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ON CHERN-LAGRANGE COMPLEX CONNECTION

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ABSTRACT. In this note we make a report about the utility of a new nonlinear connection in complex Lagrange space, which generalize the well-known Chern-Finsler complex connection.

A significant result establish its relation with a complex nonlinear connection previously obtained by us from the variational problem.

1. INTRODUCTION.

Giving up the homogeneity condition of the fundamental function in a complex Finsler space, in last years we have made an approach in a new type of geometry, called by us the complex Lagrange geometry([9],[10],[11]).

In brief we shall recall here the basic concepts of this geometry.

Let M be a complex manifold, $\dim_C M = n$, $(U,(z^i))$ complex coordinates in a local chart. The complexified bundle of the real tangent bundle TM is decomposed in any $z \in U$ according to $(1,0)-$ vectors and respectively to $(0,1)-$ vectors, $T_C M = T'M \oplus T''M$.

The bundle $\pi_T : T'M \to M$ is holomorphic and $T''M$ is its conjugate. The geometric support of complex Lagrange geometry is the complex manifold $T'M$, $\dim_C T'M = 2n$, and the induced complex coordinates in a local chat are denoted by $u = (z^i, \eta^i)$.

In its turn, the complexified $T_C(T'M)$ is decomposed in $T_C(T'M) = T'(T'M) \oplus T''(T'M)$, where $T''(T'M) = \overline{T'(T'M)}$ and, therefore, our attention is focused on the $(1,0)-$type vectors, by conjugation are obtained the vectors from $T''_u(T'M)$.

Let $V(T'M) = \{\xi \in T'(T'M) \;/\; \pi_{T*}(\xi) = 0\}$ be the vertical bundle of $T'M$ and $\left\{\frac{\partial}{\partial \eta^i} = \dot{\partial}_i\right\}_{i=1,n}$ a local base in vertical distribution $V_u(T'M)$. An horizontal bundle is a supplementary subbundle of $V(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. This determines a distribution $N : u \to H_u(T'M)$, called *complex nonlinear connection*, in brief (*c.n.c.*). A local base in the horizontal distribution $H_u(T'M)$ is denoted by:

$$\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N_i^j \frac{\partial}{\partial \eta^j} \tag{1.1}$$

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Usually, the functions N_i^j are called the coefficients of $(c.n.c.)$ and if they are transformed under the rule:

$$N_k^{\prime i}\frac{\partial z^{\prime k}}{\partial z^j}=\frac{\partial z^{\prime i}}{\partial z^k}N_j^k-\frac{\partial^2 z^{\prime i}}{\partial z^j\partial z^k}\eta^k \tag{1.2}$$

then the base $\left\{\frac{\delta}{\delta z^i}=\delta_i\right\}_{i=1,n}$, called the *adapted* base of N_i^j $(c.n.c.)$, satisfies the following rule of transformation:

$$\frac{\delta}{\delta z^i}=\frac{\partial z^{\prime j}}{\partial z^i}\frac{\delta}{\delta z^{\prime j}} \tag{1.3}$$

A $(c.n.c.)$ determines the following decomposition of $T_C(T'M)$:

$$T_C(T'M)=H(T'M)\oplus V(T'M)\oplus\overline{H(T'M)}\oplus\overline{V(T'M)} \tag{1.4}$$

and the corresponding local adapted base in any $u\in T'M$ is obtain by conjugation, briefly being denoted by $\left\{\delta_i,\dot{\partial}_i,\delta_{\bar{i}},\dot{\partial}_{\bar{i}}\right\}$.

Let D be a derivative law on $T_C(T'M)$ which preserves the four distributions from (1.4). In [9] a particular class of this kind of derivative law is considered, called $N-(c.l.c.)$,($N-$ complex linear connection), characterized by the fact that in addition the complex and tangent structures associated to the $(c.n.c.)$ are preserved. Locally, a $N-(c.l.c.)$ is determined only by the following set of coefficients $\left(L_{jk}^i,\ C_{jk}^i,\ L_{\bar{j}k}^{\bar{i}},\ C_{\bar{j}k}^{\bar{i}}\right)$ and their conjugates, where:

$$D_{\delta_k}\delta_j=L_{jk}^i\delta_i\ ;\qquad D_{\dot{\partial}_k}\dot{\partial}_j=C_{jk}^i\dot{\partial}_i$$

$$D_{\delta_k}\delta_{\bar{j}}=L_{\bar{j}k}^{\bar{i}}\delta_{\bar{i}}\ ;\qquad D_{\dot{\partial}_k}\dot{\partial}_{\bar{j}}=C_{\bar{j}k}^{\bar{i}}\dot{\partial}_{\bar{i}} \tag{1.5}$$

Definition 1.1. *A complex Lagrangian on $T'M$ is a differentiable function L : $T'M\to R$ under the condition $g_{i\bar{j}}=\partial^2L/\partial\eta^i\partial\bar{\eta}^j$ is a nondegenerate metric. The pair (M,L) is called a complex Lagrange space.*

If the function $L:(z,\eta)\to L(z,\eta)$ is absolutely homogeneous of two degree in respect to η, i.e. $L(z,\lambda\eta)=|\lambda|^2L(z,\eta)$, the pair (M,L) is said to be a *complex Finsler space*. In a complex Finsler space , the Finsler metric $g_{i\bar{j}}$ satisfies in addition the following formulas which are consequences of Euler theorem concerning the homogeneity:

$$\frac{\partial L}{\partial\eta^i}\eta^i=L\ ;\ g_{i\bar{j}}\eta^i=\frac{\partial L}{\partial\bar{\eta}^j}\ ;\ \frac{\partial g_{k\bar{j}}}{\partial\eta^i}\eta^i=\frac{\partial g_{i\bar{j}}}{\partial\eta^k}\eta^i=0\ ;\ g_{i\bar{j}}\eta^i\bar{\eta}^j=L \tag{1.6}$$

The geometry of a complex Lagrange space, particularly complex Finsler one, depends on the choice of the $(c.n.c.)N$. Certainly, such geometry must necessary contain only geometrical objects related to the space metric. Consequently, the $(c.n.c.)$ is required to be expressed only on complex Lagrange function L.

Initially this problem was solved by us using the variational problem for a complex geodesic([9])

Theorem 1.1. *Let H^j be given by :*

$$H^j = \frac{1}{2} g^{\bar{h}j} \frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^h} \eta^k \tag{1.7}$$

Then $\overset{c}{N}{}_i^j = \frac{\partial H^j}{\partial \eta^i}$ *are the coefficients of a complex nonlinear connection, determined only by the complex Lagrangian L.*

In particular , when $L = g_{i\bar{j}} \eta^i \bar{\eta}^j$ is a complex Finsler metric, and denoting by $\gamma^i_{jk} = \frac{1}{2} g^{\bar{l}i} (\frac{\partial g_{j\bar{l}}}{\partial z^k} + \frac{\partial g_{k\bar{l}}}{\partial z^j})$ the first of the Christoffel symbols corresponding to the Hermitian metric $g_{i\bar{j}}$ ([3], [6]), from the above theorem it results:

Theorem 1.2. *The functions:*

$$\overset{c}{N}{}_i^j = \frac{1}{2} \frac{\partial \gamma^j_{00}}{\partial \eta^i}, \quad where\ \gamma^i_{00} = \gamma^i_{jk} \eta^j \eta^k \tag{1.8}$$

are the coefficients of one (c.n.c.) on $T'M$, called the Cartan complex nonlinear connection of (M, L) Finsler space.

For a given $(c.n.c.) N_i^j$, a metric $N-(c.l.c.)$ in respect to the hermitian structure on $T'M$

$$G = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j \tag{1.9}$$

is given by ([9]):

$$\overset{c}{L}{}^i_{jk} = \frac{1}{2} g^{\bar{l}i} (\frac{\delta g_{j\bar{l}}}{\delta z^k} + \frac{\delta g_{k\bar{l}}}{\delta z^j}) \quad ; \quad \overset{c}{C}{}^i_{jk} = \frac{1}{2} g^{\bar{l}i} (\frac{\partial g_{j\bar{l}}}{\partial \eta^k} + \frac{\partial g_{k\bar{l}}}{\partial \eta^j}) = g^{\bar{l}i} \frac{\partial g_{j\bar{l}}}{\partial \eta^k}$$

$$\overset{c}{L}{}^{\bar{i}}_{\bar{j}k} = \frac{1}{2} g^{\bar{i}l} (\frac{\delta g_{l\bar{j}}}{\delta z^k} - \frac{\delta g_{k\bar{j}}}{\delta z^l}) \quad ; \quad \overset{c}{C}{}^{\bar{i}}_{\bar{j}k} = \frac{1}{2} g^{\bar{i}l} (\frac{\partial g_{l\bar{j}}}{\partial \eta^k} - \frac{\partial g_{k\bar{j}}}{\partial \eta^l}) = 0 \tag{1.10}$$

and is called *canonical* $N-(c.l.c.)$

In the particular case of complex Finsler spaces, as a rule, is studied a special connection called the Chern-Finsler complex connection ([1],[2]..):

$$N_i^j = g^{\bar{m}j} \frac{\partial g_{l\bar{m}}}{\partial z^i} \eta^l \ ; \quad L^i_{jk} = g^{\bar{m}i} \frac{\delta g_{j\bar{m}}}{\delta z^k} \ ; \ C^i_{jk} = g^{\bar{m}i} \frac{\partial g_{j\bar{m}}}{\partial \eta^k} \tag{1.11}$$

and $L^{\bar{i}}_{\bar{j}k} = C^{\bar{i}}_{\bar{j}k} = 0$, which being of $(1,0)-$type, assume some facilities in calculus.

On the other hand, the canonical $N-(c.l.c.)$, although is not of $(1,0)-$type, has the advantage of $h-(h,h)$ and $v-(v,v)$ vanishing torsions.

2. The Chern-Lagrange complex connection.

Having in mind the dualism of Lagrangian-Hamiltonian principles from classical mechanic, recently we have made an intensive approach of the geometry of complex Hamilton spaces ([12]). Firstly, by direct hand calculus and then by geometrical reasons, we have succeeded in finding one (*c.n.c.*) in a complex Hamilton space with a very simple expression and also its utility was proved. Then we have asked to find its back image by the complex Legendre transformation in complex Lagrange space . The purpose of the note is not to describe this technique which is quite ample. We shall make here just an analyze of the obtained result.

Theorem 2.1. *The following functions* $\overset{KL}{N_i^j}$ *are the coefficients of a* (*c.n.c.*) *in the complex Lagrange space* (M, L) :

$$\overset{KL}{N_i^j} = g^{\bar{k}j}\frac{\partial^2 L}{\partial z^i \partial \bar{\eta}^k} \tag{2.1}$$

called the complex Chern-Lagrange nonlinear connection.

By direct calculus is proved that the coefficients $\overset{KL}{N_i^j}$ verify the (1.2) law of transformation.

Proposition 2.1. *The brackets of the adapted base of Chern-Lagrange* (*c.n.c.*) *are:*

$$\begin{aligned} [\delta_j, \delta_k] &= 0 \ ; \quad [\delta_j, \delta_{\bar{k}}] = \delta_{\bar{k}}(\overset{KL}{N_j^i})\dot{\partial}_i - \delta_j(\overline{\overset{KL}{N_j^i}})\dot{\partial}_{\bar{i}} \\ \left[\delta_j, \dot{\partial}_k\right] &= \dot{\partial}_k(\overset{KL}{N_j^i})\ \dot{\partial}_i \ ; \quad \left[\delta_j, \dot{\partial}_{\bar{k}}\right] = \dot{\partial}_{\bar{k}}(\overset{KL}{N_j^i})\dot{\partial}_i \\ \left[\dot{\partial}_j, \dot{\partial}_k\right] &= 0 \ ; \quad \left[\dot{\partial}_j, \dot{\partial}_{\bar{k}}\right] = 0 \end{aligned} \tag{2.2}$$

and $\overline{[X, Y]} = \left[\bar{X}, \bar{Y}\right]$.

Proposition 2.2. *If* $\overset{1}{N_i^j}$ *is a given* (*c.n.c.*) *on* $T'M$ *then* $\overset{2}{N_i^j} = \frac{1}{2}\dfrac{\partial \overset{1}{N_0^j}}{\partial \eta^i}$, *where* $\overset{1}{N_0^j} = \overset{1}{N_k^j}\eta^k$, *is a* (*c.n.c.*) *too, called the spray of* $\overset{1}{N_i^j}$.

The proof consist in verifying of (1.2) law of transformation.

As a result, from a given (*c.n.c.*) $\overset{1}{N_i^j}$ we can obtain a sequence of (*c.n.c.*). A question is when this sequence becomes constant. Usual calculus prove that it is happening if and only if:

$$\overset{1}{N_i^j} = \frac{\partial \overset{1}{N_k^j}}{\partial \eta^i}\eta^k \tag{2.3}$$

Theorem 2.2. *In a complex Lagrange space the* $\overset{c}{N}{}_i^j$ *(c.n.c.) given by (1.7) is the spray of* $\overset{KL}{N}{}_i^j$ *(c.n.c.).*

Also, taking into account the first (2.2) formula of brackets it is obvious our interest for the $\overset{KL}{N}{}_i^j$ (*c.n.c.*).

Proposition 2.3. *In a complex Finsler space the* $\overset{c}{N}{}_i^j$ *(c.n.c.) and* $\overset{KL}{N}{}_i^j$ *(c.n.c.) coincides..*

Therefore, the Chern-Lagrange (*c.n.c.*) is a natural generalization of Chern-Finsler (*c.n.c.*) on a Lagrange complex space.

In respect to adapted bases of Chern-Lagrange (*c.n.c.*) we can consider the $\overset{KL}{N}{}_i^j$ $-$(*c.l.c.*) which has $h-(h,h)$ and $v-(v,v)$ zero torsions.

Remark. Similar reasons gives the notion of spray for a nonlinear connection in a real Lagrange space, where a nonlinear connection depending only on the fundamental function of the space is well-known, [8],pag.160:

$$N_j^i = \frac{\partial G^i}{\partial y^j} \ with \ G^i = \frac{1}{4} g^{ih} \left(\frac{\partial^2 L}{\partial y^h \partial x^k} y^k - \frac{\partial L}{\partial x^h} \right)$$

The existence of one nonlinear connection $\overset{1}{N}{}_j^i$ such that N_j^i be the spray of $\overset{1}{N}{}_j^i$ is still now an open problem.

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