Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry, 25–30 July, 2000, Debrecen, Hungary

LOCAL AND GLOBAL SOLUTIONS OF THE NAVIER-STOKES EQUATION

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ABSTRACT. A brief report is given on our recent results [5,6] proving existence of (smooth) global solutions of the $3D$ nonisothermal Navier-Stokes equation (NS), and (non)uniqueness of such solutions for (smooth) boundary value problems.

1. GEOMETRY OF THE NAVIER-STOKES EQUATION AND LOCAL AND GLOBAL EXISTENCE THEOREMS

The non-isothermal Navier-Stokes equation (NS), for incompressible fluids, on the Galilean space-time M results a 74-dimensional submanifold of the second order jet-derivative space $J\mathcal{D}^2(W)$ [2], where $W \equiv J\mathcal{D}(M) \times_M T^0_0 M \times_M T^0_0 M$, with $J\mathcal{D}(M) \equiv$ first order jet-derivative space for motions, i.e., first derivative of sections $m : T \to M$ of the Galilean affine fiber bundle $\tau : M \to T$, where T represents the time axis. $T^0_0 M \equiv M \times \mathbf{R}$. A section $s : M \to W$, of $\pi : W \to M$ is a triplet $s = (v, p, \theta)$, where $v \equiv$ velocity vector field, $p \equiv$ isotropic pressure field, $\theta \equiv$ temperature field. More precisely, with respect to an inertial frame ψ, (NS) is defined by the following equations:

$$(1) \qquad (NS) \subset J\mathcal{D}^2(W) : \left\{ \begin{array}{c} \mathrm{div}(\rho v) = 0 \\ \mathrm{div}[\rho v \otimes v - P] - \rho B = 0, \quad P = -pg + 2\chi\dot{e} \\ \rho\frac{\delta e}{\delta t} = < P, \dot{e} > -\mathrm{div}(q), \qquad q = \nu \mathrm{grad}(\theta) \end{array} \right\}$$

with $\dot{e}$ the infinitesimal strain, $g =$ the vertical metric field of M, $\chi =$ viscosity, $v =$ velocity field, $\nu =$ thermal conductivity, $e =$ internal energy, $B =$ body volume force. To the above equations we must add the thermodynamic constitutive equations for e, χ and ν. We can assume $e = e(\theta)$, $\nu = \nu(\theta)$ and $\chi = \chi(\theta)$. For the sake of semplicity we shall assume that ν and χ are constant and $C_p \equiv (\partial\theta.e) =$ constant. Furthermore, we shall assume the body volume force conservative: $B = -\mathrm{grad}(f)$. Let us consider a coordinate system adapted to the frame ψ: $M : (x^\alpha)$, $0 \leq \alpha \leq 3$; $W : (x^\alpha, \dot{x}^i, p, \theta)$, $1 \leq i \leq 3$; $J\mathcal{D}(W) : (x^\alpha, \dot{x}^i, p, \theta, \dot{x}^i_\beta, p_\beta, \theta_\beta)$, $0 \leq \beta \leq$

Received by the editors 26 July 2000, revised 20 May 2001.
Work partially supported by grants MURST "Geometry of PDEs and Applications" and GNFM/INDAM.

3. Then, (NS) can be written in the following form:

(2)
$$(NS)\left\{\begin{array}{l} F^0\equiv\dot{x}^k G^j_{jk}+\dot{x}^i_s\delta^s_i=0 \\ F^j\equiv\dot{x}^s R^j_s+\dot{x}^s\dot{x}^i\rho G^j_{is}+\dot{x}^s\dot{x}^j_s\rho+\rho\dot{x}^j_0+\dot{x}^k_s S^{js}_k+\dot{x}^k_{is}T^{jis}_k+p_i g^{ij}+\rho(\partial x_i.f)g^{ij}=0, \quad 1\leq j\leq 3 \\ F^4\equiv\theta_0\rho C_p+\dot{x}^k\theta_k\theta_{is}\bar{E}^{is}\rho C_p+\dot{x}^k\dot{x}^p W_{kp}+\dot{x}^k\dot{x}^s_p\overline{W}^p_{ks}+\dot{x}^k_i\dot{x}^s_p Y^{ip}_{ks}=0 \end{array}\right\}$$

with

$$(3)\qquad \left\{\begin{array}{l} R^j_s\equiv-2\chi[G^i_{ik}(\partial x_s.g^{kj})+G^j_{ki}(\partial x_s.g^{ki})+(\partial x_i.\partial x_s.g^{ij})] \\ S^{js}_k\equiv-2\chi[-G^i_{ik}g^{js}-G^i_{ip}g^{ps}\delta^j_k-2G^j_{ik}g^{is}+\delta^j_k(\partial x_p.g^{ps})] \\ T^{jis}_k\equiv 2\chi(g^{sj}\delta^i_k+g^{is}\delta^j_k) \\ \bar{E}^{is}\equiv-\nu g^{is} \\ W_{kp}\equiv-2\chi G^j_{ik}g_{js}(\partial x_p.g^{si}) \\ \overline{W}^p_{ks}\equiv 2\chi[G^j_{ik}g_{js}g^{ip}+G^p_{sk}+(\partial x_k.g^{pe})g_{se}] \\ Y^{ip}_{ks}\equiv 2\chi[g_{ks}g^{ip}+\delta^p_k\delta^i_s] \end{array}\right\},1\leq i,j,k,p,s\leq 3$$

Here G^i_{jp} are the spatial components of the connection coefficients of the canonical connection on the Galilean space-time. Then, (NS) results an algebraic submanifold of $J\mathcal{D}^2(W)$. The following theorem gives an important structure property of the Navier-Stokes equation.[1]

Theorem 1. *Equation (NS) is an involutive but not formally integrable, and neither completely integrable, PDE of second order on the fiber bundle $\pi : W \to M$. Into (NS) we can distinguish an important sub-PDE $\widehat{(NS)}$:*

$$(4)\qquad \begin{array}{ccccccc} \widehat{(NS)} & \subset & (NS) & \subset & J\mathcal{D}^2(W) & \subset & J^2_4(W) \\ \boxed{70} & < & \boxed{74} & < & \boxed{79} & = & \boxed{79} \end{array}$$

where the framed numbers denote the dimensions of the corresponding overstanding objects.[2] $\widehat{(NS)}$ is an involutive formally integrable PDE, but with zero characteristic distribution. (Also the characteristic distribution of (NS) is zero.) Of course for the set of solutions $\mathcal{S}ol(-)$ of these equations one has the following inclusions: $\mathcal{S}ol\widehat{(NS)} \subset \mathcal{S}ol(NS)$. Furthermore, one has the following exact commutative diagram:

$$(5)\qquad \begin{array}{ccccc} 0 & \to & \widehat{(NS)}_{+1} & \to & J\mathcal{D}^3(W) \\ & & \downarrow & & \downarrow \\ 0 & \to & \widehat{(NS)} & \to & J\mathcal{D}^2(W) \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

[1] This and the following theorems in this section will be stated without proofs. These, and other informations, can be found in refs.[3,4,5].

[2] $J^2_4(W)$ is the second order jet space for 4-dimensional submanifolds of W [1–5]. $J\mathcal{D}^2(W)$ is an open submanifold of $J^2_4(W)$.

where the index "+1" denotes "first prolongation". The local expression of $\widehat{(NS)}$ *is the following:*

$$
(6)\qquad \widehat{(NS)}\left\{\begin{array}{l}
F^0\equiv\dot{x}^kG^j_{jk}+\dot{x}^i_s\delta^s_i=0\\
F^0_\alpha\equiv\dot{x}^k(\partial x_\alpha.G^j_{jk})+\dot{x}^k_\alpha G^j_{jk}+\dot{x}^i_{s\alpha}\delta^s_i=0\\
F^j\equiv\dot{x}^sR^j_s+\dot{x}^s\dot{x}^i\rho G^j_{is}+\dot{x}^s\dot{x}^j_s\rho+\rho\dot{x}^j_0+\dot{x}^k_sS^{js}_k+\dot{x}^k_{is}T^{jis}_k+p_ig^{ij}+\rho(\partial x_i.f)g^{ij}=0\\
F^4\equiv\theta_0\rho C_p+\dot{x}^k\theta_k\theta_{is}\bar{E}^{is}\rho C_p+\dot{x}^k\dot{x}^pW_{kp}+\dot{x}^k\dot{x}^s_p\overline{W}^p_{ks}+\dot{x}^k_i\dot{x}^s_pY^{ip}_{ks}=0\\
0\leq\alpha\leq3,\quad 1\leq j\leq3
\end{array}\right\}
$$

where the symbols of the coefficients are like before. (NS) *is an affine fiber bundle over the affine submanifold* $(C)\subset J\mathcal{D}(W)$*. More precisely, we can write* $(NS)=\bigcup_{\bar{q}\in(C)}(NS)_{\bar{q}}$*, where* $(NS)_{\bar{q}}$ *is a* 46*-dimensional affine submanifold of* $\pi^{-1}_{2,1}(\bar{q})$ *with associated vector space the symbol* $(g_2)_q$ *of* (NS) *at any point* $q\in(NS)_{\bar{q}}$*. Furthermore,* (C) *is a fiber bundle over* W*. One has the following exact commutative diagrams:*

(7)

$$
\begin{array}{ccccccc}
J\mathcal{D}^2(W) & \supset & (NS) & \to & W & \to & 0\\
\downarrow\pi_{2,1} & & \downarrow & & \| & & \\
J\mathcal{D}(W) & \supset & (C) & \to & W & \to & 0\\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}
\qquad 0\to g_2\to vT(NS)\to\pi^*_{2,1}vT(C)\to0
$$

The Cartan distribution $\mathbf{E}_2\widehat{(NS)}$ *of* $\widehat{(NS)}$ *is the Cauchy characteristic distribution associated to the contact ideal* $\mathcal{C}_2\subset\Omega^\bullet(J^2_4(W))$ *generated by the following differential forms:*

$$
(8)\qquad \left\{\begin{array}{l}
\epsilon^0\equiv dF^0=(\partial x_\alpha.G^j_{jk})\dot{x}^kdx^\alpha+G^j_{jk}d\dot{x}^k+d\dot{x}^k_s\delta^s_k\\
\epsilon^0_\alpha\equiv dF^0_\alpha=[(\partial x_\beta\partial x_\alpha.G^j_{jk})\dot{x}^k+(\partial x_\beta.G^j_{jk})\dot{x}^k_\alpha]dx^\beta+(\partial x_\alpha.G^j_{jk})d\dot{x}^k+G^j_{jk}d\dot{x}^k_\alpha+d\dot{x}^i_{i\alpha}\\
\epsilon^j\equiv dF^j=A^j_\alpha dx^\alpha+B^j_id\dot{x}^i+\rho d\dot{x}^j_0+S^{js}_kd\dot{x}^k_s+T^{jis}_kd\dot{x}^k_{is}+g^{kj}dp_k\\
\epsilon^4\equiv dF^4=\bar{A}_\beta dx^\beta+\bar{B}_kd\dot{x}^k+\bar{C}^\alpha d\theta_\alpha+\bar{D}^s_pd\dot{x}^p_s+\bar{E}^{is}d\theta_{is}\\
\omega^j\equiv d\dot{x}^j-\dot{x}^j_\alpha dx^\alpha,\quad \omega^j_\alpha\equiv d\dot{x}^j_\alpha-\dot{x}^j_{\alpha\beta}dx^\beta\\
\omega^4\equiv dp-p_\alpha dx^\alpha,\quad \omega^4_\alpha\equiv dp_\alpha-p_{\alpha\beta}dx^\beta\\
\omega^5\equiv d\theta-\theta_\alpha dx^\alpha,\quad \omega^5_\alpha\equiv d\theta_\alpha-\theta_{\alpha\beta}dx^\beta\\
1\leq j\leq3,\quad 0\leq\alpha\leq3
\end{array}\right\}
$$

with

$$
(9)\qquad \left\{\begin{array}{l}
A^j_\alpha\equiv\dot{x}^s(\partial x_\alpha.R^j_s)+\dot{x}^i\dot{x}^s\rho(\partial x_\alpha.G^j_{is})+\dot{x}^k_s(\partial x_\alpha.S^{js}_k)+\dot{x}^k_{is}(\partial x_\alpha.T^{jis}_k)\\
+\rho(\partial x_\alpha.g^{ij})(\partial x_i.f)+\rho g^{ij}(\partial x_\alpha\partial x_j.f)+(\partial x_\alpha.g^{ij})p_i\\
B^j_i\equiv\dot{x}^k2\rho G^j_{ik}+\dot{x}^j_i\rho+R^j_i\\
\bar{A}_\beta\equiv\theta_{is}(\partial x_\beta.\bar{E}^{si})+\dot{x}^k\dot{x}^p(\partial x_\beta.W_{kp})+(\partial x_\beta.\bar{W}^p_{ks})+\dot{x}^k\dot{x}^s_p(\partial x_\beta.Y^{ip}_{ks})\\
\bar{B}_k\equiv\theta_k\rho C_p+\dot{x}^p(W_{kp}+W_{pk})+\dot{x}^s_p\bar{W}^p_{ks}\\
\bar{C}^\alpha\equiv\dot{x}^\alpha\rho C_p\ \ (\dot{x}^0=1)\\
\bar{D}^p_s\equiv\dot{x}^k\bar{W}^p_{ks}+2\dot{x}^k_iY^{ip}_{ks}\\
1\leq i,j,k,p,s\leq3,\quad 0\leq\alpha,\beta\leq3
\end{array}\right\}
$$

Therefore

$$\mathcal{C}_2 =< \omega^I, \omega^I_\alpha, \epsilon^J, \epsilon^0_\alpha >, \quad 1 \leq I \leq 5, \quad 0 \leq J \leq 4, \quad 0 \leq \alpha \leq 3. \tag{10}$$

The Cartan distribution $\mathbf{E}_2\widehat{(NS)}$ *is generated by vector fields of the following type:*

(11)

$$\zeta = X^\alpha[\partial x_\alpha + \dot{x}^j_\alpha \partial \dot{x}_j + p_\alpha \partial p + \theta_\alpha \partial \theta + \dot{x}^j_{\alpha\beta} \partial \dot{x}^\beta_j + p_{\alpha\beta} \partial p^\beta + \theta_{\beta\alpha} \partial \theta^\beta] + \dot{X}^j_{\alpha\beta} \partial \dot{x}^{\alpha\beta}_j + Y_{\alpha\beta} \partial p^{\alpha\beta} + Z_{\alpha\beta} \partial \theta^{\alpha\beta}$$

where the components X^α, $\dot{X}^j_{\alpha\beta}$, $Y_{\alpha\beta}$, $Z_{\alpha\beta}$ *satisfy the following equations:*

$$\left\{ \begin{array}{l} X^\alpha[A^j_\alpha + B^j_i \dot{x}^i_\alpha + \rho \dot{x}^j_{0\alpha} + S^{js}_k \dot{x}^k_{s\alpha} + g^{kj} p_{\alpha k}] + T^{jis}_k \dot{X}^k_{is} = 0 \\ X^\alpha[\bar{A}_\alpha + \bar{B}_k \dot{x}^k_\alpha + \bar{C}^s \theta_{\alpha s} + \rho C_p \theta_{0\alpha} + \bar{D}^s_k \dot{x}^k_{s\alpha}] + \bar{E}^{is} Z_{is} = 0 \\ X^\beta[(\partial x_\beta \partial x_\alpha . G^j_{jk}) \dot{x}^k + (\partial x_\beta . G^j_{jk}) \dot{x}^k_\alpha + \dot{x}^k_\beta (\partial x_\alpha . G^j_{jk}) + \dot{x}^k_{\beta\alpha} G^j_{jk}] + \dot{X}^s_{\alpha s} = 0 \\ 1 \leq j \leq 3, 0 \leq \alpha \leq 3 \end{array} \right\}. \tag{12}$$

Therefore, $\mathbf{E}_2\widehat{(NS)}$ *is a distribution of dimension* 46*:* $\dim \mathbf{E}_2\widehat{(NS)} = 46$*. (The Cartan distribution* $\mathbf{E}_2(NS)$ *on* (NS) *is of dimension* 50*.)*

Theorem 2. *(***Existence of local regular solutions for** (NS)*). For any initial condition of* (NS) *belonging to the augmented equation* $\widehat{(NS)}$*, i.e., for any point* $q \in \widehat{(NS)} \subset (NS)$*, we can construct a (smooth) regular solution of* $(NS) \subset J^2_4(W)$*, that is a* 4*-dimensional (smooth) submanifold* $V \subset W$ *such that its second holonomic prolongation* $V^{(2)}$ *is contained into* (NS) *and* $q \in V^{(2)}$*.*

Theorem 3. *(***Cauchy problem for** $(NS) \subset J^2_4(W)$*). Any (regular) integral submanifold* $N \subset \widehat{(NS)} \subset (NS) \subset J^2_4(W)$ *of dimension* 3 *that satisfies the initial conditions and the transversality conditions, with respect to a time-like vector field* $\zeta \in \mathfrak{s}(\mathcal{C}_2)$*, (where* $\mathcal{C}_2$ *is the ideal encoding* $\widehat{(NS)}$ *given in* Theorem 1*, and* $\mathfrak{s}$ *denotes "infinitesimal symmetry"), generates a (regular, smooth) solution of* (NS) *if* N *has no frozen singularities* [1].

In order to pass from local existence theorems to global ones and characterize the topology of these global solutions, it is necessary to consider the integral bordism group in dimension 3 of (NS). In the following we give a short summary of these concepts. (For more details and proofs see refs.[2,3,5]. See also refs.[7] for some further applications.)

We shall say that a p-dimensional, $p \leq 3$, compact closed smooth integral manifold $N \subset (NS)$ is **admissible** if the following conditions hold: (i) The set $\Sigma(N)$ of singular points of N has no open subsets and has no frozen singularities; (ii) For N passes at least one integral manifold V of dimension $p+1$ such that its set $\Sigma(V)$ of singular points has no open subset; (iii) $\Sigma(N)$ can be solved by means of integral deformations. From Theorem 3 it follows that the set of such admissible p-dimensional, $p \leq 3$, compact closed smooth integral manifolds is not empty. We define **integral bordism** in (NS) the usual bordism [2], but with the additional requirement that the manifolds contained in (NS) must be admissible integral ones. We denote by $\Omega^{(NS)}_p$ the set of integral bordism classes $[N]_{(NS)}$ of

(NS) with $0 \leq p \leq 3$. The operation of taking disjoint union $\dot{\bigcup}$ defines a sum $+$ on $\Omega_p^{(NS)}$ so that it becomes an abelian group (**integral p-bordism group** of (NS)). We call **bar singular chain complex** of (NS), with **coefficients into an abelian group** G, the chain complex $\{\bar{C}_p((NS);G),\bar{\partial}\}$ where $\bar{C}_p((NS);G)$ is the G-module of formal linear combinations with coefficients in G, $\sum_i \lambda_i c_i$, where c_i is a singular p-chain $f:\Delta^p \to (NS)$ that extends on a neighborhood $U \subset \mathbf{R}^{p+1}$, such that f on U is differentiable and $Tf(\Delta^p) \subset \mathbf{E}_2(NS)$. Denote by $\bar{H}_p((NS);G)$ the corresponding homology (**bar singular homology with coefficients in G of** (NS)). Let $\{\bar{C}^p((NS);G) \equiv Hom_{\mathbf{Z}}(\bar{C}_p((NS);\mathbf{Z}),G),\bar{\delta}\}$ be the corresponding dual complex and let $\bar{H}^p((NS);G)$ be the associated homology spaces (**bar singular cohomology with coefficients into G of** (NS)). Let us denote by ${}^G\Omega_{p,s}^{(NS)}$ the corresponding bordism groups in the singular case. Let us denote also by ${}^G[N]_{(NS)}$ the equivalence classes of integral singular bordisms. In the following, for semplicity, we will consider the case $G = \mathbf{R}$ only and we will omit the apex G in the symbols.

Theorem 4. *(***Integral bordisms groups in** (NS)*). In the following table we summarize the calculated integral bordism groups of (NS).*

Table: Integral bordism groups in the Navier-Stokes equation

p	$\Omega_p^{(NS)}$	$\Omega_p^{(NS)+\infty}$	$\Omega_{p,s}^{(NS)}$	$\Omega_{p,s}^{(NS)+\infty}$
0	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{R}$	$\mathbf{R}$
1	0	0	0	0
2	$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	0
3	0	0	0	0

Theorem 5. *(***Tunnel effects in the Navier-Stokes equation***). In the set of p-dimensional, $2 \leq p \leq 4$, integral manifolds of (NS) one has manifolds that change their sectional topology (***tunnel effect***). In particular, in the set $\mathcal{S}ol(NS)$, of all solutions of the Navier-Stokes equation, there are also solutions V that present (multi)tunnel effects. This means that V are 4-dimensional amissible integral manifolds contained into (NS) on which there exist Morse functions $f: V \to [a,b]$, with $\partial V = f^{-1}(a)\dot{\bigcup} f^{-1}(b)$ where a and b are regular values, such that there are disjoint k-cells $e_i^k \subset V \setminus N_1$ and disjoint $(4-k)$-cells $(e_*)_i^{4-k} \subset V \setminus N_0$, $1 \leq i \leq \nu_k$ =number of critical points with index k, $k = 0,\cdots,4$, such that:* (i) *$e_i^k \cap N_0 = \partial e_i^k$;* (ii) *$(e_*)_i^{4-k} \cap N_1 = \partial(e_*)_i^{4-k}$;* (iii) *there is a deformation retraction of V onto $N_0 \cup \{\cup_{i,k} e_i^k\}$;* (iv) *there is a deformation retraction of V onto $N_1 \cup \{\cup_{i,k}(e_*)_i^{4-k}\}$;* (v) *$(e_*)_i^{4-k} = q_i \in V$; $(e_*)_i^{4-k} \pitchfork e_i^k$, ($\pitchfork$ denotes transverse). These solutions are not, in general, representable are image of second derivative of sections of the fiber bundle $\pi: W \equiv J\mathcal{D}(M) \times_M T_0^0 M \times_M T_0^0 M \to M$.*

2. (Non) uniqueness theorems for boundary value problems

Definition 6. *We define* **global (smooth) solution** *of* (NS) *any (smooth) solution of* (NS) *that is defined in any space-like region inside the boundary and for any* $t \geq t_0$*, where* t_0 *is an initial time.*

Theorem 7. (**Existence of global (smooth) solutions of** (NS)). *For a generic (smooth) boundary value problem contained into* $\widehat{(NS)} \subset (NS)$ *exist global (smooth) solutions of* (NS).

Proof. A boundary value problem for (NS) can be directly implemented in the manifold $(NS) \subset J\mathcal{D}^2(W) \subset J^2_4(W)$ by requiring that a 3-dimensional compact space-like (for some $t = t_0$), admissible integral manifold $B_{t_0} \subset \widehat{(NS)} \subset (NS)$ propagates in (NS) in such a way that the boundary ∂B describes a fixed 3-dimensional time-like integral manifold $Y \subset \widehat{(NS)} \subset (NS)$.[3] Y is not, in general, a closed (smooth) manifold. However, we can solder Y with two other compact 3-dimensional integral manifols X_i, $i = 1, 2$, in such a way that the result is a closed 3-dimensional (smooth) integral manifold $Z \subset \widehat{(NS)} \subset (NS)$. More precisely, we can take $X_1 \equiv B$, so that $\widetilde{Z} \equiv X_1 \bigcup_{\partial B} Y$ is a 3-dimensional compact integral manifold such that $\partial\widetilde{Z} \equiv C$ is a 2-dimensional space-like integral manifold. We can assume that C is an orientable manifold. Then from Theorem 4 it follows that $\partial X_2 = C$, for some space-like compact 3-dimensional integral manifold $X_2 \subset \widehat{(NS)}$. Set $Z \equiv \widetilde{Z} \bigcup_C X_2$. Therefore, one has $Z = X_1 \bigcup_{\partial B} Y \bigcup_C X_2$. Then, from Theorem 4 it follows that there exists a 4-dimensional integral (smooth) manifold $V \subset \widehat{(NS)} \subset (NS)$ such that $\partial V = Z$. It follows that the integral manifold V is a solution of our boundary value problem between the times t_0 and t_1, where t_0 and t_1 are the times corresponding to the boundaries where are soldered X_i, $i = 1, 2$ to Y. Now, this process can be extended for any $t_2 > t_1$. So we are able to find (smooth) solutions for any $t > t_0$. Therefore we are able to find global (smooth) solutions. Remark that in order to assure the smoothness of the global solutions so built it is enough to develop such constructions in the infinity prolongation $(NS)_{+\infty}$ of (NS). □

Theorem 8. *Boundary conditions do not assure, in general, the uniqueness of solutions.*

Proof. We shall use the following lemmas. (For their proofs see ref.[6].)

Lemma 9. *For any admissible* 3*-dimensional space-like integral manifold there exist (global) solutions of* (NS)*, but these are not unique.*

Lemma 10. *For the Navier-Stokes equation with viscosity* $\chi \neq 0$*, the Reynolds number identifies characteristic nunbers associated to* 3*-dimensional, (resp.* 1*-dimensional closed), space-like submanifolds* $B_t \subset V$ *of solutions* $V \subset (NS)$*,*

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[3] We shall require that the boundary ∂B_{t_0} of B is orientable.

but these numbers are not, in general, characteristic for the solutions V, i.e., the Reynolds number does not identify 1*-conservation law and neither conservation laws for* (NS).[4]

Lemma 11. *In the set $\mathcal{S}ol(NS)$ of solutions of the Navier-Stokes equation (NS), there are also ones where are present different domains, some corresponding to stable flows and other to unstable flows.*

Let us consider, now, a boundary value problem for $(NS) \subset J\mathcal{D}^2(W) \subset J^2_4(W)$ defined by means of a 3-dimensional compact space-like (for some $t = t_0$), admissible integral manifold $B \subset (NS)$ propagating in (NS) in such a way that the boundary ∂B describes a fixed 3-dimensional time-like integral manifold $Y \subset \widehat{(NS)}$. Now, from Theorem 7 and Lemma 9, it follows that we can find many "characteristic vectors" that agree on the boundary Y, i.e., many global (smooth) solutions that satisfy fixed boundary conditions. In fact, let us consider Y diffeomorphic to a manifold $X \subset W$ by means of the projection $\pi_{2,0} : J^2_4(W) \to W$. Let us assume, for example, that X is topologically equivalent to S^3: $X \simeq S^3$. As W is an affine fiber bundle over M of dimension 9, it follows that there are many 4-dimensional disks D^4 such that $\partial D^4 = S^3$. In other words, there are many compact smooth 4-dimensional manifolds $Z \subset W$ such that $\partial Z = X$. If we consider their second holonomic prolongation $Z^{(2)} \subset J^2_4(W)$ one has $\partial Z^{(2)} = Y$. In general $Z^{(2)}$ is not contained into (NS). However, as (NS) is a deformation retract of $J^2_4(W)$, we can deform $Z^{(2)}$ in such a way to become a regular integral manifold $V \subset (NS)$ with $\partial V = Y$. This is equivalent to say that (NS) is of (p)-homotopy type, $0 \leq p \leq 3$. (For a detailed proof see ref.[5].) Of course, as $\dim(NS) = 74$ and $\dim \mathbf{E}_2(NS) = 50$, at different Z there correspond, in general, different integral manifolds V such that $\partial V = Y$. Furthermore, taking into account the triviality of the first integral bordism group $\Omega_1^{(NS)}$ of (NS), we can have also solutions that produce some singular points in the corresponding flows. In fact, let us consider a loop in B_{t_0}. Its propagation could produce a tunnel effect forming, at some instant $t > t_0$, n loops, since the first integral bordism group, $\Omega_1^{(NS)}$, of (NS) is trivial: $\Omega_1^{(NS)} = 0$. This corresponds to the production of a tunnel effect in the integral flow lines of the fluid. These are, of course, compatible with the boundary conditions. Sources of such tunnel effects are eventual unstability of flows. In fact, as Lemma 10 and Lemma 11 say, a 3-dimensional space-like submanifold $B_t \subset V$, where V is a solution of (NS), can evolve following other solutions $V' \neq V$, passing for B_t, when the solution V at B_t becomes unstable, (e.g., if the characteristic Reynolds number $R_e(t)$ of B_t exceds the critical Reynolds number). (For details see ref.[6].) Therefore, we can conclude that, since (NS) is a PDE that satisfies

[4]Recall that a p-conservation law, $0 \leq p \leq 3$, is a function on the integral bordism groups $\Omega_p^{(NS)}$ or $\Omega_p^{(NS)_{+\infty}}$. A conservation law for (NS) is a differential 3-form α on $J\mathcal{D}^\infty(W)$ such that $\alpha|_{V^{(\infty)}} = 0$, for any solution $V^{(\infty)} \subset (NS)_{+\infty}$. Any conservation law identifies a 3-conservation law. (For more details see refs.[2,3,5].)

the (p)-homotopy principle, $0 \leq p \leq 3$, and standing the triviality of the bordism groups $\Omega_1^{(NS)}$ and $\Omega_1^{(NS)_{+\infty}}$, (as well as $\Omega_{1,s}^{(NS)}$ and $\Omega_{1,s}^{(NS)_{+\infty}}$), boundary conditions do not assure, in general, the uniqueness of solutions. □

Remark. From Theorem 8 it follows as a by-product that the Navier-Stokes equation produces solutions which exihibit finite-time singularities. But this does not contradicts Theorem 7 that states that there are global smooth solutions for any "*initial condition*" in the sub-equation $\widehat{(NS)}$ of the Navier-Stokes equation. In fact a global smooth solution may be eventually unstable and, therefore, will not be possible to completely observe it in experiments. (Here "completely" means "for any time".) □

Example 12. *(***Incompressible Newtonian fluid: Isothermal nonsteady state laminar flow in circular pipe***). Let us assume: (a) the pipe is horizontal and very long* (L)*; (b) there is not flow for* $t<0$*; for* $t=0$ *it is applied a gradient of pressure:* $\frac{P_0-P_L}{L}$*; (c) the field of velocity has the following structure:* $(v_j)=(v_r=0, v_\theta=0, v_z=v_z(r,t))$*. Then the motion equation and continuity equations give the following equation:*

$$\left\{ \begin{array}{l} \rho(\partial t \cdot v_z)=\frac{P_0-P_L}{L}+\chi\frac{1}{r}[\partial r\cdot(r(\partial r\cdot v_z))] \\ \text{boundary conditions:}\left\{ \begin{array}{l} v_z(r,0)=0, \quad 0\leq r\leq R \\ v_z(0,t)=\text{finite} \\ v_z(R,t)=0. \end{array} \right\} \end{array} \right\} \tag{13}$$

As a consequence we get:

$$v_z(r,t)=\frac{(P_0-P_L)R^2}{4\chi L}[(1-(\tfrac{r}{R})^2)-8\sum_{n=1}^{\infty}\frac{J_0(\alpha_n\frac{r}{R})}{\alpha^3{}_n J_1(\alpha_n)}e^{-\alpha_n^2\tau}] \tag{14}$$

where $\tau\equiv\chi t/\rho R^2$*,* $\alpha_n=$ *zeros of Bessel function* J_0*,* $J_i\equiv$ *Bessel functions with* $i\geq 0$*.*[5] *Therefore, it should appear, from above calculations, that the solution is unique for the fixed boundary conditions. But all depends on the fact that we have just at the beginning fixed a flow type (laminar flow). Of course many other flow types can occur with the same boundary conditions! In fact, it is well known that laminar flows are unstable for high Reynolds numbers. Of course these unstable and turbulent flows respect the same boundary conditions than laminar flows.*

The Euler equation, (E), for incompressible fluids can be considered the limit case of the Navier-Stokes equation (NS) for zero viscosity ($\chi = 0$) and zero thermal conductivity ($\nu = 0$), (or, equivalently, for isothermal case). Of course, for such an equation continue to hold Theorem 2, Theorem 3, Theorem 4, Theorem 5, Theorem 7 and Theorem 8 by simply substituting (NS) and $\widehat{(NS)}$ with (E), but there are not more informations coming from Reynolds number as it is always ∞-valued. In fact, in the case of the Euler equation Lemma 10 does not work more, (as $\chi = 0$). We have, instead, an important 1-conservation law, absent in the Navier-Stokes

[5]For details on the calculations see ref.[6].

equation. However, this conservation law does not guarantees, in general, the conservation of the local vorticity. In fact, we have the following theorem.

Theorem 13. *The vorticity (or spin) identifies a* 1*-conservation law (***strenght of the vortex***) $f : \overline{\Omega}_1^{(E)+\infty} \to \mathbf{R}$ on the first causal integral bordism group $\overline{\Omega}_1^{(E)+\infty}$ of the Euler equation.*[6] *This is, in general, a partial obstruction to the change of the local vorticity.*[7] *It becomes a full obstruction in the $D = 2$ case.*[8]

Proof. See ref.[6]. □

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[6]Emphasize that such a conservation law does not exist for the Navier-Stokes equation.

[7]This interpretes the well known phenomenon called **enstrophy**[6].

[8]In this case, a global smooth solution that at some instant t_0 has zero vorticity, will remain irrotational for any $t>t_0$. (This agrees with a well known behaviour of the Euler equation.)