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PROJECTIVE AND INDUCTIVE LIMITS OF DIFFERENTIAL TRIADS

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ABSTRACT. We prove that in the category of differential triads projective and inductive systems have limits.

1. INTRODUCTION

The geometry of differential manifolds is a very effective machinery to deal with problems in many fields of pure mathematics and numerous applications. In particular, it is the underlying mathematical theory for the contemporary (non-quantum) mechanics, relativity and cosmology.

However, this machinery does not work when the smooth manifold structure breaks down, for instance, when singularities appear, as the big bang or the black holes. On the other hand, the differential manifold structure is so strong an assumption, that it is not preserved under the most common operations on manifolds. For example, one cannot pull-back or push-out an atlas by a continuous map. An arbitrary subset of a manifold is not a manifold. The limit of a projective system of manifolds is not, in general, a manifold. In some cases, as in the theory of jets, it is a manifold, but it is infinite dimensional. Thus, the category of smooth, finite dimensional manifolds is not closed for projective limits. The same is true for inductive limits of manifolds.

These deficiencies led many authors to introduce several generalizations of the notion of manifolds (see, for instance, *differential spaces* [1, 5, 8, 9]), so that the differential mechanism is preserved but the manifold structure is not needed. Among these generalizations, the most recent and most general is that of *differential triads*, introduced by A. Mallios in [2], by replacing the the assumptions on the local structure of the space X (charts, atlases) with assumptions on the existence of an (algebraic) derivation on an arbitrary sheaf $\mathcal{A}$ of algebras on X, which plays the rôle of the structural sheaf of germs of smooth functions. Differential triads generalize smooth manifolds (and differential spaces) and also include (non-smooth) spaces with very general, non-functional, structural sheaves. As it happens, any sheaf of algebras may be regarded as the structural sheaf of a differential triad (Example 2.1(ii)).

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Analogously to the considerations on the space X, one may replace the usual smooth maps between manifolds with an algebraically defined class of "differentiable maps", or "morphisms of differential triads", giving rise to a category, denoted by $\mathcal{DT}$, in which the category of manifolds is embedded [6].

Our aim in this paper is to prove that $\mathcal{DT}$ is closed for the projective and inductive limits of differential triads. To this end, we first prove that a projective (resp. inductive) system of differential triads over the same base space has a limit in $\mathcal{DT}$ (Propositions 3.2 and 3.3). Then, for a projective (resp. inductive) system of differential triads over a projective system of base spaces, we construct a new projective (resp. inductive) system of differential triads over the projective (resp. inductive) limit of the base spaces, and we prove that their limit satisfies the universal property of the projective (resp. inductive) limit for the initially given family (Theorems 4.4 and 4.5).

2. Preliminaries

Throughout the paper, $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$; if X is a topological space, τ_X denotes its topology. First we recall the main concept in our framework, introduced by A. Mallios (see [2], or [3, Vol. II]):

Let X be a topological space, $\mathcal{A}$ a sheaf of unital, commutative, associative $\mathbb{K}$-algebras over X and Ω an $\mathcal{A}$-module; i.e., Ω is a sheaf of $\mathbb{K}$-vector spaces over X, so that $\Omega(U)$ is an $\mathcal{A}(U)$-module, for every $U \in \tau_X$. Besides, let $\partial \equiv (\partial_U)_{U\in\tau_X} : \mathcal{A} \to \Omega$ be a sheaf morphism. The triplet $\delta = (\mathcal{A}, \partial, \Omega)$ is said to be a *differential triad*, if

i) ∂ is $\mathbb{K}$-linear, and

ii) ∂ satisfies the *Leibniz condition*: for every $(\alpha, \beta) \in \mathcal{A} \times_X \mathcal{A}$,

$$\partial(\alpha\beta) = \alpha\partial(\beta) + \beta\partial(\alpha).$$

Examples 2.1. (i) Every $\mathcal{C}^k$-manifold X $(k \geq 1)$ defines a differential triad: take as $\mathcal{A}$ the sheaf of germs of local $\mathbb{R}$-valued $\mathcal{C}^k$-functions on X, as Ω the sheaf of germs of local $\mathcal{C}^{k-1}$-differential 1-forms and as ∂ the sheafification of the usual differential d. Thus, the concept of a differential triad generalizes that of a manifold. On the other hand, manifolds are far from being alone in the new context, as the following example shows:

(ii) Let $(X, \mathcal{A})$ be an *algebraized space*, that is, X is a topological space and $\mathcal{A}$ is a sheaf of (commutative, associative, unital) $\mathbb{K}$-algebras over X. We denote by $\mathcal{X}(\mathcal{A})$ the set of all sheaf morphisms $\xi : \mathcal{A} \to \mathcal{A}$, which are $\mathbb{K}$-linear and satisfy the Leibniz condition (of course, for each ξ, $(\mathcal{A}, \xi, \mathcal{A})$ is a differential triad). $\mathcal{X}(\mathcal{A})$ is an $\mathcal{A}$-module. We consider the dual $\mathcal{A}$-module $\Omega := \mathcal{X}(\mathcal{A})^*$ and the sheaf morphism $\partial : \mathcal{A} \to \Omega$, given by

$$\partial\alpha : \mathcal{X}(\mathcal{A}) \to \mathcal{A} : \xi \mapsto (\partial\alpha)(\xi) := \xi(\alpha), \qquad \forall\alpha \in \mathcal{A}.$$

Then $(\mathcal{A}, \partial, \Omega)$ is a differential triad. Thus, *every algebraized space defines a differential triad.*

Some other examples are found in [3].

Let $f : X \to Y$ be a continuous map and let $\mathcal{S}h_X$, $\mathcal{S}h_Y$ denote the categories of sheaves over X, Y, respectively. We denote by $f_* : \mathcal{S}h_X \to \mathcal{S}h_Y$ the *push-out functor* and by $f^* : \mathcal{S}h_Y \to \mathcal{S}h_X$ the *pull-back functor* induced by f.

Definition 2.2. Let $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X)$, $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y)$ be differential triads over the topological spaces X, Y, respectively. A *morphism from δ_X to δ_Y* is a triplet $(f, f_{\mathcal{A}}, f_\Omega)$, where

(i) $f : X \to Y$ is continuous;

(ii) $f_{\mathcal{A}} : \mathcal{A}_Y \to f_*(\mathcal{A}_X)$ is a unit preserving morphism of sheaves of $\mathbb{K}$-algebras over Y.

(iii) $f_\Omega : \Omega_Y \to f_*(\Omega_X)$ is an $f_{\mathcal{A}}$-morphism, i.e., it is a morphism of sheaves of $\mathbb{K}$-vector spaces over Y, with

$$f_\Omega(\alpha\omega) = f_{\mathcal{A}}(\alpha) f_\Omega(\omega), \quad \forall\, (\alpha, \omega) \in \mathcal{A}_Y \times_Y \Omega_Y;$$

(iv) the following diagram is commutative

$$\begin{tikzcd}
\mathcal{A}_Y \arrow[r, "f_{\mathcal{A}}"] \arrow[d, "\partial_Y"'] & f_*(\mathcal{A}_X) \arrow[d, "f_*(\partial_X)"] \\
\Omega_Y \arrow[r, "f_\Omega"'] & f_*(\Omega_X)
\end{tikzcd}$$

Following the classical terminology, we say that a continuous $f : X \to Y$ is *differentiable*, if it can be completed to a morphism of differential triads $(f, f_{\mathcal{A}}, f_\Omega)$.

Differential triads and their morphisms form a category [6], denoted in the sequel by $\mathcal{DT}$. The identity morphism of a triad $(\mathcal{A}, \partial, \Omega)$ over X is $(id_X,\ id_{\mathcal{A}},\ id_\Omega)$ and the composition of $(f, f_{\mathcal{A}}, f_\Omega)$ and $(g, g_{\mathcal{A}}, g_\Omega)$ is given by

$$(g \circ f,\ (g \circ f)_{\mathcal{A}} = g_*(f_{\mathcal{A}}) \circ g_{\mathcal{A}},\ (g \circ f)_\Omega = g_*(f_\Omega) \circ g_\Omega). \tag{2.1}$$

The differential triads over a fixed topological space X and the morphisms of the form $(id_X, f_{\mathcal{A}}, f_\Omega)$ constitute a subcategory of $\mathcal{DT}$, denoted by $\mathcal{DT}_X$.

Clearly, if the $\mathcal{C}^k$-manifolds X, Y $(k \geq 1)$ are endowed with the differential triads $(\mathcal{A}_X, \partial_X, \Omega_X)$, $(\mathcal{A}_Y, \partial_Y, \Omega_Y)$ induced by their manifold structure, then every $\mathcal{C}^k$-map $f : X \to Y$ is differentiable, with

$$f_{\mathcal{A}}(\alpha) := \alpha \circ f\,, \quad \forall\, \alpha \in \mathcal{A}_Y(V)\,,$$
$$f_\Omega(\omega) := \omega \circ df\,, \quad \forall\, \omega \in \Omega_Y(V),$$

for any $V \in \tau_Y$. Thus, the category of $\mathcal{C}^k$-manifolds is embedded in $\mathcal{DT}$.

If $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X) \in \mathcal{DT}_X$, $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y) \in \mathcal{DT}_Y$ and $f : X \to Y$ is continuous, then it is clear that the *push-out of* δ_X *by* f

$$f_*(\delta_X) := (f_*(\mathcal{A}_X), f_*(\partial_X), f_*(\Omega_X))$$

is a differential triad over Y and the *pull-back of* δ_Y *by* f

$$f^*(\delta_Y) := (f^*(\mathcal{A}_Y), f^*(\partial_Y), f^*(\Omega_Y))$$

is a differential triad over X. The following two results are already known ([7, Theorems 3.1 and 3.4]).

Theorem 2.3. *Let* $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X) \in \mathcal{DT}_X$ *and let* $f : X \to Y$ *be a continuous map. If* Y *is endowed by the push-out* $f_*(\delta_X)$ *of* δ_X *by* f, *then there is a morphism* $(f, f_\mathcal{A}, f_\Omega) : \delta_X \to f_*(\delta_X)$ *in* $\mathcal{DT}$, *i.e.,* f *becomes differentiable. Besides,* $f_*(\delta_X)$ *satisfies the following universal property: if* $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y) \in \mathcal{DT}_Y$ *and* $(f, \tilde{f}_\mathcal{A}, \tilde{f}_\Omega) : \delta_X \to \delta_Y$ *is a morphism, then there exists a unique morphism* $(id_Y, g_\mathcal{A}, g_\Omega) : f_*(\delta_X) \to \delta_Y$, *so that*

$$(f, \tilde{f}_\mathcal{A}, \tilde{f}_\Omega) = (id_Y, g_\mathcal{A}, g_\Omega) \circ (f, f_\mathcal{A}, f_\Omega). \tag{2.2}$$

Theorem 2.4. *Let* $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y) \in \mathcal{DT}_Y$ *and let* $f : X \to Y$ *be a continuous map. If* X *is endowed by the pull-back* $f^*(\delta_Y)$ *of* δ_Y *by* f, *then there is a morphism* $(f, f_\mathcal{A}, f_\Omega) : f^*(\delta_Y) \to \delta_Y$ *in* $\mathcal{DT}$, *i.e.,* f *becomes differentiable. Besides,* $f^*(\delta_Y)$ *satisfies the following universal property: if* $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X) \in \mathcal{DT}_X$ *and* $(f, \tilde{f}_\mathcal{A}, \tilde{f}_\Omega) : \delta_X \to \delta_Y$ *is a morphism, then there exists a unique morphism* $(id_X, g_\mathcal{A}, g_\Omega) : \delta_X \to f^*(\delta_Y)$, *so that*

$$(f, \tilde{f}_\mathcal{A}, \tilde{f}_\Omega) = (f, f_\mathcal{A}, f_\Omega) \circ (id_X, g_\mathcal{A}, g_\Omega). \tag{2.3}$$

3. Projective and inductive limits over X

In this section we prove that projective and inductive systems of differential triads over the same base space X have limits in $\mathcal{DT}_X$.

Lemma 3.1. *(i) Let* $(\mathcal{S}_i; f_{ij})_{i \geq j \in I}$ *be a projective (resp. inductive) system of sheaves and sheaf morphisms over* X. *Then the projective (resp. inductive) limit of the system exists in* $\mathcal{S}h_X$.

(ii) Let $(\mathcal{S}_i; f_{ij})_{i \geq j \in I}$, $(\mathcal{T}_i; g_{ij})_{i \geq j \in I}$ *be projective (resp. inductive) systems of sheaves and sheaf morphisms over* X *and let* $(\mathcal{S}, \{f_i\}_{i \in I})$, $(\mathcal{T}, \{g_i\}_{i \in I})$ *be their projective (resp. inductive) limits. If* $(h_i : \mathcal{S}_i \to \mathcal{T}_i)_{i \in I}$ *is a morphism of projective (resp. inductive) systems, then there exists a sheaf morphism* $h : \mathcal{S} \to \mathcal{T}$, *with* $g_i \circ h = h_i \circ f_i$ *(resp.* $h \circ f_i = g_i \circ h_i$*), for every* $i \in I$.

Proof. (i) For every open $U \subseteq X$, $(\mathcal{S}_i(U); f_{ij}^U)_{i \geq j \in I}$ is a projective system. Besides, if V is open with $V \subseteq U$, the family of the restriction maps

$$\rho_{iU}^V : \mathcal{S}_i(U) \to \mathcal{S}_i(V)$$

is a morphism of projective systems. We set

$$S(U) := \varprojlim \mathcal{S}_i(U) \ , \qquad U \in \tau_X \ ;$$

$$\rho_U^V := \varprojlim \rho_{iU}^V : S(U) \longrightarrow S(V)\ , \qquad V \subseteq U \in \tau_X .$$

$(S(U), \rho_U^V)_{V\subseteq U\in\tau_X}$ is a presheaf over X, generating a sheaf $\mathcal{S}$. Since $S(U)$ is the projective limit of $(\mathcal{S}_i(U); f_{ij}^U)_{i\geq j\in I}$, there is a family of maps

$$f_i^U : S(U) \longrightarrow \mathcal{S}_i(U)\ , \qquad i \in I\ ,$$

so that

$$f_j^U = f_{ij}^U \circ f_i^U\ , \qquad \forall\ U \in \tau_X\ ,\ i \geq j \in I. \tag{3.1}$$

For every $i \in I$, $(f_i^U)_{U\in\tau_X}$ is a presheaf morphism. Let f_i denote the induced sheaf morphism. Then $(\mathcal{S}, \{f_i\}_{i\in I})$ is the projective limit of $(\mathcal{S}_i; f_{ij})_{i\geq j\in I}$.

(ii) Each h_i corresponds to a presheaf morphism $(h_i^U)_{U\in\tau_X}$. For a fixed U, $(h_i^U : \mathcal{S}_i(U) \to \mathcal{T}_i(U))_{i\in I}$ is a morphism of projective systems, hence there is a morphism $h^U : \mathcal{S}(U) \to \mathcal{T}(U)$, so that $h_i^U \circ f_i^U = g_i^U \circ h^U$, for every $i \in I$. Then, $(h^U)_{U\in\tau_X}$ is a presheaf morphism. Let h be the corresponding sheaf morphism. The required equality $h_i \circ f_i = g_i \circ h$ is obtained from the respective equality for the presheaf morphisms.

Analogous reasoning holds for inductive systems. □

Proposition 3.2. *Let X be a topological space, $(I, \leq)$ a directed set and*

$$((\mathcal{A}_i, \partial_i, \Omega_i); (id_X, f_{ij\mathcal{A}}, f_{ij\,\Omega}))_{i\geq j\in I}$$

a projective system in $\mathcal{DT}_X$. Then there exist a differential triad $(\mathcal{A}, \partial, \Omega)$ over X and a family of morphisms

$$(id_X, f_{i\mathcal{A}}, f_{i\Omega}) : (\mathcal{A}, \partial, \Omega) \to (\mathcal{A}_i, \partial_i, \Omega_i),\ \ i \in I$$

that satisfy the universal property of the projective limit in $\mathcal{DT}_X$.

Proof. If $((\mathcal{A}_i, \partial_i, \Omega_i); (id_X, f_{ij\mathcal{A}}, f_{ij\,\Omega}))_{i\geq j\in I}$ is a projective system of differential triads, then $(\mathcal{A}_i; f_{ij\mathcal{A}})_{i\geq j\in I}$ and $(\Omega_i; f_{ij\,\Omega})_{i\geq j\in I}$ are *inductive* systems of sheaves over X and $(\partial_i)_{i\in I}$ is a morphism of inductive systems. According to the previous lemma, the inductive limit

$$(\mathcal{A}, \partial, \Omega) := (\varinjlim \mathcal{A}_i, \varinjlim \partial_i, \varinjlim \Omega_i)$$

exists, along with morphisms $f_{i\mathcal{A}} : \mathcal{A}_i \to \mathcal{A}$ and $f_{i\Omega} : \Omega_i \to \Omega$, $i \in I$, so that

$$\partial \circ f_{i\mathcal{A}} = f_{i\Omega} \circ \partial_i\ , \quad \forall\ i \in I,$$

that is, for every $i \in I$, $(id_X, f_{i\mathcal{A}}, f_{i\Omega}) : \delta \to \delta_i$ is a morphism in $\mathcal{DT}_X$. The universal property of the projective limit is readily checked. □

It is clear that the dual result also holds true. That is, we have

Proposition 3.3. *Let X be a topological space, $(I, \leq)$ a directed set and*

$$((\mathcal{A}_i, \partial_i, \Omega_i); (id_X, f_{ij\mathcal{A}}, f_{ij\,\Omega}))_{i\leq j\in I}$$

an inductive system in $\mathcal{DT}_X$. Then there are a differential triad $(\mathcal{A}, \partial, \Omega)$ over X and a family of morphisms

$$(id_X, f_{i\mathcal{A}}, f_{i\Omega}) : (\mathcal{A}_i, \partial_i, \Omega_i) \to (\mathcal{A}, \partial, \Omega),\ \ i \in I$$

that satisfy the universal property of the inductive limit in $\mathcal{DT}_X$.

4. Projective and inductive limits in $\mathcal{DT}$

In this section we consider a projective system

$$(\delta_i = (\mathcal{A}_i, \partial_i, \Omega_i); (f_{ij}, f_{ij\mathcal{A}}, f_{ij\,\Omega}))_{i \geq j \in I}$$

of differential triads δ_i over the base spaces X_i, $i \in I$, I directed, and of morphisms of differential triads $(f_{ij}, f_{ij\mathcal{A}}, f_{ij\,\Omega}) : \delta_i \to \delta_j$, $i \geq j \in I$. Our aim is to construct a differential triad $\delta = (\mathcal{A}, \partial, \Omega)$ over some space X and a family of morphisms

$$(f_i,\, f_{i\mathcal{A}},\, f_{i\Omega}) : \delta \to \delta_i,\ i \in I,$$

that satisfy the universal property of the projective limit in $\mathcal{DT}$.

To this end, we consider the projective limit

$$(X := \varprojlim X_i, \{f_i\}_{i \in I})$$

of the projective system $(X_i; f_{ij})_{i \geq j \in I}$ of topological spaces and continuous maps. X will be the base space of the required differential triad.

Next, we consider the pull-backs $f_i^*(\delta_i)$, $i \in I$, which constitute a family of differential triads over X. We have the following.

Lemma 4.1. *Let $(\delta_i = (\mathcal{A}_i, \partial_i, \Omega_i); (f_{ij}, f_{ij\mathcal{A}}, f_{ij\,\Omega}))_{i \geq j \in I}$ be a projective system in $\mathcal{DT}$ and let X_i be the base space of δ_i. If $(X, \{f_i\}_{i \in I})$ is the projective limit of $(X_i; f_{ij})_{i \geq j \in I}$, then there exist connecting sheaf morphisms*

$$g_{ij\mathcal{A}} : f_j^*(\mathcal{A}_j) \to f_i^*(\mathcal{A}_i),$$

making $(f_i^*(\mathcal{A}_i);\ g_{ij\mathcal{A}})_{i \geq j \in I}$ an inductive system of sheaves of algebras over X. Similarly, there exist connecting sheaf morphisms

$$g_{ij\,\Omega} : f_j^*(\Omega_j) \to f_i^*(\Omega_i),$$

so that $(f_i^*(\Omega_i);\ g_{ij\,\Omega})_{i \geq j \in I}$ is an inductive system of $f_i^*(\mathcal{A}_i)$-modules, $i \in I$.

In order to prove Lemma 4.1 (and its dual, needed for the analogous considerations for inductive limits), we need some additional information about the pull-bach and the push-out functors, induced by a continuous $f : X \to Y$. The crucial feature is the existence of a natural transformation $\phi^f : id \to f_* f^*$ between the covariant functors id, $f_* f^* : \mathcal{S}h_Y \to \mathcal{S}h_Y$, and a natural transformation $\psi^f : f^* f_* \to id$ between the covariant functors $f^* f_*$, $id : \mathcal{S}h_X \to \mathcal{S}h_X$. For details we refer to [10, 7.11]. We note that, for every $\mathcal{B} \in \mathcal{S}h_Y$ and $V \in \tau_Y$,

$$f_* f^*(\mathcal{B})(V) \equiv f^*(\mathcal{B})(f^{-1}(V)) \equiv \Gamma_f(f^{-1}(V), \mathcal{B}),$$

where $\Gamma_f(f^{-1}(V), \mathcal{B})$ is the set of all continuous maps $\alpha : f^{-1}(V) \to \mathcal{B}$, with $\alpha(x) \in \mathcal{B}_{f(x)}$, for every $x \in f^{-1}(V)$, and $\phi^f_{\mathcal{B}}$ is given by

$$\phi^f_{\mathcal{B}V} : \mathcal{B}(V) \to \Gamma_f(f^{-1}(V), \mathcal{B}) : \beta \mapsto \beta \circ f.$$

Regarding ψ^f, let $\mathcal{A} \in \mathcal{S}h_X$ and

$$(x,s) \in f^*f_*(\mathcal{A}) \equiv \{(x,s) \in X \times f_*(\mathcal{A}) : s \in f_*(\mathcal{A})_{f(x)}\}.$$

Then there exist $V \in \tau_Y$ and $\bar{s} \in f_*(\mathcal{A})(V) = \mathcal{A}(f^{-1}(V))$, so that $r^V_{f(x)}(\bar{s}) = s$, $(r^V_{V'})_{V' \subseteq V \in \tau_Y}$ being the restrictions of the presheaf $(f_*(\mathcal{A})(V))_{V \in \tau_Y}$. We have

$$\psi^f_{\mathcal{A}}(x,s) := \rho^{f^{-1}(V)}_x(\bar{s}),$$

where $(\rho^U_{U'})_{U' \subseteq U \in \tau_X}$ are the restrictions of the presheaf $(\mathcal{A}(U))_{U \in \tau_X}$. The above functors interact with each other, in the following way (see [7])

$$f_*(\psi^f_{\mathcal{A}}) \circ \phi^f_{f_*(\mathcal{A})} = id_{f_*(\mathcal{A})}, \tag{4.1}$$

$$\psi^f_{f^*(\mathcal{B})} \circ f^*(\phi^f_{\mathcal{B}}) = id_{f^*(\mathcal{B})}. \tag{4.2}$$

Regarding the way ϕ and ψ behave with respect to the composition of maps, we have

Lemma 4.2. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then for every $\mathcal{A} \in \mathcal{S}h_X$ and $\mathcal{C} \in \mathcal{S}h_Z$,*

$$\psi^{g\circ f}_{\mathcal{A}} = \psi^f_{\mathcal{A}} \circ f^*(\psi^g_{f_*(\mathcal{A})}), \tag{4.3}$$

$$\phi^{g\circ f}_{\mathcal{C}} = g_*(\phi^f_{g^*(\mathcal{C})}) \circ \phi^g_{\mathcal{C}}. \tag{4.4}$$

Proof. Let $(x,s) \in (g\circ f)^*(g \circ f)_*(\mathcal{A})$. By definition, there are $W \in \tau_Z$ and $\bar{s} \in (g \circ f)_*(\mathcal{A})(W) = \mathcal{A}((g\circ f)^{-1}(W)$ such that $r^W_{g(f(x))}(\bar{s}) = s$, where $(r^W_{W'})_{W' \subseteq W \in \tau_Z}$ are the restrictions of $(g \circ f)_*(\mathcal{A})$. Then

$$\psi^{g\circ f}_{\mathcal{A}}(x,s) = \rho^{(g\circ f)^{-1}(W)}_x(\bar{s}),$$

where $(\rho^U_{U'})_{U' \subseteq U \in \tau_X}$ are the restrictions of $\mathcal{A}$. Now (x,s) coincides with $(x, (f(x), p^W_{g(f(x))}(\bar{s})) \in f^*g^*g_*f_*(\mathcal{A})$, where $(p^W_{W'})_{W' \subseteq W \in \tau_Z}$ are the restrictions of $g_*f_*(\mathcal{A})$. Thus,

$$\begin{aligned}
\psi^f_{\mathcal{A}} \circ f^*(\psi^g_{f_*(\mathcal{A})})(x,s) &= \psi^f_{\mathcal{A}} \circ f^*(\psi^g_{f_*(\mathcal{A})})(x, (f(x), p^W_{g(f(x))}(\bar{s}))) = \\
&= \psi^f_{\mathcal{A}}(\psi^g_{f_*(\mathcal{A})}(f(x), p^W_{g(f(x))}(\bar{s}))) = \\
&= \psi^f_{\mathcal{A}}(x, \pi^{g^{-1}(W)}_{f(x)}(\bar{s})) = \rho^{f^{-1}(g^{-1}(W))}_x(\bar{s}),
\end{aligned}$$

$(\pi^V_{V'})_{V' \subseteq V \in \tau_Y}$ being the restrictions of $f_*(\mathcal{A})$, and (4.3) is proved.

Let now $W \in \tau_Z$ and $a \in \mathcal{C}(W)$. Then

$$g_*(\phi^f_{g^*(\mathcal{C})})_W \circ \phi^g_{\mathcal{C}W}(a) = \phi^f_{g^*(\mathcal{C})f^{-1}(W)}(a \circ g) = a \circ g \circ f = \phi^{g\circ f}_{\mathcal{C}W}(a),$$

completing the proof. ∎

Proof of Lemma 4.1. Let $i \leq j \in I$ and consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f_j"'] & X_i \arrow[d, "f_{ij}"] \\ & X_j \end{tikzcd}$$

Over X, we have the pull-backs $f_i^*(\mathcal{A}_i)$ and $f_j^*(\mathcal{A}_j) = f_i^* f_{ij}^*(\mathcal{A}_j)$. For each $i \in I$, the sheaf morphism $f_{ij\mathcal{A}} : \mathcal{A}_j \to f_{ij*}(\mathcal{A}_i)$ induces a morphism

$$f_{ij}^*(f_{ij\mathcal{A}}) : f_{ij}^*(\mathcal{A}_j) \to f_{ij}^* f_{ij*}(\mathcal{A}_i).$$

We consider the composition

$$\psi_{\mathcal{A}_i}^{f_{ij}} \circ f_{ij}^*(f_{ij\mathcal{A}}) : f_{ij}^*(\mathcal{A}_j) \to \mathcal{A}_i$$

and we pull it back via f_i, obtaining

$$g_{ij\mathcal{A}} := f_i^*(\psi_{\mathcal{A}_i}^{f_{ij}} \circ f_{ij}^*(f_{ij\mathcal{A}})) : f_j^*(\mathcal{A}_j) = f_i^* f_{ij}^*(\mathcal{A}_j) \to f_i^*(\mathcal{A}_i).$$

We prove that the family $(f_i^*(\mathcal{A}_i);\ g_{ij\mathcal{A}})_{i \geq j \in I}$ is an inductive system of sheaves of algebras over X. To this end, we prove that

$$g_{ij\mathcal{A}} \circ g_{jk\mathcal{A}} = g_{ik\mathcal{A}}, \quad \forall\ i \geq j \geq k \in I.$$

In fact,

$$\begin{aligned} g_{ij\mathcal{A}} \circ g_{jk\mathcal{A}} &= f_i^*(\psi_{\mathcal{A}_i}^{f_{ij}} \circ f_{ij}^*(f_{ij\mathcal{A}})) \circ f_j^*(\psi_{\mathcal{A}_j}^{f_{jk}} \circ f_{jk}^*(f_{jk\mathcal{A}})) = \\ &= f_i^*(\psi_{\mathcal{A}_i}^{f_{ij}} \circ f_{ij}^*(f_{ij\mathcal{A}}) \circ f_{ij}^*(\psi_{\mathcal{A}_j}^{f_{jk}} \circ f_{jk}^*(f_{jk\mathcal{A}}))) = \\ &= f_i^*(\psi_{\mathcal{A}_i}^{f_{ij}} \circ f_{ij}^*(f_{ij\mathcal{A}} \circ \psi_{\mathcal{A}_j}^{f_{jk}} \circ f_{jk}^*(f_{jk\mathcal{A}}))). \end{aligned}$$

Since $\psi^{f_{jk}} : f_{jk}^* f_{jk*} \to id$ is a natural transformation, we have

$$f_{ij\mathcal{A}} \circ \psi_{\mathcal{A}_j}^{f_{jk}} = \psi_{f_{ij*}(\mathcal{A}_i)}^{f_{jk}} \circ f_{jk}^* f_{jk*}(f_{ij\mathcal{A}}),$$

hence

$$\begin{aligned} g_{ij\mathcal{A}} \circ g_{jk\mathcal{A}} &= f_i^*(\psi_{\mathcal{A}_i}^{f_{ij}} \circ f_{ij}^*(\psi_{f_{ij*}(\mathcal{A}_i)}^{f_{jk}} \circ f_{jk}^* f_{jk*}(f_{ij\mathcal{A}}) \circ f_{jk}^*(f_{jk\mathcal{A}}))) = \\ &= f_i^*(\psi_{\mathcal{A}_i}^{f_{ij}} \circ f_{ij}^*(\psi_{f_{ij*}(\mathcal{A}_i)}^{f_{jk}} \circ f_{jk}^*(f_{jk*}(f_{ij\mathcal{A}}) \circ f_{jk\mathcal{A}}))). \end{aligned}$$

By virtue of (2.1), the expression $f_{jk*}(f_{ij\mathcal{A}}) \circ f_{jk\mathcal{A}}$ in the last equality coincides with $(f_{jk} \circ f_{ij})_{\mathcal{A}} = f_{ik\mathcal{A}}$, consequently, we have

$$\begin{aligned} g_{ij\mathcal{A}} \circ g_{jk\mathcal{A}} &= f_i^*(\psi_{\mathcal{A}_i}^{f_{ij}} \circ f_{ij}^*(\psi_{f_{ij*}(\mathcal{A}_i)}^{f_{jk}} \circ f_{jk}^*(f_{ik\mathcal{A}}))) = \\ &= f_i^*(\psi_{\mathcal{A}_i}^{f_{ij}} \circ f_{ij}^*(\psi_{f_{ij*}(\mathcal{A}_i)}^{f_{jk}}) \circ f_{ik}^*(f_{ik\mathcal{A}})). \end{aligned}$$

Combining the last equality with (4.3), we obtain

$$g_{ij\mathcal{A}} \circ g_{jk\mathcal{A}} = f_i^*(\psi_{\mathcal{A}_i}^{f_{ik}} \circ f_{ik}^*(f_{ik\mathcal{A}})) = g_{ik\mathcal{A}}.$$

Analogously, we define

$$g_{ij\,\Omega} := f_i^*(\psi^{f_{ij}}_{\Omega_i} \circ f_{ij}^*(f_{ij\,\Omega}))\,; \qquad i \geq j \in I.$$

Then $(f_i^*(\Omega_i);\, g_{ij\,\Omega})_{i\geq j\in I}$ is an inductive system of $f_i^*(\mathcal{A}_i)$-modules. □

Lemma 4.3. *With the assumptions of Lemma 4.1,*

$$(f_i^*(\delta_i);\ (id_X, g_{ij\mathcal{A}}, g_{ij\,\Omega}))_{i\geq j\in I}$$

is a projective system of differential triads over X.

Proof. We prove that every

$$(id_X, g_{ij\mathcal{A}}, g_{ij\,\Omega}) : f_i^*(\delta_i) \to f_j^*(\delta_j)$$

is a morphism: It suffices to prove condition (iv) of Definition 2.2. We have

$$f_i^*(\partial_i) \circ g_{ij\mathcal{A}} = f_i^*(\partial_i) \circ f_i^*(\psi^{f_{ij}}_{\mathcal{A}_i} \circ f_{ij}^*(f_{ij\mathcal{A}})) = f_i^*(\partial_i \circ \psi^{f_{ij}}_{\mathcal{A}_i} \circ f_{ij}^*(f_{ij\mathcal{A}})).$$

Since $\psi^{f_{ij}}$ is a natural transformation,

$$\partial_i \circ \psi^{f_{ij}}_{\mathcal{A}_i} = \psi^{f_{ij}}_{\Omega_i} \circ f_{ij}^*(f_{ij*}(\partial_i)),$$

hence

$$\begin{aligned} f_i^*(\partial_i) \circ g_{ij\mathcal{A}} &= f_i^*(\psi^{f_{ij}}_{\Omega_i} \circ f_{ij}^*(f_{ij*}(\partial_i)) \circ f_{ij}^*(f_{ij\mathcal{A}})) = \\ &= f_i^*(\psi^{f_{ij}}_{\Omega_i} \circ f_{ij}^*(f_{ij*}(\partial_i) \circ f_{ij\mathcal{A}})). \end{aligned}$$

Condition (iv) for the morphism $(f_{ij}, f_{ij\mathcal{A}}, f_{ij\mathcal{A}})$ implies that

$$f_{ij*}(\partial_i) \circ f_{ij\mathcal{A}} = f_{ij\,\Omega} \circ \partial_j$$

yielding, in turn,

$$\begin{aligned} f_i^*(\partial_i) \circ g_{ij\mathcal{A}} &= f_i^*(\psi^{f_{ij}}_{\Omega_i} \circ f_{ij}^*(f_{ij\,\Omega} \circ \partial_j)) = \\ &= f_i^*(\psi^{f_{ij}}_{\Omega_i} \circ f_{ij}^*(f_{ij\,\Omega})) \circ f_j^*(\partial_j) = g_{ij\,\Omega} \circ f_j^*(\partial_j), \end{aligned}$$

and the proof is complete. □

Theorem 4.4. *Let $(\delta_i = (\mathcal{A}_i, \partial_i, \Omega_i); (f_{ij}, f_{ij\mathcal{A}}, f_{ij\,\Omega}))_{i\geq j\in I}$ be a projective system in $\mathcal{DT}$ and let X_i be the base space of δ_i. There exists a differential triad $\delta = (\mathcal{A}, \partial, \Omega)$ over the projective limit X of the base spaces, satisfying the universal property of the projective limit in $\mathcal{DT}$.*

Proof. By Lemma 4.3, there exists a projective system

$$(f_i^*(\delta_i);\, (id_X, g_{ij\mathcal{A}},\, g_{ij\,\Omega}))_{i\geq j\in I}$$

of differential triads over X. Let

$$(\delta = (\mathcal{A}, \partial, \Omega),\, \{(id_X, g_{i\mathcal{A}}, g_{i\Omega})\}_{i\in I})$$

be the limit of this system in $\mathcal{DT}_X$ (see Proposition 3.2). For every $i \in I$, we consider the composition

$$(f_i, h_{i\mathcal{A}}, h_{i\Omega}) := (f_i, \phi^{f_i}_{\mathcal{A}_i}, \phi^{f_i}_{\Omega_i}) \circ (id_X, g_{i\mathcal{A}}, g_{i\Omega}),$$

of the morphisms

$$(id_X, g_{i\mathcal{A}}, g_{i\Omega}) : \delta \to f_i^*(\delta_i)\,,$$
$$(f_i, \phi^{f_i}_{\mathcal{A}_i}, \phi^{f_i}_{\Omega_i}) : f_i^*(\delta_i) \to \delta_i\,.$$

That is, we set

$$h_{i\mathcal{A}} := f_{i*}(g_{i\mathcal{A}}) \circ \phi^{f_i}_{\mathcal{A}_i}\,, \qquad h_{i\Omega} := f_{i*}(g_{i\Omega}) \circ \phi^{f_i}_{\Omega_i} \tag{4.5}$$

(cf. (2.1)). We will prove that $((\mathcal{A}, \partial, \Omega), \{(f_i, h_{i\mathcal{A}}, h_{i\Omega})\}_{i\in I})$ is the projective limit of the initial projective system $(\delta_i; (f_{ij}, f_{ij\mathcal{A}}, f_{ij\,\Omega}))_{i\geq j\in I}$.

First we prove that, for every $i \geq j \in I$,

$$(f_{ij}, f_{ij\mathcal{A}}, f_{ij\,\Omega})) \circ (f_i, h_{i\mathcal{A}}, h_{i\Omega}) = (f_j, h_{j\mathcal{A}}, h_{j\Omega}).$$

In virtue of (2.1), it suffices to prove that

$$h_{j\mathcal{A}} = f_{ij*}(h_{i\mathcal{A}}) \circ f_{ij\mathcal{A}}\,, \qquad h_{j\Omega} = f_{ij*}(h_{i\Omega}) \circ f_{ij\Omega}.$$

For the first equality we have (see (4.5))

$$\begin{aligned} h_{j\mathcal{A}} &= f_{j*}(g_{j\mathcal{A}}) \circ \phi^{f_j}_{\mathcal{A}_j} = (f_{ij} \circ f_i)_*(g_{i\mathcal{A}} \circ g_{ij\mathcal{A}}) \circ \phi^{f_{ij}\circ f_i}_{\mathcal{A}_j} = \\ &= f_{ij*} f_{i*}(g_{i\mathcal{A}}) \circ f_{ij*} f_{i*}(g_{ij\mathcal{A}}) \circ \phi^{f_{ij}\circ f_i}_{\mathcal{A}_j} \end{aligned}$$

and

$$f_{ij*}(h_{i\mathcal{A}}) \circ f_{ij\mathcal{A}} = f_{ij*} f_{i*}(g_{i\mathcal{A}}) \circ f_{ij*}(\phi^{f_i}_{\mathcal{A}_i}) \circ f_{ij\mathcal{A}}.$$

Thus, it suffices to prove that

$$f_{ij*} f_{i*}(g_{ij\mathcal{A}}) \circ \phi^{f_{ij}\circ f_i}_{\mathcal{A}_j} = f_{ij*}(\phi^{f_i}_{\mathcal{A}_i}) \circ f_{ij\mathcal{A}}.$$

Taking into account the definition of $g_{ij\mathcal{A}}$, along with (4.4) and the naturality of ϕ^{f_i} and of $\phi^{f_{ij}}$, we obtain

$$\begin{aligned} f_{ij*} f_{i*}(g_{ij\mathcal{A}}) \circ \phi^{f_{ij}\circ f_i}_{\mathcal{A}_j} &= f_{ij*}[f_{i*} f_i^*(\psi^{f_{ij}}_{\mathcal{A}_i} \circ f_{ij}^*(f_{ij\mathcal{A}})) \circ \phi^{f_i}_{f_{ij}^*(\mathcal{A}_j)}] \circ \phi^{f_{ij}}_{\mathcal{A}_j} = \\ &= f_{ij*}[\phi^{f_i}_{\mathcal{A}_i} \circ (\psi^{f_{ij}}_{\mathcal{A}_i} \circ f_{ij}^*(f_{ij\mathcal{A}}))] \circ \phi^{f_{ij}}_{\mathcal{A}_j} = \\ &= f_{ij*}(\phi^{f_i}_{\mathcal{A}_i}) \circ f_{ij*}(\psi^{f_{ij}}_{\mathcal{A}_i}) \circ f_{ij*} f_{ij}^*(f_{ij\mathcal{A}}) \circ \phi^{f_{ij}}_{\mathcal{A}_j} = \\ &= f_{ij*}(\phi^{f_i}_{\mathcal{A}_i}) \circ f_{ij*}(\psi^{f_{ij}}_{\mathcal{A}_i}) \circ \phi^{f_{ij}}_{f_{ij*}(\mathcal{A}_i)} \circ f_{ij\mathcal{A}} \end{aligned}$$

and the required equality is a result of (4.1). The analogous equality for $h_{i\Omega}$ is proved in a similar way.

Finally, we prove that $((\mathcal{A}, \partial, \Omega), \{(f_i, h_{i\mathcal{A}}, h_{i\Omega})\}_{i\in I})$ is unique. In fact, let $(\tilde{\mathcal{A}}, \tilde{\partial}, \tilde{\Omega})$ be a differential triad over some topological space $\tilde{X}$ and

$$(\tilde{h}_i, \tilde{h}_{i\mathcal{A}}, \tilde{h}_{i\Omega}) : (\tilde{\mathcal{A}}, \tilde{\partial}, \tilde{\Omega}) \to (\mathcal{A}_i, \partial_i, \Omega_i)\,, \quad i \in I$$

a family of morphisms, with

$$(f_{ij}, f_{ij\mathcal{A}}, f_{ij\Omega}) \circ (\tilde{h}_i, \tilde{h}_{i\mathcal{A}}, \tilde{h}_{i\Omega}) = (\tilde{h}_j, \tilde{h}_{j\mathcal{A}}, \tilde{h}_{j\Omega}),$$

for every $i \geq j \in I$. Since $(X, \{f_i\}_{i\in I})$ is the projective limit of $(X_i; f_{ij})_{i\geq j\in I}$, there exists a continuous map $h : \tilde{X} \to X$, with $f_i \circ h = \tilde{h}_i$, for every $i \in I$. We set

$$k_{i\mathcal{A}} := \psi^{f_i}_{h_*(\tilde{\mathcal{A}})} \circ f_i^*(\tilde{h}_{i\mathcal{A}}) , \qquad k_{i\Omega} := \psi^{f_i}_{h_*(\tilde{\Omega})} \circ f_i^*(\tilde{h}_{i\Omega}) .$$

We check that

$$(h, k_{i\mathcal{A}}, k_{i\Omega}) : (\tilde{\mathcal{A}}, \tilde{\partial}, \tilde{\Omega}) \to f_i^*(\mathcal{A}_i, \partial_i, \Omega_i) \tag{4.6}$$

is a morphism and

$$(id_X, g_{ij\mathcal{A}}, g_{ij\Omega}) \circ (h, k_{i\mathcal{A}}, k_{i\Omega}) = (h, k_{j\mathcal{A}}, k_{j\Omega}),$$

for every $i \geq j \in I$. The family (4.6) can be viewed as a family

$$(id_X, k_{i\mathcal{A}}, k_{i\Omega}) : (h_*(\tilde{\mathcal{A}}), h_*(\tilde{\partial}), h_*(\tilde{\Omega})) \to f_i^*(\mathcal{A}_i, \partial_i, \Omega_i)$$

of morphisms in $\mathcal{DT}_X$. Since $(\mathcal{A}, \partial, \Omega)$ is the projective limit of the system $(f_i^*(\mathcal{A}_i, \partial_i, \Omega_i), (id_X, g_{ij\mathcal{A}}, g_{ij\Omega}))$ in $\mathcal{DT}_X$, there exists a unique

$$(id_X, h_{\mathcal{A}}, h_{\Omega}) : (h_*(\tilde{\mathcal{A}}), h_*(\tilde{\partial}), h_*(\tilde{\Omega})) \to (\mathcal{A}, \partial, \Omega),$$

so that

$$(id_X, g_{i\mathcal{A}}, g_{i\Omega}) \circ (id_X, h_{\mathcal{A}}, h_{\Omega}) = (id_X, k_{i\mathcal{A}}, k_{i\Omega}) , \quad \forall i \in I.$$

Then $(id_X, h_{\mathcal{A}}, h_{\Omega})$ corresponds to a unique morphism

$$(h, h_{\mathcal{A}}, h_{\Omega}) : (\tilde{\mathcal{A}}, \tilde{\partial}, \tilde{\Omega}) \to (\mathcal{A}, \partial, \Omega)$$

satisfying

$$(f_i, h_{i\mathcal{A}}, h_{i\Omega}) \circ (h, h_{\mathcal{A}}, h_{\Omega}) = (\tilde{h}_i, \tilde{h}_{i\mathcal{A}}, \tilde{h}_{i\Omega}), \quad \forall i \in I,$$

and the proof is complete. □

It is now trivially checked that dual arguments give the following

Theorem 4.5. *Let $((\mathcal{A}_i, \partial_i, \Omega_i), (f_{ij}, f_{ij\mathcal{A}}, f_{ij\,\Omega}))_{i\geq j\in I}$ be an inductive system in $\mathcal{DT}$ and let X_i be the base space of $(\mathcal{A}_i, \partial_i, \Omega_i)$. There exists a differential triad $(\mathcal{A}, \partial, \Omega)$ over the inductive limit X of the base spaces, satisfying the universal property of the inductive limit in $\mathcal{DT}$.*

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