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ON THE GEOMETRICAL THEORY OF HIGHER-ORDER HAMILTON SPACES

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ABSTRACT. One investigates the geometrical properties of the Hamilton spaces of order $k \geq 1$, the natural presymplectic and Poisson structures and Hamilton-Jacobi equations, [2],[9]. An $\mathcal{L}$-duality between the Lagrange spaces of order k and Hamilton spaces of the same order is pointed out.

INTRODUCTION

The notion of Hamilton space was introduced by the author in [3],[4]. It was defined as a pair $H^n = (M, H(x,p))$, for M a C^∞-manifold of dimension n and $H : (x,p) \in T^{*k}M \longrightarrow H(x,p) \in I\!R$ a regular Hamiltonian. H^n has a canonical symplectic structure and a canonical Poisson structure. The Hamilton spaces appear as dual, via Legendre transformation, of the Lagrange spaces $L^n = (M, L(x,y))$, [3].

The notion of Lagrange space of order $k \geq 1$, $L^{(k)n} = (M, L(x, y^{(1)}, .., y^{(k)}))$ was defined by author some years ago. Its geometry was showed in the book [7].

A definition of the notion of higher-order Hamilton space $H^{(k)n}$ is difficult to get. This is due to the fact that the space $H^{(k)n}$ must have some important properties, which extend those of $H^{(1)n} = H^n$:

a) $dim\ H^{(k)n} = dim\ L^{(k)n}$.

b) $H^{(k)n}$ has a canonical presymplectic structure.

c) $H^{(k)n}$ has at least one Poisson structure.

d) The spaces $H^{(k)n}$ and $L^{(k)n}$ to be diffeomorphic via Legendre transformation.

In the paper [5] we solved the above mentioned problem.

Now, in the lecture at the "Colloquium on Differential Geometry", July 2000, Debrecen, I should like to present an abstract of the paper [5], published this year by the *International Journal of Theoretical Physics.* Some new results concerning the $\mathcal{L}$-duality of the spaces $L^{(k)n}$ and $H^{(k)n}$ will be provided. The proofs are omitted.

1. THE "DUAL" BUNDLE OF T^kM-BUNDLE.

Let M be a real C^∞-manifold, n-dimensional and (T^kM, π^k, M) its k-accelerations bundle ($k \in I\!N^*$). It can be identified with k-osculator bundle $(Osc^k M, \pi^*, M)$. A point $u \in T^kM$ has the coordinates $(x, y^{(1)}, .., y^{(k)})$, $x \in M$ and $y^{(1)}, .., y^{(k)}$ are the "higher order accelerations". The local coordinates of u are $(x^i, y^{(1)i}, .., y^{(k)i})$. The indices $i, j, h, ..$ run over the set $\{1, .., n\}$ and summation convention will be used.

We define "the dual" of (T^kM, π^k, M) as being $(T^{*k}M, \pi^{*k}, M)$ where $T^{*k}M$ is the following fibred product:

$$T^{*k}M = T^{k-1}M \times_M T^*M \tag{1.1}$$

Clearly, $(T^{k-1}M, \pi^{k-1}, M)$ is the $k-1$-acceleration bundle and (T^*M, π^*, M) is the cotangent bundle of the base manifold M.

$T^{*k}M$ is a C^∞-differentiable manifold and $dim\, T^{*k}M = dim\, T^kM = (k+1)n$. A point $u \in T^{*k}M$ is of the form $u = (x, y^{(1)}, .., y^{(k-1)}, p)$, $\pi^{*k}(u) = x$ and u has the coordinate $(x^i, y^{(1)i}, .., y^{(k-1)i}, p_i)$.

For $k = 1, T^{*1}M$ is identified with T^*M.

The following diagram is commutative:

$$\begin{tikzcd}
& T^{*k}M \arrow[dl, "\pi^{*k}_{k-1}"'] \arrow[dd, "\pi^{*k}"] \arrow[dr, "\overline{\pi}^*"] & \\
T^{k-1}M \arrow[dr, "\pi^{k-1}"'] & & T^*M \arrow[dl, "\pi^*"] \\
& M &
\end{tikzcd}$$

The changes of local coordinates on $T^{*k}M$ can be easily written, [8]. We consider the following differential forms

$$\begin{aligned} \omega &= p_i dx^i \\ \theta &= d\omega = dp_i \wedge dx^i. \end{aligned} \tag{1.2}$$

Theorem 1.1. 1°. *The forms ω and θ are globally defined on the manifold $T^{*k}M$.*

2°. $d\theta = 0,\ rank||\theta|| = 2n.$

3°. *θ is a canonical presymplectic structure on $T^{*k}M, k > 1$.*

The proof is not difficult.

Let us consider the bracket:

$$\{f, g\} = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i}, \quad \forall f, g \in \mathcal{F}(T^{*k}M). \tag{1.3}$$

We have

Theorem 1.2. 1°. *The bracket $\{f, g\}$ has a geometrical meaning.*

2°. *$\{f, g\}$ is a Poisson structure on $T^{*k}M$.*

Indeed, one proves by a staightforward calculus, using the changes of local coordinates of $T^{*k}M$, that these brackets are conserved. Then it is shown that

$\{f, g\}$ is R-linear in every argument,

$\{f, g\} = -\{g, f\}$ and Jacobi identity holds,

the mapping $\{f, \cdot\} : \mathcal{F}(T^{*k}M) \longrightarrow \mathcal{F}(T^{*k}M)$ is a derivation in the function algebra $\mathcal{F}(T^{*k}M)$.

q.e.d.

Remark 1.1. *The following brackets*

$$\{f, g\}_\alpha = \frac{\partial f}{\partial y^{(\alpha)i}}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial y^{(\alpha)i}}\frac{\partial f}{\partial p_i}, \quad (\alpha = 1, .., k-1),$$

are Poisson structures on $T^{*k}M$.

2. Hamiltonian system of order k. The spaces $H^{(k)n}$.

A mapping $H : T^{*k}M \longrightarrow I\!R$ is called a differentiable Hamiltonian of order k, if H is a C^∞-function on $\widetilde{T^{*k}M} = T^{*k}M \setminus \{0\}$ and continuous on the null section of π^{*k}.

Definition 2.1. *An Hamilton system of order k is a triple $(T^{*k}M, \theta, H)$, where θ is a presymplectic structure on $T^{*k}M$ and H is a differentiable Hamiltonian of order k.*

In the case $k = 1$, and θ a symplectic structure, the triple $(T^{*k}M, \theta, H)$ is a classical Hamilton system.

Let us consider the section Σ_0 of the projection

$$\pi_2^* : (x, y^1, .., y^{k-1}, 0) \in T^{*k}M \longrightarrow (x, 0, .., 0, p) \in T^{*k}M.$$

Σ_0 is an imersed submanifold of the manifold $T^{*k}M$. The restrictions $\theta_o = \theta_{|\Sigma_0}$, $H_o = H_{|\Sigma_0}$ together of Σ_0 determine an Hamiltonian system of order 1, $(\Sigma_0, \theta_0, H_0)$. In this case, θ_0 is a symplectic structure on Σ_0.

It is not difficult to prove the following theorem:

Theorem 2.1. 1°. *The triple $(\Sigma_0, \theta_0, H_0)$ is an Hamiltonian system, θ_0 being a symplectic structure on the manifold Σ_0.*

2°. *There exists an unique vector field X_{H_0} on Σ_0 with the property*

$$i_{H_0}\theta_0 = -dH_0. \tag{2.1}$$

3°. *The integral curve of the vector field X_{H_0} are given by the canonical equations (Hamilton - Jacobi eq.):*

$$\frac{dx^i}{dt} = \frac{\partial H_0}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial H_0}{\partial x^i}. \tag{2.2}$$

4°. *The following equations hold:*

$$\{f, g\} = \theta(X_f, X_g), \quad \forall f, g \in \mathcal{F}(\Sigma_0). \tag{2.3}$$

Now, for a differentiable Hamiltonian $H(x, y^{(1)}, .., y^{(k-1)}, p)$, we consider its Hessian with respect to p_i. Its matrix has the elements:

$$g^{ij} = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j}. \tag{2.4}$$

We can prove that g^{ij} is a distinguished tensor field (shortly a d-tensor) on $T^{*k}M$, symmetric and contravariant.

We say that H is regular if

$$rank||g^{ij}|| = n = dim\ M \quad \text{on } \widetilde{T^{*k}M}. \tag{2.5}$$

Definition 2.2. *An Hamilton space of order k, ($k \in I\!N^{|ast}$) is a pair $H^{(k)n} = (M, H(x, y^{(1)}, .., y^{(k-1)}, p))$ formed by a C^∞-manifold M, n-dimensional and a regular Hamiltonian of order k, H with the property that the d-tensor field g^{ij} has a constant signature on $\widetilde{T^{*k}M}$.*

In the paper [5], we proved the existence of the Hamilton spaces of order k over the paracompact manifolds M.

In order to prove the duality between the Lagrange spaces of order k,

$$L^{(k)n} = (M, L(x, y^{(1)}, .., y^{(k-1)}, y^{(k)}))$$

and the Hamilton spaces of order k,

$$H^{(k)n} = (M, H(x, y^{(1)}, .., y^{(k-1)}, p))$$

we consider the Legendre mapping, defined by

$$\mathcal{L}eg : L^{(k)n} \longrightarrow H^{(k)n}$$

given by

$$\mathcal{L}eg : (x, y^{(1)}, .., y^{(k-1)}, y^{(k)}) \in T^kM \longrightarrow (x, y^{(1)}, .., y^{(k-1)}, p) \in T^{*k}M \tag{2.6}$$

where

$$p_i = \frac{1}{2} \frac{\partial L}{\partial y^{(k)i}} = \varphi_i(x, y^{(1)}, .., y^{(k-1)}, y^{(k)}). \tag{2.7}$$

We obtain:

Theorem 2.2. *The mapping $\mathcal{L}eg$, (2.6), (2.7) is a local diffeomorphism of the manifolds T^kM and $T^{*k}M$.*

Indeed, the determinant of the Jacobian matrix of the mapping $\mathcal{L}eg$ coincides with the determinant of matrix $||a_{ij}||$, where $a_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}}$. This is different of zero.

q.e.d.

Concluding, the properties a)-d) enunciated in the introduction hold.

The geometry of the higher-order Hamilton spaces $H^{(k)n}$ can be investigsted as a natural extension of the geometry of Hamilton spaces H^n.

3. $\mathcal{L}$-DUALITY BETWEEN THE SPACES $L^{(k)n}$ AND $H^{(k)n}$.

Assuming that the Lagrange space of order k,

$$L^{(k)n} = (M, L(x, y^{(1)}, .., y^{(k)}))$$

is given and a nonlinear connection $\overset{\circ}{N}$ on the manifold $T^{k-1}M$ is apriori given, too, we can determine a regular Hamiltonian such that the pair

$$H^{(k)n} = (M, L(x, y^{(1)}, .., y^{(k-1)}, p))$$

is an Hamilton space of order k. The application $\mathcal{L} : L^{(k)n} \longrightarrow H^{(k)n}$ will be called $\mathcal{L}$-duality.

Let us consider the local inverse $\mathcal{L}eg^{-1}$ of the Legendre transformation (2.6):

$$\mathcal{L}eg^{-1} : (x, y^{(1)}, .., y^{(k-1)}, p) \in T^{*k}M \longrightarrow (x, y^{(1)}, .., y^{(k-1)}, y^{(k)i}) \in T^k M$$

where

$$y^{(k)i} = \xi^i(x, y^{(1)}, .., y^{(k-1)}, p). \tag{3.1}$$

It follows:

$$\frac{\partial \xi^i}{\partial p_j} = a^{ij} \tag{3.2}$$

where a^{ij} is the contravariant tensor of the fundamental tensor of space $L^{(k)n}$.

Let us consider an apriori given nonlinear connection $\overset{\circ}{N}$ on $T^{k-1}M$, having the dual coefficients $\underset{(1)}{M}{}^i{}_j, .., \underset{(k-1)}{M}{}^i{}_j$ depending, evidently, by $(x, y^{(1)}, .., y^{(k-1)})$). Then the k-Liouville d-vector field $z^{(k)i}$ on T^kM is well defined:

$$kz^{(k)i} = ky^{(k)i} + (k-1)\underset{(1)}{M}{}^i{}_s y^{(k-1)s} + \cdots + \underset{(k-1)}{M}{}^i{}_s y^{(1)s}.$$

Consequently, the d-vector field

$$\check{z}^{(k)i} = z^{(k)i}(x, y^{(1)}, .., y^{(k-1)}, \xi^i(x, y^{(1)}, .., y^{(k-1)}, p)) \tag{3.3}$$

can be considered.

We define the function

(3.4)
$$H(x, y^{(1)}, .., y^{(k-1)}, p) = 2p_i \check{z}^{(k)i} - L(x, y^{(1)}, .., y^{(k-1)}, \xi^i(x, y^{(1)}, .., y^{(k-1)}, p)).$$

We can prove that H is an Hamiltonian defined on an open set of the manifold $T^{*k}M$.

So, the construction is a local one.

The following theorem holds:

Theorem 3.1. *The pair $H^{(k)n} = (M, H)$ with H from (3.4) is an Hamilton space having the fundamental tensor*

$$g^{ij}(x, y^{(1)}, .., y^{(k-1)}, p) = a^{ij}(x, y^{(1)}, .., y^{(k-1)}, \xi^i(x, y^{(1)}, .., y^{(k-1)}, p)).$$

We can use this $\mathcal{L}$-duality to transform the main geometrical object fields of the space $L^{(k)n}$ in the main geometrical object fields of the space $H^{(k)n}$.

In the case $k = 1$, we obtain the classical $\mathcal{L}$-duality between the Lagrange space $L^n = (M, L(x, y))$ and Hamilton spaces $H^n = (M, H(x, p))$.

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