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ON K-CONTACT η-EINSTEIN MANIFOLDS

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ABSTRACT. The object of the present paper is to study a K-contact η-Einstein manifold satisfying certain conditions on the curvature tensor.

1. INTRODUCTION

Let (M^n, g) be a contact Riemannian manifold with contact from η, associated vector field ξ, $(1,1)$-tensor field φ and associated Riemannian metric g. If ξ is a killing vector field, then M^n is called a K-contact Riemannian manifold [1], [2]. A K-contact Riemannian manifold is called Sasakian [1], if the relation

$$(\nabla_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X \tag{1.1}$$

holds, where ∇ denotes the operator of covariant differentiation with respect of g.

Recently, M. C. CHAKI and M. TARAFDAR [3] studied a Sasakian manifold M^n $(n > 3)$ satisfying the relation $R(X,Y) \,.\, C = 0$, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold and C is the Weyl conformal curvature tensor of type $(1,3)$. Generalizing the result of CHAKI and TARAFDAR, N. GUHA and U. C. DE [4] proved that if a K-contact manifold with charcteristic vector field ξ belonging to the k-nullity distribution satisfies the condition $R(\xi, X) \,.\, C = 0$, then $C(\xi, X)Y = 0$ for any vector fields X, Y. In Section 3 of the present paper we prove, without assuming that ξ belongs to the K-nullity distribution, that a K-contact η-Einstein manifold (M^n, g) $(n > 3)$ satisfying the condition $R(X, \xi) \,.\, C = 0$ is a space of constant curvature.

In [5] S. TANNO studied a K-contact manifold satisfying the condition $R(X, \xi) \,.\, S = 0$, where S is the Ricci tensor of type $(0,2)$. But the condition $R(X, \xi) \,.\, S = 0$ does not imply the condition $S(X, \xi) \,.\, R = 0$. In Section 4 we prove that if a K-contact manifold M^n still satisfies the relation $S(X, \xi) \,.\, R = 0$ than it is an η-Einstein manifold.

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2. Preliminaries

In a K-contact Riemannian manifold the following relations hold: [1], [2], [6]

$$a)\quad \varphi\xi = 0, \qquad b)\quad \eta(\xi) = 1 \qquad c)\quad g(X,\xi) = \eta(X) \tag{2.1}$$

$$\varphi^2 X = -X + \eta(X)\xi \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

$$\nabla_X \xi = -\varphi X \tag{2.4}$$

$$g(R(\xi,X)Y),\xi) = \eta(R(\xi,X)Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.5}$$

$$R(\xi,X)\xi = -X + \eta(X)\xi \tag{2.6}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{2.7}$$

$$(\nabla_X \varphi)(Y) = R(\xi,X)Y \tag{2.8}$$

for any vector fields X, Y.

A K-contact manifold M^n is said to be η-Einstein if its Ricci tensor S is of the form $S = ag + b\eta \otimes \eta$, where a, b are smooth functions on M.

3. K-contact η-Einstein manifolds satisfying $R(X,\xi)\,.\,C = 0$

Let us consider a K-contact η-Einstein manifold $M^n (n > 3)$ satisfying the relation

$$R(X,\xi)\,.\,C = 0. \tag{3.1}$$

In this case we have

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{3.2}$$

Putting $X = Y = \xi$ in (3.2) and then using (2.7) and (2.1) b, we get

$$n - 1 = a + b. \tag{3.3}$$

Also (3.2) implies that

$$r = an + b. \tag{3.4}$$

From (3.3) and (3.4) we have

$$a = \frac{r}{n-1} - 1, \qquad b = n - \frac{r}{n-1}. \tag{3.5}$$

Again from (3.2) we obtain

$$QX = \left(\frac{r}{n-1} - 1\right) X + \left(n - \frac{r}{n-1}\right) \eta(X)\xi, \tag{3.6}$$

where Q denotes the Ricci operator, i.e. $g(QX,Y) = S(X,Y)$.

By definition the Weyl conformal curvature tensor C is given by

$$\begin{aligned}(3.7)\qquad C(X,Y)Z = R(X,Y)Z &- \frac{1}{n-2}\big[g(Y,Z)QX - g(X,Z)QY \\ &+ S(Y,Z)X - S(X,Z)Y\big] \\ &+ \frac{r}{(n-1)(n-2)}\big[g(Y,Z)X - g(X,Z)Y\big].\end{aligned}$$

Using (3.2) and (3.6) in (3.7), we get

$$\begin{aligned}(3.8)\qquad C(X,Y)Z = R(X,Y)Z &+ \left[\frac{2}{(n-1)} - \frac{r}{(n-2)}\right]\big[g(Y,Z)X - g(X,Z)Y\big] \\ &- \left[\frac{n}{n-1} - \frac{r}{(n-1)(n-2)}\right]\big[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\big].\end{aligned}$$

Now (3.1) gives us by definition

$$\begin{aligned}(3.9)\qquad R(X,\xi)C(U,V)W &- C(R(X,\xi)U,V)W - C(U,R(X,\xi)V)W \\ &- C(U,V)R(X,\xi)W = 0, \quad \text{for all} \quad X,U,V,W.\end{aligned}$$

Substitution of U and W by ξ in (3.9) yields

$$\begin{aligned}(3.10)\qquad R(X,\xi)C(\xi,V)\xi &- C(R(X,\xi)\xi,V)\xi - C(\xi,R(X,\xi)V)\xi \\ &- C(\xi,V)R(X,\xi)\xi = 0.\end{aligned}$$

From (3.8) we get by virtue of (2.1) (b), (2.1) (c) and (2.6),

$$(3.11)\qquad C(\xi,V)\xi = 0, \quad \text{for any vector field } V.$$

Hence by virtue of (3.11) we have

$$(3.12)\qquad R(X,\xi)C(\xi,V)\xi = 0.$$

Again in view of (2.6) we get

$$C(R(X,\xi)\xi,V)\xi = C(X,V)\xi - \eta(X)C(\xi,V)\xi$$

which implies by means of (3.11) that

$$(3.13)\qquad C(R(X,\xi)\xi,V) = C(X,V)\xi.$$

From (3.8) we obtain

$$(3.14)\qquad C(X,V)\xi = R(X,V)\xi - \eta(V)X + \eta(X)V$$

for any vector fields X and V. By virtue of (3.13) and (3.14) we have

$$(3.15)\qquad C(R(X,\xi)\xi,V)\xi = R(X,V)\xi - \eta(V)X + \eta(X)V,$$

and by virtue of (3.11) we get

$$(3.16)\qquad C(\xi,R(X,\xi)V)\xi = 0.$$

Finally using (2.6), we have

$$C(\xi,V)R(X,\xi)\xi = C(\xi,V)X - \eta(X)C(\xi,V)\xi,$$

from which it follows by means of (3.11) and (3.14) that

$$C(\xi, V)R(X, \xi)\xi = R(\xi, V)X - g(X, V)\xi + \eta(X)V. \tag{3.17}$$

Applying (3.12), (3.15), (3.16) and (3.17) in (3.10) we obtain

$$R(X, V)\xi + R(\xi, V)X - g(X, V)\xi - \eta(V)X + 2\eta(X)V = 0. \tag{3.18}$$

Interchanging X and V in (3.18) we have

$$R(V, X)\xi + R(\xi, X)V - g(X, V)\xi - \eta(X)V + 2\eta(V)X = 0. \tag{3.19}$$

Subtracting (3.19) from (3.18) and then using Bianchi's first identity, we get

$$R(X, V)\xi = \eta(V)X - \eta(X)V,$$

from which it follows that

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X. \tag{3.20}$$

In view of (1.1), (2.8) and (3.20), we obtain that the manifold is Sasakian and hence by the result of CHAKI and TARAFDAR [3], the manifold is a space of constant curvature 1. Thus we have the following

Theorem 1. *A K-contact η-Einstein manifold (M^n, g) $(n > 3)$ satisfying the condition $R(X, \xi) \,.\, C = 0$ is a space of constant curvature 1.*

A contact Riemannian manifold satisfying the condition $R(X, \xi) \,.\, C = 0$ have been studied by C. BAIKOUSSIS and T. KOUFOGIORGOS [7].

4. K-CONTACT MANIFOLDS SATISFYING THE CONDITION $S(X, \xi) \,.\, R = 0$

We consider a K-contact Riemannian manifold M^n satisfying the condition

$$(S(X, \xi) \,.\, R(U, V)W = 0. \tag{4.1}$$

Now by definition we have

$$\begin{aligned}(S(X, .\xi) \,.\, R(U, V)W = ((X \wedge_s \xi) \,.\, R)(U, V)W &= (X \wedge_s \xi)R(U, V)W \\ &+ R((X, \wedge_S \xi)U, V)W + R((U, (X \wedge_s \xi)V)W \\ &+ R(U, V)(X \wedge_s \xi)W,\end{aligned} \tag{4.2}$$

where the endomorphism $X \wedge_s Y$ is defined by

$$(X \wedge_s y)Z = S(Y, Z)X - S(X, Z)Y. \tag{4.3}$$

Using the definition of (4.3) in (4.2), we get by virtue of (2.7)

$$\begin{aligned}S(X, \xi) \,.\, R(U, V)W = (n-1)\big[&\eta(R(U, V)W)X + \eta(U)R(X, V)W \\ &+ \eta(V)R(U, X)W + \eta(W)R(U, V)X\big] \\ &- S(X, R(U, V)W)\xi - S(X, U)R(\xi, V)W \\ &- S(X, V)R(U, \xi)W - S(X, W)R(U, V)\xi\end{aligned} \tag{4.4}$$

and by virtue of (4.1) and (4.4) we have

$$
\begin{aligned}
&(n-1)\big[\eta(R(U,V)W)X+\eta(U)R(X,V)W+\eta(V)R(U,X)W \\
&+\eta(W)R(U,V)W)\big]-S(X,R(U,V)W)\xi-S(X,U)R(\xi,V)W \\
&-S(X,V)R(U,\xi)W-S(X,W)R(U,V)\xi=0.
\end{aligned} \tag{4.5}
$$

Taking the inner product on both sides of (4.5) by ξ, we obtain

$$
\begin{aligned}
&(n-1)\big[\eta(R(U,V)W)\eta(X)+\eta(U)\eta(R(X,V)W \\
&\quad+\eta(V)\eta(R(U,X)W)+\eta(W)\eta(R(U,V)X)\big] \\
&\quad-S(X,R(U,V)W)-S(X,U)\eta(R(\xi,V)W) \\
&\quad-S(X,V)\eta(R(U,\xi)W)-S(X,W)\eta(R(U,V)\xi)=0.
\end{aligned} \tag{4.6}
$$

Putting $U=W=\xi$ in (4.6) and using (2.5)–(2.7) we get

$$S(X,V)=-(n-1)g(X,V)+2(n-1)\eta(X)\eta(V)$$

which means that the manifold is η-Einstein.

Thus we have the following

Theorem 2. *A K-contact Riemannian manifold (M^n,g) satisfying the condition $S(X,\xi)\,.\,R=0$ is an η-Einstein manifold.*

From Theorem 1 and Theorem 2 we immediately have:

Theorem 3. *A K-contact Riemannians manifold (M^n,g) stisfying the conditions $S(X,\xi)R=0$ and $R(X,\xi)C=0$ is a space of constant curvature 1.*

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