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ASPECTS OF TIME-DEPENDENT SECOND-ORDER DIFFERENTIAL EQUATIONS: BERWALD-TYPE CONNECTIONS

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1. Introduction: time-dependent second-order equations

At one of the previous editions of this conference, M. Crampin gave a talk on the construction of a linear connection associated to an arbitrary system of second-order equations (Sode for short) [4]. Some people in the audience, with expertise in Finsler geometry, made the comment that this had to be essentially the Berwald connection. However, Crampin's story was about a connection on some pullback bundle (which the original Berwald construction is not) and, more importantly, it was about time-dependent Sodes, i.e. it had to do with the affine bundle structure of a jet space, rather than the vector bundle structure of a tangent bundle. Moreover, at about the same time, a few other constructions of such linear Sode-connections were published independently by Massa and Pagani [7] and by Byrnes [3], and these are all quite different! So, the least one can say is that it is far from obvious how the qualification "Berwald-type connection" could be attributed to all of these constructions.

The purpose of the present contribution is precisely to explain a general framework for understanding the subtle differences between the above mentioned connections and for describing accurately what "Berwald-type" means in a time-dependent context. As such, it gives a survey of an elaborate study on these matters [8] which will be published elsewhere.

We begin by recalling the basic features about modelling time-dependent Sodes. Consider the first jet bundle $J^1\pi$ of a bundle $\pi : E \to \mathbb{R}$.

A Sode-field Γ is a vector field on $J^1\pi$ with the properties $\langle \Gamma, dt \rangle = 1$ and $S(\Gamma) = 0$, where S is the vertical endomorphism:

$$S = \theta^i \otimes \frac{\partial}{\partial v^i}, \qquad \theta^i = dx^i - v^i dt.$$

Locally, Γ is of the form

$$\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} + f^i(t,x,v) \frac{\partial}{\partial v^i}.$$

Γ defines a horizontal distribution on $J^1\pi$ which we will indicate most of the time by the corresponding horizontal projector field P_H. We have:

$$P_H = \tfrac{1}{2}(I - \mathcal{L}_\Gamma S + dt \otimes \Gamma),$$

and

$$\operatorname{Im} P_H = \mathrm{sp}\left\{\Gamma, H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial v^j}\right\} \quad \text{where} \quad \Gamma_i^j = -\tfrac{1}{2}\frac{\partial f^j}{\partial v^i}.$$

An accurate description of the natural decomposition of $\mathcal{X}(J^1\pi)$ which originates from this horizontal distribution, inevitably brings the bundle $\pi_1^{0*}(\tau_E) \to J^1\pi$ of the diagram below into the picture.

$$\begin{tikzcd}
\pi_1^{0*}(\tau_E) \arrow[r] \arrow[d] & TE \arrow[d, "\tau_E"] & \\
J^1\pi \arrow[r, "\pi_1^0"'] & E \arrow[r, "\pi"'] & \mathbb{R}
\end{tikzcd}$$

Observe first that there exists a canonical section of $\pi_1^{0*}(\tau_E) \to J^1\pi$, denoted by

$$\mathbf{T} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}.$$

The $C^\infty(J^1\pi)$-module of such sections (which are called *vector fields along* π_1^0), will be denoted by $\mathcal{X}(\pi_1^0)$. It has the natural decomposition:

$$\mathcal{X}(\pi_1^0) \equiv \overline{\mathcal{X}}(\pi_1^0) \oplus \langle \mathbf{T} \rangle.$$

In other words, for each $X \in \mathcal{X}(\pi_1^0)$, we write

$$X = \overline{X} + \langle X, dt \rangle \mathbf{T}, \quad \text{with} \quad \overline{X} = X^i(t,x,v)\frac{\partial}{\partial x^i}.$$

Looking at the larger module $\mathcal{X}(J^1\pi)$ now, we have $\Gamma = \mathbf{T}^H$ and there is a corresponding decomposition:

$$\begin{aligned}
\mathcal{X}(J^1\pi) &\equiv \mathcal{X}(\pi_1^0)^H \oplus \overline{\mathcal{X}}(\pi_1^0)^V \\
&\equiv \overline{\mathcal{X}}(\pi_1^0)^H \oplus \overline{\mathcal{X}}(\pi_1^0)^V \oplus \langle \Gamma \rangle
\end{aligned}$$

Typically, for $\xi \in \mathcal{X}(J^1\pi)$ we will write (as in [6])

$$\xi = {\xi_H}^H + {\overline{\xi}_V}^V = {\overline{\xi}_H}^H + {\overline{\xi}_V}^V + \langle \xi, dt \rangle \Gamma,$$

with $\xi_H \in \mathcal{X}(\pi_1^0)$ and $\overline{\xi}_H, \overline{\xi}_V \in \overline{\mathcal{X}}(\pi_1^0)$. The horizontal and vertical lift operations from $\overline{\mathcal{X}}(\pi_1^0)$ to $\mathcal{X}(J^1\pi)$ are given by:

$$\overline{X}^V = X^i \frac{\partial}{\partial v^i}, \qquad \overline{X}^H = X^i H_i.$$

The fact that horizontal vector fields on $J^1\pi$ further decompose into a component along Γ and an element of $\overline{\mathcal{X}}(\pi_1^0)^H$ has an effect on most tensorial quantities of interest. For example, we have

$$P_H = P_{\overline{H}} + dt \otimes \Gamma = \theta^i \otimes H_i + dt \otimes \Gamma.$$

Roughly speaking, the complexity of the time-dependent picture (as compared to the autonomous framework) originates precisely from the fact that there is a certain freedom in "fixing the time-component", or better the "Γ-component". Note in passing that we cannot incorporate the framework for time-dependent second-order equations as proposed in [2, 9] in our comparative discussion, because it takes the choice of a trivialization of $J^1\pi$ for granted, which means that time and space coordinates are kept strictly separated. As a result, some of the constructions of these authors do not have an intrinsic meaning in our set-up.

2. An associated linear connection on $J^1\pi$

An interesting, though rather peculiar, construction of a linear connection associated to a SODE was given by Massa and Pagani [7]. For completeness, let us recall that by linear connection on $J^1\pi$ we mean an operator $\nabla_\xi : \mathcal{X}(J^1\pi) \to \mathcal{X}(J^1\pi)$, defined for each $\xi \in \mathcal{X}(J^1\pi)$ and having the properties

$$\begin{aligned} \nabla_\xi(F\eta) &= F\,\nabla_\xi\eta + \xi(F)\,\eta, \\ \nabla_{F\xi}\eta &= F\,\nabla_\xi\eta. \qquad F \in C^\infty(J^1\pi) \end{aligned}$$

The construction of such a ∇ in [7] is very indirect. The idea is to narrow down the list of candidates by gradually introducing extra requirements on the ∇ under construction, until in fact there is only one left. It is only at the very last stage that a particular SODE Γ comes into the picture to which the connection then can be said to be associated. We briefly summarize the main steps in that construction here.

The first fundamental requirements are that we should have:

$$\nabla_\xi dt = 0, \qquad \nabla_\xi S = 0,$$

$$\nabla_{\overline{X}^V}\overline{Y}^V = 0, \quad \forall \overline{X}, \quad \forall \text{ basic } \overline{Y},$$

where the last condition is just a technical way of expressing that the connection should preserve parallel transport in the fibres.

For the second stage, let T be the torsion of the as yet undetermined ∇, i.e. $T(\xi,\eta) = \nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta]$, and let us define operators P and Q by

$$P(\eta) = T(\gamma, S(\eta)), \qquad Q(\eta) = S(T(\gamma,\eta)) + \langle\eta, dt\rangle\gamma$$

where γ is an *arbitrary* SODE. Massa and Pagani show that these are projection operators which, as the notation indicates, do not depend on the choice of γ. The additional requirements now are that P and Q must be complementary and P must be parallel, i.e.

$$P + Q = I \qquad \text{and} \qquad \nabla_\xi P = 0.$$

Next, let *curv* denote the curvature of the as yet undetermined ∇:

$$curv(\xi,\eta) = \nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]}.$$

Require now further that

$$curv(\gamma, \overline{X}^V) = 0, \quad \forall \overline{X}, \ \forall \text{ SODE } \gamma.$$

At this stage, it is a theorem that under the above requirements, ∇ will be completely determined as soon as we know for any pre-assigned SODE Γ, the value of $\nabla_\gamma \Gamma$ for arbitary SODEs γ.

The final step in the construction of Massa and Pagani then consists in fixing the remaining freedom by requiring simply that for a given Γ $\nabla_\gamma \Gamma = 0$, from which it actually follows that

$$\nabla_\xi \Gamma = 0, \qquad \forall \xi \in \mathcal{X}(J^1\pi).$$

A quite remarkable feature of this construction is that the projector P, which afterall was defined in terms of the torsion of the linear connection under construction, in the end turns out to coincide with the operator $P_{\overline{H}}$ which (together with Γ) determines the horizontal distribution of the non-linear connection coming from Γ.

3. An associated linear connection on ${\pi_1^0}^*(\tau_E) \to J^1\pi$

By way of contrast with the preceding section, let us now recall the direct construction of a linear connection, as presented by Crampin *et al* [6].

Given the SODE Γ with its P_H, define the operator $\mathrm{D}: \mathcal{X}(J^1\pi) \times \mathcal{X}(\pi_1^0) \to \mathcal{X}(\pi_1^0)$ by

$$\mathrm{D}_\xi X = [P_H(\xi), X^V]_V + [P_V(\xi), X^H]_H + P_H(\xi)(\langle X, dt\rangle)\mathbf{T}.$$

It is easy to verify that D is a linear connection on ${\pi_1^0}^*(\tau_E) \to J^1\pi$, i.e. we have

$$\begin{aligned} \mathrm{D}_\xi(FX) &= F\,\mathrm{D}_\xi X + \xi(F)X, \\ \mathrm{D}_{F\xi}X &= F\,\mathrm{D}_\xi X \qquad F \in C^\infty(J^1\pi). \end{aligned}$$

For brevity, a connection on the bundle ${\pi_1^0}^*(\tau_E) \to J^1\pi$ will be called simply a connection on ${\pi_1^0}^*(\tau_E)$ in what follows.

Coming back to our introduction now, it will no doubt be clear that understanding how the two different constructions so far described are related, is not an entirely trivial matter. In particular, we wish to identify a scheme which will allow to qualify both of these connections as being of Berwald type. Note that as a prerequisite, we will have to establish some sort of mechanism for comparing connections on ${\pi_1^0}^*(\tau_E)$ with connections on $J^1\pi$.

We have found excellent guidance for our comparative study in recent work on Finsler and Berwald-type connections within the *autonomous* framework by Anastasiei [1], Szilasi [11] and Crampin [5]. The extra dimension which comes with the *time-dependent* framework apparently leaves us a choice in "fixing the time-component". It turns out that in order to accomodate all existing constructions within an overall scheme, we need to introduce equivalence classes of connections. The final question thus inevitably will be: how should one select an optimal representative of the class of Berwald-type connections?

4. Finsler- and Berwald-type connections

Most of what follows can be developed starting from an arbitary horizontal distribution on $J^1\pi$ (see [8]). To fix the idea, however, we will limit ourselves here to the situation where the data are: a given SODE Γ on $J^1\pi$ and the corresponding horizontal distribution represented by P_H.

Only connections (either on $J^1\pi$ or on $\pi_1^{0*}(\tau_E)$) with the following properties will be taken into account and will characterize what we call connections of Finsler type:

D on $\pi_1^{0*}(\tau_E)$	∇ on $J^1\pi$
$\mathrm{D}_\xi(\overline{\mathcal{X}}(\pi_1^0)) \subset \overline{\mathcal{X}}(\pi_1^0)$	$\nabla_\xi\left(\overline{\mathcal{X}}(\pi_1^0)^H\right) \subset \overline{\mathcal{X}}(\pi_1^0)^H$ $\nabla_\xi\left(\overline{\mathcal{X}}(\pi_1^0)^V\right) \subset \overline{\mathcal{X}}(\pi_1^0)^V$ $\nabla_\xi J\|_{\overline{\mathcal{X}}(J^1\pi)} = 0$

Here J is the degenerate almost complex structure coming from the horizontal distribution: $J(\overline{X}^H) = \overline{X}^V$, $J(\overline{X}^V) = -\overline{X}^H$, $J(\Gamma) = 0$.

So, under these assumptions, (P_H, ∇) is called a *Finsler pair*, and we use the same terminology also for the couple (P_H, D). This may seem a little odd in the latter case, since no horizontal distribution is needed to express the simple assumption on D. However, we need a horizontal distribution when we want to introduce for example a notion of torsion for D (see later) and also when we want to "raise" a given D to a corresponding ∇ (or class of ∇'s) on $J^1\pi$.

Let us first describe the mechanism of raising and lowering connections which will be useful for our purposes.

- For a given pair (P_H, D), we construct a class of ∇ by putting
$$\nabla_\xi \overline{X}^H = (\mathrm{D}_\xi \overline{X})^H, \quad \nabla_\xi \overline{X}^V = (\mathrm{D}_\xi \overline{X})^V, \quad \nabla_\xi \Gamma = K(\xi),$$
where K is a type (1,1) tensor field on $J^1\pi$ which is left free to choose.

  Note that there exists a natural direct formula for constructing a particular ∇ out of a given pair (P_H, D). It is given by
$$\nabla_\xi \eta = (\mathrm{D}_\xi \eta_H)^H + (\mathrm{D}_\xi \overline{\eta}_V)^V$$
and corresponds to making the choice $K(\xi) = (\mathrm{D}_\xi \mathbf{T})^H$ within the above general scheme.
- Conversely, for a given Finsler pair (P_H, ∇), we construct a class of D by putting
$$\mathrm{D}_\xi \overline{X} = (\nabla_\xi \overline{X}^H)_H = (\nabla_\xi \overline{X}^V)_V, \qquad \mathrm{D}_\xi \mathbf{T} = L(\xi),$$
where the $C^\infty(J^1\pi)$-linear map $L : \mathcal{X}(J^1\pi) \to \mathcal{X}(\pi_1^0)$ again is left arbitrary.

D or ∇ now are said to be of *Berwald type* if $\forall \overline{X} \in \overline{\mathcal{X}}(\pi_1^0)$, we have
$$\mathrm{D}_\xi \overline{X} = [P_H(\xi), \overline{X}^V]_V + [P_V(\xi), \overline{X}^H]_H.$$

Clearly, this definition says nothing about the action of D on $\mathbf{T}$. Hence, when the connection we start from is a ∇, the defining relation for being of Berwald type expresses a requirement on any of the D's which correspond to ∇ in the above scheme.

That the direct construction of a D in the preceding section yields a connection of Berwald type is now quite trivial of course. It is shown in detail in [8] that the same is true for the ∇ of Massa and Pagani.

One way of comparing different constructions of Berwald-type connections now, is to look, in some sense, at the difference in the choice of K. More precisely, this can be done as follows: if a D on $\pi_1^{0*}(\tau_E)$ is the starting point, we take the natural direct formula for a corresponding ∇ explained above and read from its action on Γ directly what the tensor field K does. Applied to the D of the previous section, this gives $K(\xi) = \overline{\xi}_V{}^H$.

If, on the other hand, a ∇ on $J^1\pi$ is were we start from, we can look at any of the corresponding D-connections in its restriction to $\overline{\mathcal{X}}(\pi_1^0)$, and then look for the tensor field K which is needed to restore the original ∇. Applied to the ∇ of Massa and Pagani, we get $K = 0$.

At this point, we can mention another ∇ on $J^1\pi$, associated to a given time-dependent SODE, which was constructed independently by Byrnes [3]. It is also a connection of Berwald type in the sense of our present definition and one can verify that the corresponding choice of the tensor field K this time is: $K(\xi) = \overline{\xi}_V{}^H - \Phi(\overline{\xi}_H)^V$, where Φ is the so-called *Jacobi endomorphism* of Γ (see e.g. [6]).

From this first point of comparison, the construction of Byrnes may look like a rather artificial way to proceed, but there is another way of describing the differences which will make it look less exotic.

Note in passing that working with a connection D on $\pi_1^{0*}(\tau_E)$ (where the fibre dimension is $n+1$), is clearly more 'economical' than working with a corresponding ∇ on $J^1\pi$ (with fibre dimension $2n+1$). Roughly speaking, leaving the time-component apart, passing from a D to a ∇ somehow 'doubles the number of formulas'! However, ∇ is needed to give meaning to the notion of *torsion*.

Looking at the torsion is now the second way by which we will compare the three constructions described so far.

A local basis of vector fields on $J^1\pi$ is of the form $\{\Gamma, \overline{X}_i{}^H, \overline{X}_i{}^V\}$, where $\{\overline{X}_i\}$ is a local basis for $\overline{\mathcal{X}}(\pi_1^0)$. The image of the torsion tensor T, when acting on pairs of such vector fields, in turn can be decomposed into horizontal and vertical components. When all such decompositions are consistently taken into account, it turns out that T is completely determined by nine in general non-vanishing type (1,2) tensors along π_1^0. We can call these the 'torsion tensors' for D and they are defined as follows (with notations which match those of [11, 5] for the autonomous case):

$\mathcal{A}(\overline{X},\overline{Y}) = T(\overline{X}^{H},\overline{Y}^{H})_{H}$	$\mathcal{A}_{\mathbf{T}}(\overline{X}) = T(\Gamma,\overline{X}^{H})_{H}$
$\mathcal{R}(\overline{X},\overline{Y}) = T(\overline{X}^{H},\overline{Y}^{H})_{V}$	$\mathcal{R}_{\mathbf{T}}(\overline{X}) = T(\Gamma,\overline{X}^{H})_{V}$
$\mathcal{B}(\overline{X},\overline{Y}) = T(\overline{X}^{H},\overline{Y}^{V})_{H}$	$\mathcal{B}_{\mathbf{T}}(\overline{X}) = T(\Gamma,\overline{X}^{V})_{H}$
$\mathcal{P}(\overline{X},\overline{Y}) = T(\overline{X}^{H},\overline{Y}^{V})_{V}$	$\mathcal{P}_{\mathbf{T}}(\overline{X}) = T(\Gamma,\overline{X}^{V})_{V}$
$\mathcal{S}(\overline{X},\overline{Y}) = T(\overline{X}^{V},\overline{Y}^{V})_{V}$	

Now, for a D of Berwald type, we have

$$\mathcal{B} = \mathcal{P} = \mathcal{S} = 0$$

and in fact (due to the SODE nature of the P_H under consideration) also

$$\mathcal{A} = 0.$$

$\mathcal{R}$ generically will not be zero, since it is essentially the *curvature* of P_H. Thus we see from the left column in the table that for an *autonomous* Γ, Berwald-type means maximally vanishing torsion!

For the *time-dependent* situation, a comparison of the three linear connections under consideration leads to the following conclusions.

- The construction of Byrnes continues the idea of maximally vanishing torsion by fixing the freedom in the time-component exactly in such a way that also
$$\mathcal{A}_{\mathbf{T}} = \mathcal{R}_{\mathbf{T}} = \mathcal{B}_{\mathbf{T}} = \mathcal{P}_{\mathbf{T}} = 0.$$
- For the D of Crampin *et al* (raised to a ∇ by the natural direct formula), we have
$$\mathcal{A}_{\mathbf{T}} = \mathcal{B}_{\mathbf{T}} = \mathcal{P}_{\mathbf{T}} = 0, \quad \text{but} \quad \mathcal{R}_{\mathbf{T}} \neq 0.$$
- In the case of Massa and Pagani finally:
$$\text{only} \;\; \mathcal{A}_{\mathbf{T}} = \mathcal{P}_{\mathbf{T}} = 0 \quad \text{while} \quad \mathcal{B}_{\mathbf{T}} = -I|_{\overline{\mathcal{X}}(\pi_1^0)}.$$

From this point of view, one might say that it is the construction of Massa and Pagani which is the more exotic one! In any event, it is not yet clear from these arguments whether one of the three connections deserves preference over the other.

5. A SIDE STEP

Let U be a type (1,1) tensor field along π_1^0.

Given any horizontal distribution P_H, one can define various lifted tensors on $J^1\pi$, denoted by $U^{H;H}$, $U^{H;V}$, $U^{V;H}$, $U^{V;V}$ respectively, as follows (see [10]):

$$\begin{array}{ll} U^{H;H}(X^H) = U(X)^H, & U^{H;H}(\overline{X}^V) = 0, \\ U^{H;V}(X^H) = U(X)^V, & U^{H;V}(\overline{X}^V) = 0, \\ U^{V;H}(X^H) = 0, & U^{V;H}(\overline{X}^V) = U(\overline{X})^H, \\ U^{V;V}(X^H) = 0, & U^{V;V}(\overline{X}^V) = U(\overline{X})^V. \end{array}$$

The reason why it is forced upon us to look at such tensor fields is that any $\mathcal{U}$ on $J^1\pi$ has a unique decomposition into the form:

$$\mathcal{U} = U_1^{H;H} + U_2^{H;V} + U_3^{V;H} + U_4^{V;V},$$

with $U_2(\mathcal{X}(\pi_1^0)) \in \overline{\mathcal{X}}(\pi_1^0)$, $U_3(\mathbf{T}) = 0$, $U_4(\mathcal{X}(\pi_1^0)) \in \overline{\mathcal{X}}(\pi_1^0)$, and $U_4(\mathbf{T}) = 0$.

Proposition: If (P_H, ∇) is a Finsler pair and D is any associated connection on ${\pi_1^0}^*(\tau_E)$, we have

$$\nabla_\xi \mathcal{U} = 0 \qquad \Longleftrightarrow \qquad \mathrm{D}_\xi U_i = 0,$$

provided that

$$\nabla_\xi \mathbf{T}^H = (\mathrm{D}_\xi \mathbf{T})^H \quad \text{and} \quad \mathrm{D}_\xi \mathbf{T} \in \langle \mathbf{T} \rangle.$$

We discover with this result two quite natural conditions, which in fact have a simple and elegant interpretation. The first condition means that the procedure for raising a D to a corresponding ∇ is taken to be the natural one: $\nabla_\xi \eta = (\mathrm{D}_\xi \eta_H)^H + (\mathrm{D}_\xi \overline{\eta}_V)^V$. With the extra condition $\mathrm{D}_\xi \mathbf{T} \in \langle \mathbf{T} \rangle$, taken together with the restriction on D we started from in the previous section, we will have that D fully respects the natural decomposition

$$\mathcal{X}(\pi_1^0) \equiv \overline{\mathcal{X}}(\pi_1^0) \oplus \langle \mathbf{T} \rangle.$$

We shall take the hint which comes from this side step into account for deciding about the optimal choice of a Berwald-type connection now.

6. AN OPTIMAL REPRESENTATIVE IN THE BERWALD CLASS

Let us come back now to the question whether one of the three constructions of a Berwald-type connection explained before, deserves preference over the others. Closer analysis, in part inspired by the observations of the preceding section, have brought us to the conclusion that none of them is completely satisfactory. Certainly, insisting on maximally vanishing torsion, also in the $\mathbf{T}$-components, does not seem to have any essential advantage in the time-dependent framework. Instead, it looks much more interesting to have, not only

D on $\pi_1^{0*}(\tau_E)$	∇ on $J^1\pi$
$\mathrm{D}_\xi(\overline{\mathcal{X}}(\pi_1^0)) \subset \overline{\mathcal{X}}(\pi_1^0)$	$\nabla_\xi\left(\overline{\mathcal{X}}(\pi_1^0)^H\right) \subset \overline{\mathcal{X}}(\pi_1^0)^H$ $\nabla_\xi\left(\overline{\mathcal{X}}(\pi_1^0)^V\right) \subset \overline{\mathcal{X}}(\pi_1^0)^V$
but also	
$\mathrm{D}_\xi\mathbf{T} \in \langle\mathbf{T}\rangle$	$\nabla_\xi\Gamma \in \langle\Gamma\rangle$.

From this perspective, only the ∇ of Massa and Pagani (which happens to have the most non-zero torsion components) would seem to be satisfactory. That construction, however, as reported in Section 2, clearly suffers from the fact that it is very indirect. In addition, for reasons of 'economy' in the number of connection components, what we really prefer is a connection D on $\pi_1^{0*}(\tau_E)$.

At this point, let us look again at the direct construction formula for $\mathrm{D}_\xi X$ in Section 3. The first two terms of the defining relation are identical to those for the autonomous situation. In fact, the construction of Crampin *et al* originated from copying the formula from the autonomous case and adding a correction term to make sure that D_ξ has the right derivation property for a linear connection.

There is, however, another way of doing this! Indeed, if we replace X by $\overline{X}$ in the first two terms, these still reduce to the same formula in case there is no extra time variable. But the correction term which is needed then is different. We thus come to the following new direct construction of a linear connection on $\pi_1^{0*}(\tau_E)$:

$$\mathrm{D}_\xi X = [P_H(\xi), \overline{X}^V]_V + [P_V(\xi), \overline{X}^H]_H + \xi(\langle X, dt\rangle)\mathbf{T}.$$

It immediately follows that with this D we have: $\mathrm{D}_\xi\mathbf{T} = 0$. Making a choice for $\mathrm{D}_\xi\mathbf{T}$ is the only freedom we have in selecting a representative of the class of Berwald-type connections we introduced, so obviously, the new construction amounts to making the simplest possible choice.

If for any reason, we want to have a corresponding ∇ on jet at our disposal, we can stick to natural 'raising formula' mentioned before, namely: $\nabla_\xi\eta = (\mathrm{D}_\xi\eta_H)^H + (\mathrm{D}_\xi\overline{\eta}_V)^V$. It then follows that also $\nabla_\xi\Gamma = 0$ and in fact, the resulting ∇ then turns out to coincide with the connection of Massa and Pagani!

7. Generalization of other well-known connections in Finsler geometry

We briefly sketch finally how the connections of Cartan, Chern-Rund and Hashiguchi can be generalized to the present framework. Such a generalization merely requires having one extra geometrical object as part of the data, namely a symmetric type (0,2) tensor field along π_1^0. As before, we will consider the case here that the horizontal distribution we start from comes from a SODE Γ, but everything works just as well for any other given horizontal distribution. In almost every step of the constructions which follow, there is freedom again in fixing a $\mathbf{T}$-component, but having now our optimal Berwald-type connection in mind, we will choose such components to be zero also wherever possible. More importantly, however, there is another type of freedom which requires making a choice. Indeed, as we learn from [9] in the context of autonomous, so-called generalized Lagrange spaces, the construction of a metrical connection is unique to within selecting certain torsion components. Following these authors we will fix the analogous torsion components in our time-dependent picture to be zero as well.

So, let g be a symmetric type (0,2) tensor field along π_1^0, with the properties:

$$g(\mathbf{T},\cdot)=0, \quad \text{and } g|_{\overline{\mathcal{X}}(\pi_1^0)} \text{ is non-singular.}$$

Define type (1,2) tensor fields C_V and C_H along π_1^0 by requiring firstly that

$$\begin{aligned} g(C_V(\overline{X},\overline{Y}),\overline{Z}) &= \mathrm{D}_{\overline{X}^V} g\,(\overline{Y},\overline{Z}) + \mathrm{D}_{\overline{Y}^V} g\,(\overline{X},\overline{Z}) - \mathrm{D}_{\overline{Z}^V} g\,(\overline{X},\overline{Y}),\\ g(C_H(X,\overline{Y}),\overline{Z}) &= \mathrm{D}_{X^H} g\,(\overline{Y},\overline{Z}) + \mathrm{D}_{\overline{Y}^H} g\,(X,\overline{Z}) - \mathrm{D}_{\overline{Z}^H} g\,(X,\overline{Z}),\end{aligned}$$

and by fixing the remaining freedom as follows:

$$C_V(\,.\,,\mathbf{T}) = C_V(\mathbf{T},\,.\,) = 0, \qquad C_H(\,.\,,\mathbf{T}) = 0.$$

Let D be the 'optimal' Berwald-type connection of the preceding section. Then, for any other connection $\hat{\mathrm{D}}$ on ${\pi_1^0}^*(\tau_E)$, we know that

$$\hat{\mathrm{D}}_\xi X - \mathrm{D}_\xi X = \delta(\xi, X)$$

defines a tensorial object δ. Splitting ξ, as by now familiar, into its horizontal and vertical components, we can introduce type (1,2) tensor fields δ^V and δ^H along π_1^0 by putting

$$\begin{aligned} &\delta^V(\overline{Z},X) = \delta(\overline{Z}^V,X), \quad \delta^V(\mathbf{T},X)=0,\\ &\delta^H(Z,X) = \delta(Z^H,X).\end{aligned}$$

Since $\mathrm{D}_\xi\mathbf{T}=0$, we shall require to have the property $\hat{\mathrm{D}}_\xi\mathbf{T}=0$ as well, for which the conditions are: $\delta^V(Z,\mathbf{T})=0$ and $\delta^H(Z,\mathbf{T})=0$. Any new connection can now be constructed from the Berwald-type D by making a choice for the non-zero components of δ^V and δ^H. We thus arrive at the following concepts:

- The *Cartan-type connection* on ${\pi_1^0}^*(\tau_E)$ is determined by

$$\delta^V = \tfrac{1}{2}C_V, \qquad \delta^H = \tfrac{1}{2}C_H.$$

- The *Hashiguchi-type connection* on ${\pi_1^0}^*(\tau_E)$ is determined by

$$\delta^V = \tfrac{1}{2}C_V, \qquad \delta^H = 0.$$

- The *Chern-Rund-type connection* on ${\pi_1^0}^*(\tau_E)$ is determined by

$$\delta^V = 0, \qquad \delta^H = \tfrac{1}{2}C_H.$$

One easily proves that the following properties hold true, which are the analogues of the well-known properties of the corresponding connections in classical Finsler geometry: (i) for the Cartan-type connection, we have $\hat{\mathrm{D}}_\xi g=0$; (ii) in the Hashiguchi case: $\hat{\mathrm{D}}_{\overline{X}^V} g=0$; (iii) for the Chern-Rund-type connection: $\hat{\mathrm{D}}_{X^H} g=0$.

As is customary: making the connection more metrical, also in this more general set-up, is at the expense of introducing more torsion.

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