ R_{1313}=2,\ R_{1414}=-1, R_{2424}=2, R_{2323}=-1,\\ R_{3434} &= 2,\ R_{1234}=-1,\ R_{1324}=1,\ R_{1423}=2. \end{aligned}$$

Theorems 3.1 and 3.2 classify the IP algebraic curvature tensors if $m \geq 4$ and if $m \neq 7$. The corresponding classification of IP metrics is given by the following result:

Theorem 3.3. (Gilkey [21], Gilkey, Leahy and Sadofsky [23]; Ivanov and Petrova [27]) *Let M be an IP Riemannian manifold of dimension m. Assume $m \geq 4$. If $m = 7$, we further assume that rank $R = 2$. Exactly one and only one of the following assertions is valid for M:*

1. *M has constant sectional curvature.*
2. *M is locally a warped product: $ds_M^2 = dt^2 + f(t)ds_N^2$ of a connected open interval $I \subset \mathbb{R}$ with a Riemannian manifold N of dimension $m-1$ which has constant sectional curvature $\mathcal{K} \neq 0$. Furthermore, the warping function f is given by $f(t) = \mathcal{K}t^2 + At + B$, where A and B are auxiliary constants so that $4\mathcal{K}B - A^2 \neq 0$ and that $f(t) > 0$ on I.*

§4 Main results in the Pseudo-Riemannian setting

The results discussed in §3 are in the Riemannian setting where $(p, q) = (0, m)$; the fact that the metric in question is positive definite is used at several crucial points in the argument. We shall present some analogous results in the Lorentzian setting $(p, q) = (1, m-1)$ if $m \geq 10$ and in the higher signature setting $(p, q) = (2, m-2)$ if $q \geq 11$. We refer to [39] for further details.

Theorem 4.1. *Let R be an algebraic curvature tensor of rank r on $\mathbb{R}^{p,q}$.*

1. *If $p = 1$ and if $q \geq 9$, then $r \leq 2$.*
2. *If $p = 2$ and if $q \geq 11$, then $r \leq 4$. Furthermore, if q and $2+q$ are not powers of 2, then $r \leq 2$.*
3. *There exists a rank 4 IP algebraic curvature tensor if $(p, q) = (2, 2)$.*

Theorem 4.1 bounds the rank of an IP algebraic curvature tensor. In the rank 2 Lorentzian setting, we have a trichotomy:

Theorem 4.2. *Let R be a rank 2 Lorentzian IP algebraic curvature tensor and let $m \geq 4$. Exactly one and only one of the following assertions is valid for R:*

1. *For all $\pi \in Gr_2^+(\mathbb{R}^{1,m-1})$, we have that Range$(R(\pi))$ is spacelike and that $R(\pi)$ has two nontrivial purely imaginary eigenvalues. Thus R is spacelike.*
2. *For all $\pi \in Gr_2^+(\mathbb{R}^{1,m-1})$, we have that Range$(R(\pi))$ is of type $(1,1)$ and that $R(\pi)$ has two nontrivial real eigenvalues. Thus R is mixed.*
3. *For all $\pi \in Gr_2^+(\mathbb{R}^{1,m-1})$, we have that Range$(R(\pi))$ is degenerate with a positive semi-definite metric and that $R(\pi)$ has only the zero eigenvalue. Thus R is null.*

The following theorem shows that most rank 2 Lorentzian IP algebraic curvature tensors are spacelike.

Theorem 4.3. *Let R be a rank 2 Lorentzian IP algebraic curvature tensor and let $m \geq 4$.*

1. *If R is mixed, then $m = 4$, 5, 8, or 9.*
2. *If R is null, then $m = 5$ or 9.*

Theorems 4.1 4.2, and 4.3 show that in the Lorentzian setting, the rank of a nontrivial IP algebraic curvature tensor is 2 and the tensor in question is spacelike if $m \geq 10$. We have the following classification of rank 2 IP algebraic curvature tensors which are spacelike or timelike.

Theorem 4.4.

1. *If (C, ϕ) is an admissible pair, then $R_{C,\phi}$ is a rank 2 IP algebraic curvature tensor which is spacelike if $\varepsilon = 1$ and timelike if $\varepsilon = -1$.*
2. *Let R be an IP algebraic curvature tensor on $\mathbb{R}^{p,q}$. Suppose that $q = 6$ or that $q \geq 9$. Suppose that R is spacelike or timelike and that R has rank 2. Then there exists an admissible pair (C, ϕ) so that $R = R_{C,\phi}$.*

Let ϕ be an involutive isometry of $\mathbb{R}^{p,q}$. We generalize the construction of IP metrics given in Theorem 3.3 as follows. Let $\varepsilon = \pm 1$. Let $I \subset \mathbb{R}$ be a connected open interval. Let N be the germ of a pseudo-Riemannian manifold of constant sectional curvature $\mathcal{K} \neq 0$. Let A and B be auxiliary constants so that $4\mathcal{K}B - \varepsilon A^2 \neq 0$ and that $f_\varepsilon(t) := \varepsilon \mathcal{K} t^2 + At + B > 0$ on I. Let $M := I \times N$ and let

$$g_M := \varepsilon dt^2 + f_\varepsilon(t) g_N \tag{4.4.a}$$

define a rank 2 IP metric on M. We have the following classification of IP algebraic curvature tensors and rank 2 IP metrics in the Lorentzian setting provided $m \geq 10$.

Theorem 4.5. *Assume that $m \geq 10$.*

1. *Let R be an IP algebraic curvature tensor on $\mathbb{R}^{1,m-1}$. R is nontrivial if and only if there exists an admissible pair (C, ϕ) with ϕ an involutive isometry of $\mathbb{R}^{1,m-1}$ so that $R = R_{C,\phi}$.*
2. *If g_M is a rank 2 Lorentzian IP metric, then exactly one and only one of the following assertions is valid for g_M:*
   (2a) *g_M is a metric of constant sectional curvature $C \neq 0$.*
   (2b) *g_M is locally isometric to a warped product metric of the form given in (4.4.a).*

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