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REGULARIZATION OF DIFFERENTIAL OPERATORS ON A HILBERT SPACE AND GEOMETRIC MEANING OF ZETA-REGULARIZATION

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ABSTRACT. Let H be a Hilbert space and G a positive non-degenerate Schatten class operator on H such that whose zeta function $\zeta(G,s) = tr(G^s)$ is holomorphic at $s = 0$. By using spectra of G, we propose a regularization of differential operators on H and compute proper values and functions of the regularized Dirac operator on H with the periodic boundary condition. The answer provides determinant bundle interpretation of zeta-regularization.

1. INTRODUCTION

In the study of infinite dimensional geometry, sometimes we need to give a finite number to the quantity which is originally infinite. We say this process regularization. For example, to add infinite degree element or γ_5 to the Grassmann algebra or Clifford algebra over an infinite dimensional space V, we need to regularize the dimension of V to be finite. At present, the most powerful tool for the regularization is the zeta-regularization. Roughly speaking, zeta-regularization is done as follows. Let D be a differential operator with spectra λ_n. The zeta function $\zeta(D,s)$ of D is defined be $\sum \lambda_n^{-s}$. If $\zeta(D,s)$ allows analytic continuation to $s = 0$ and holomorphic at $s = 0$,then the finite number $\zeta(D,0)$ regularizes the number of proper values of D, which is infinite. Thanks to the excellent results of Atiyah-Patodi-Singer and Gilkey([11], for the operators on non compact space, we refer [8]), we can apply this regularization to many interesting problems of geometry and physics. But in my knowledge, the geometric meanings of zeta-regularization were never discussed. In this paper, we show determinant bundle interpretation of zeta-regularization is possible via the study of proper value problem of regularized Dirac operator on a Hilbert space with the periodic boundary condition.

The paper is organized as follows. To define the regularization of a differential operator D on a Hilbert space H, we need to associate a special Schatten class operator G to H. The regularization $: D :$ of D is defined by

$$: D : f = G^{-s} D G^s f|_{s=0}.$$

These are explained in the next section(Section 2). Since our attention in this paper is concentrated to the regularized Dirac operator, we review the definition of Clifford algebra with the infinite spinor (γ_5) over H in Section 3. To consider periodic boundary condition, H is not appropriate. Appropriate space to consider periodic boundary condition is introduced in Section 4. Then the proper value problem of the regularized Dirac operator with the periodic boundary condition is solved in Section 5. The answer provides determinant bundle interpretation and other geometric interpretations of zeta-regularization. These are explained in the last section.

Acknowledgement. The outline of this paper together with the results on regularized Laplacian are given in [6]. [6] also contains report on the results on regularized Laplacian which is a joint work with N.Tanabe.

2. Regularization of differential operators on a Hilbert space

Let H be a (real) Hilbert space, G a positive non-degenerate Schatten class operator on H such that whose zeta-function $\zeta(G,s) = tr(G^s)$ allows analytic continuation to $s=0$ and holomorphic at $s=0$. In the rest, we fix the pair H, g. This pair has the following numerical invariants.

1. $\nu = \zeta(G,0)$, the regularized dimension of H.
2. d, the place of the first pole of $\zeta(G,s)$.
3. $detG = \exp(\zeta'(G,0))$.

The meaning of d can be seen by the following example.

Example. Let H be $L^2(X,E)$, X is a compact Riemannian manifold and E is a symmetric vector bundle over X, G the Green operator of a non-degenerate self-adjoint elliptic (pseudo) differential operator D acting on the sections of E. Then the pair H, G satisfies above assumptions, because $\zeta(G,s)$ is nothing but the spectral zeta-function $\zeta(D,s)$ of D in this case. If D is a 1-st order operator, then above d is the dimension of X.

Note 1. The positivity of G is not necessary. It is assumed only for the simplicity. In fact, the most interesting geometric example is obtained in the above example assuming X is a spin manifold and take D to be the Dirac operator ([2]).

Note 2. G is a kind of metric on H , but since G has no bounded inverse, such metric does not exist in finite dimensional case. Considering the pairing $\{H, G\}$ is closely related to Connes' spectre triple ([10]).

The complete ortho-normal basis of H is fixed to be e_n, e_n is a proper function of G: $Ge_n = \mu_n e_n$. For simple, we arrange μ_n as follows. $\mu_1 \geq \mu_2 \geq .... \geq > 0$.This assumption provides limitation to the symmetry of H, G. But in this paper, we do not discuss on this problem. We introduce k-th Sobolev norm $||x||_k$ of an element x of H by

$$||x||_k = ||G^{-k}x||.$$

Here the power G^k of G is defined by

$$G^k f = \sum \mu^k (f, e_n) e_n.$$

We note that since G is positive,for an arbitrary complex number s, G^s is defined by the same way. The Sobolev space constructed from H and $||x||_k$ is denoted by W^k. By the definition of Sobolev norm, $e_{n,k}$, $e_{n,k} = \mu^k e_n$ is the (fixed) complete orth-normal basis of W^k. So the regularized dimension of W^k is ν, the regularized dimension of H. We define the coordinate of $x = \sum x_n e_n$ by $(x_1, x_2, ...)$. The coordinate $(x_{1,k}, x_{2,k}, ...)$ of an element x of W^k is defined by $x = \sum x_{n,k} e_{n,k}$. Since $x_{n,k} = \mu^k x_n$, we have

$$G^{-k}(\partial/\partial x_n)G^k f = \mu^k \partial f/\partial x_n,$$
$$G^k f(x) = f(G^k x).$$

Let D be a differential operator on H. Then we set $D(s) = G^{-s} D G^s$. Here s may be a complex number. Then we define

Definition. Let D be a differential operator on H and f a function on H such that $D(s)f$ is defined for large s and allows analytic continuation to $s = 0$ and holomorphic at $s = 0$. Then we define the regularization $: D :$ of D by

$$: D : f = D(s)f|_{s=0}.$$

Example. Let $\triangle = \sum \partial^2/\partial x_n^2$ be the Laplacian on H. Then, since $\partial^2 r/\partial x_n^2 = 1/r - x_n^2/r^3$, $r(x) = ||x||$, $\triangle r$ diverges. But since $\triangle(s)r = \zeta(G, 2s)/r - \sum \mu_n^2 x_n^2/r^3$, we have $: \triangle : r = (\nu - 1)/r$ as expected.

$\triangle$ on H does not allow polar coordinate expression. But $: \triangle :$ allows polar coordinate expression. In [5](see also [6]), we calculated proper values and functions of regularized spherical Laplacian on H. Tanabe also investigated regularized spherical Laplacian from the view point of quantum mechanics on H. He also introduce regularized momentum operator on the "sphere" of H and discuss the relation between Coulomb gauge constraint and the Riemann hypothesis. The meaning of Planck constant via the regularization is also discussed ([15]). But in this paper, we concentrate our attention to the study of regularized Dirac operator on H.

3. Clifford algebra with an infinite spinor

Clifford algebra $Cl(H)$ over H is the algebra generated by $\{e_n\}$ with the relations $e_n e_m + e_m e_n = -2\delta_{n,m}$. Here the sign determined by the later convenience. We give Hilbert space structure to $Cl(H)$ by the inner product induced from the following inner product of generators

$$(e_{i_1} e_{i_2} \cdots e_{i_p}, e_{i_1} e_{i_2} \cdots e_{i_p}) = 1,$$

$$(e_{i_1} e_{i_2} \cdots e_{i_p}, ej_1 ej_2 \cdots e_{j_q} = 0, (i_1, i_2, \cdots i_p) \neq (j_1, j_2, \cdots, j_q),$$

(cf.[4], [12]). But $Cl(H)$ lacks the infinite spinor or γ_5, that is the product of all generators. So we add infinite spinor e_∞ to $Cl(H)$ by the relation

$$e_\infty e_n = (-1)^{\nu-1} e_n e_\infty, e_\infty^2 = (-1)^{\nu(\nu-1)/2}.$$

This extended algebra is called Clifford algebra over H with an infinite spinor and denoted by $Cl(H)[e_\infty]$. As a module, we have $Cl(H)[e_\infty]$ is the direct sum of $Cl(H)$ and $Cl(H)e_\infty$.

Note. In general, $Cl(H)[e_\infty]$ is not associative. It is shown $Cl(H)[e_\infty]$ is associative if and only if the regularized dimension ν of H is an integer ([4], see also [2]). Of course, for arbitrary G, we can not expect the integrity of ν. But if G is the Green operator of an elliptic operator D, we can select a mass-term m such that $D+m$ is non-degenerate and $\zeta(D+m,0)$ becomes an integer ([3], [9], see also [7]). Hence we may assume the integrity of ν. In the rest, we assume the regularized dimension of H is an integer.

We define the Clifford algebras $Cl(W^k)$ and $Cl(W^k)[e_{\infty,k}]$ over W^k by the same way . Since these algebras are generated by $\{e_{n,k}\}$, we may consider these algebras to be the algebras generated by $\{e_n\}$ with the relations

$$e_n e_m + e_m e_n = -2\mu_n^{-2k}\delta_{n,m}, e_{\infty,k}^2 = (-1)^{\nu(\nu-1)/2}(detG)^{-2k}.$$

While the commutation relation of $e_{\infty,k}$ and e_n is same that of e_∞ and e_n.

It is known $Cl(H)$ is isomorphic to $Gr(H)$, the Grassmann algebra over H, as a module and has a representation in the algebra of bounded linear operators on $Gr(H)$ ([12]). Extending this, we can represent $Cl(W^k)[e_\infty]$ in the algebra of bounded linear operators on $Gr(W^k) \oplus Gr(W^{-k})$. In this representation, $e_{\infty,k}$ is represented by an offdiagonal matrix with the matrix elements G^{2k} and G^{-2k} ([4]).

To consider periodic boundary condition, we need to modify H and use Clifford algebras over the modified spaces. So we introduce the following spaces:

$$W^{k-0} = \bigcap_{l<k} W^l,$$

$$W^{k+0} = \bigcup_{l>k} W^l.$$

If $k=0$, we denote H^- and H^+ instead of W^{0-0} and $W\ 0+0$.They are projective limits and inductive limits of Sobolev spaces and have the following coordinate expressions:

$$W^{k-0} = \{\sum x_n e_{n,k} | \sum \mu_n^{-l} x_n^2 < \infty, \quad \textit{for some } l>0\},$$

$$W^{k+0} = \{\sum x_n e_{n,k} | \sum \mu_n^{l} x_n^2 < \infty, \quad \textit{for all } l>0\}.$$

Hence the Clifford algebras with infinite spinors over W^{k+0} and W^{k-0} are generated by $\{e_{n,k}\}$. In the rest, we do not use W^{k+0}. But we note that W^{-k+0} is the dual space of W^{k-0} by the standard Sobolev duality.

The Dirac operator D on H is $\sum e_n \partial/\partial x_n$. Similarly, the Dirac operator on W^k is given by $\sum e_{n,k}\partial/\partial x_{n,k}$.This operator can be regarded as an operator on W^{k-0} in one hand and $D(k)$ on the other hand. The regularized Dirac operator $:D:$ on

H can be regarded as an operator on H^- and its subspaces.We note that on scalar functions, we have

$$(: D :)^2 = - : \triangle : .$$

4. The periodic boundary condition

Our boundary condition to the operator $: D :$ (and $D(s)$) is

$$f|_{x_n=-\mu_n^{d/2}} = f|_{x_n=\mu_n^{d/2}}.$$

$$\partial f/\partial x_n|_{x_n=-\mu_n^{d/2}} = \partial f/\partial x_n|_{x_n=\mu_n^{d/2}}.$$

But H is not appropriate to this condition. Because $\sum \mu_n^{d/2} e_n$ does not belong to H. But this element belongs to H^-. We also introduce the following subspaces of H^-.

$$H^-(finite) = \{\sum x_n e_n \in H^- | lim_{n\to\infty} \mu_n^{-d/2} x_n \ exists\},$$

$$H^-(0) = \{\sum x_n e_n \in H^- | lim_{n\to\infty} \mu_n^{-d/2} x_n = 0\}.$$

$\sum \mu_n^{d/2} e_n$ belongs to $H^-(finite)$. By the map

$$\sum x_n e_n \mapsto \sum (x_n - t\mu_n^{d/2}) e_n + t,$$

$\lim_{n\to\infty} \mu_n^{d/2} x_n = t$, we have

$$H^-(finite) = H^-(0) \oplus \mathbf{R},$$

as a module. We consider $H^-(0)$ to be a subspace of H^-. But the topology of $H^-(finite)$ is the product space topology induced by this direct sum decomposition.

Note 1. We consider the regularized dimension of $H^-(0)$ to be ν. Then the regularized dimension of $H^-(finite)$ should be $\nu + 1$. This might causes ambiguity in the later discussions. But as we remark in the next Section, $H^-(finite)$ can be interpreted as the total space of the determinant bundle over $H^-(0)$, which justifies later discusions.

Note 2. We can define the spaces $W^{k-0}(finite)$ and $W^{k-0}(0)$ by the same way. Similar to $H^-(finite)$ and $H^-(0)$, for these spaces, we have

$$W^{k-0}(finite) = W^{k-0}(0) \oplus \mathbf{R}.$$

We denote the lattices in $H^-(finite)$ and $H^(0)$ generated by the periodic boundary condition by $\mathbf{Z}^\infty$ and $\mathbf{Z}^\infty(0)$, respectively. Explicitly, they are given by

$$\mathbf{Z}^\infty = \{\sum m_n \mu_n^{d/2} e_n | m_n \quad are\ integers\ such\ that\ m_n = m_{n+1} = \ldots = m_\infty\},$$

$$\mathbf{Z}^\infty(0) = \{\sum m_n \mu_n^{d/2} e_n | m_n \quad are\ integers\ and\ m_n = 0 \quad for\ n\ is\ large\}.$$

By definitions, we have

$$\mathbf{Z}^\infty \cong \{(m_1, m_2, \ldots)|m_n \in \mathbf{Z},\ m_n = m_{n+1} = \cdots = m_\infty,\ n \text{ is large}\},$$

$$\mathbf{Z}^\infty(0) \cong \{(m_1, m_2, \ldots)|m_n \in \mathbf{Z}, m_n = 0, n \text{ is large}\}.$$

Similar to $H^-(finite)$, $\mathbf{Z}^\infty$ is the direct sum of $\mathbf{Z}^\infty(0)$ and $\mathbf{Z}$: $\mathbf{Z}^\infty = \mathbf{Z}^\infty(0) \oplus \mathbf{Z}$. The direct sum decomposition is given by the map

$$(m_1, m_2, \ldots) \mapsto (m_1 - m_\infty, m_2 - m_\infty, \ldots, m_\infty).$$

The fundamental domains of $\mathbf{Z}^\infty$ and $\mathbf{Z}^\infty(0)$ in $H^-(finite)$ and $H^-(0)$ are denoted by Q^∞ and $Q^\infty(0)$, respectively. Then we have $Q^\infty = Q^\infty(0) \times \mathbf{I}$. Here $\mathbf{I}$ means the interval $[-1, 1]$.

Note. Strictly saying, we need to assume each m_n to be an even number for the lattice generated by the periodic boundary condition. But we use above definitions for the simplicity.

Lemma. *Let $f(x)$ be $\prod f_n(x_n)$, where $f_n(x_n)$ is either of $\sin(m_n\mu_n^{d/2}\pi x_n)$ or $\cos(m_n\mu_n^{d/2}\pi x_n)$, m_n is an integer. Then $f(x)$ vanishes on Q^∞ unless $f_n(x_n)$ is equal to $\sin(m_n\mu_n^{d/2}\pi x_n)$ except finite numbers of n, or $f_n(x_n)$ is equal to $\cos(m_n\mu_n^{d/2}\pi x_n)$ except finite numbers of n.*

Proof. First we note that the infinite product $\prod \sin(x_n)$ vanishes unless $\lim_{n\to\infty} |\sin(x_n)| = 1$. Because otherwise, $\liminf_{n\to\infty} |\sin(x_n)| = c < 1$, so there are infinitely many n such that $|\sin(x_n)| < c' < 1$.

Applying this fact to $f(x)$, we have Lemma, because on Q^∞, $\lim_{n\to\infty} m_n\mu_n^{d/2}\pi x_n$ exists.

Note. On $Q^\infty(0)$, $f(x)$ vanishes unless $f_n(x_n)$ is equal to $\cos(m_n\mu_n^{d/2}\pi x_n)$ except finite numbers of n. That is the possibility of non-vanishing of infinite product of $\sin(m_n\mu_n^{d/2}\pi x_n)$ is excluded.

The quotient spaces of $H^-(finite)$ and $H^-(0)$ by the lattices $\mathbf{Z}^\infty$ and $\mathbf{Z}^\infty(0)$ are denoted by T^∞ and $T^\infty(0)$. They are infinite dimensional tori such that $T^\infty = T^\infty(0) \times S^1$ The diameter and orientation of S^1 will be given in the next Section.

5. Proper values and proper functions of regularized Dirac operator with the periodic boundary condition

We compute proper values and functions of the regularized Dirac operator $: D :$ with the periodic boundary condition, by the method of separation of variables. $: D :$ is considered to be an operator on $H^-(finite)$.

We assume a proper function f of $: D :$ takes the form $\prod f_n(x_n)$, $f_n(x_n) = v_n(x_n) + e_n w_n(x_n)$, where $v_n(x_n)$ and $w_n(x_n)$ are scalar functions. The boundary condition imposed to $f_n(x_n)$ is

$$f_n(-\mu_n^{d/2}) = f_n(\mu_n^{d/2}), \quad \partial f/\partial x_n(-\mu_n^{d/2}) = \partial f/\partial x_n(\mu_n^{d/2}).$$

The equation $D(s)f = \lambda(s)f$ induces the equation

$$\partial v_n/\partial x_n = \mu_n^s \lambda_n w_n, \quad \partial w_n/\partial x_n = -\mu_n^s \lambda_n v_n.$$

Hence we have

$$\lambda_n = m_n \mu_n^s \pi,$$
$$v_n = A\cos(m_n \mu_n^{-d/2}\pi x_n) + B\sin(m_n \mu_n^{-d/2}\pi x_n).$$

Here m_n means an integer. Since we consider $: D :$ is defined on $H^-(finite)$, the series $\{m_1, m_2, \ldots\}$ is not arbitrary. It needs to satisfy $m_n = m_{n+1} = \cdots = m_\infty$, for n is large. Hence, if the infinite product $\prod f_n(x_n)$ has a meaning, we have

$$D(s)f = -(m_\infty \zeta(G, s-d/2) + \sum(m_n - m_\infty)\mu_n^{s-d/2})f.$$

Here, $\sum(m_n - m_\infty)\mu_n^{s-d/2}$ is a finite sum, hence always has a meaning. Therefore, if $\zeta(G, -d/2)$ is finite, we obtain

$$: D : f = -(m_\infty \zeta(G, -d/2) + \sum(m_n - m_\infty)\mu_n^{-d/2})f.$$

To search the meaning of the infinite product $\prod f_n$, we set

$$f_n = A_n \cos(m_n \mu_n^{-d/2}\pi(x_n + c_n) + \sin(m_n \mu_n^{-d/2}\pi(x_n + c_n)e_n.$$

Since we are considering infinite product, we may assume A_n is equal to 1 for all n. We also assume c_n is equal to 0, for simple. If $\lim_{n\to\infty} |\cos(m_n \mu_n^{-d/2}\pi x_n)| = 1$, then by Lemma in the Section 4, in the development of the infinite product

$$\prod(\cos(m_n \mu_n^{-d/2}\pi x_n) + \sin(m_n \mu_n^{-d/2}\pi x_n)e_n),$$

only those terms that contain finite product of e_n survive. That is this infinite product takes the value in $Cl(H^-(finite))$. Here we use the convention $(-1)^\infty = (-1)^\nu$.

If $\lim_{n\to\infty} |\sin(m_n \mu_n^{-d/2}\pi x_n)| = 1$, we rewrite $\prod f_n$ as follows.

$$\prod(\cos(m_n \mu_n^{-d/2}\pi x_n) + \sin(m_n \mu_n^{-d/2}\pi x_n)e_n)$$
$$= e_\infty \prod(\sin(m_n \mu_n^{-d/2}\pi x_n) + (-1)^{\nu-n}\cos(m_n \mu_n^{-d/2}\pi x_n)e_n).$$

Because we thought e_∞ to be the infinite product $e_1 e_2 \cdots$. Then by using Lemma, we conclude this infinite product takes the value in $Cl(H^-(finite)e_\infty$. Lemma also shows except these two cases, all terms in the expansion of the infinite product $\prod f_n$ vanishes on Q^∞.

Summarizing these, we have

Proposition. *As for the proper values and functions of $: D :$ with the periodic boundary condition, we have the following two cases.*

Case 1. $m_\infty = 0$.: In this case, proper values and functions of $: D :$ comes from finite dimensional cases. Proper functions take the values in $Cl(H^-(finite))$ and we need not to adjoin the infinite spinor e_∞ (or γ_5) to $Cl(H^-(finite))$.

Case 2. $m_\infty \neq 0.$*: In this case, proper values and functions of* $: D :$ *have no finite dimensional analogy. Proper functions take the values in* $Cl(H^-(finite))e_\infty$ *at some points of* Q^∞*. Hence we need to adjoin* e_∞ *to* $Cl(H^-(finite))$*.*

We also need to emphasize that to get proper functions of $: D :$, we only need to adjoin only one infinite spinor to $Cl(H^-(finite))$. That is, we need not to consider half infinite spinor and so on. This comes from the positivity of G. If we use the Green operator of the Dirac operator as G, we need to adjoin two half infinite spinors to $Cl(H^(finite))$. The boundary condition imposed to $: D :$ in this case might be not Hermitian. These will be discussed in future.

Note. In general, we can not exclude the possibility that $\zeta(G, -d/2)$ belongs to the set $\{\sum m_n \mu_n^{-d/2} | m_n \in \mathbf{Z}.\}$ Since this is a countable set, in geometric examples, suitable selection of mass term implies $\zeta(G, -d/2) \notin \{\sum m_n \mu_n^{-d/2}\}$. But by the integrity of ν, the regularized dimension of H, only countable mass term are allowed. So there may exist example that $\zeta(G, -d/2)$ always belongs to $\{\sum m_n \mu_n^{-d/2}\}$ if ν is an integer.

Next, we need to show the above proper values and functions are exhaust proper values and functions of $: D :$. To show this, first we note that since $||\cos(m_n \mu_n^{-d/2} \pi x_n) + \sin(m_n \mu_n^{-d/2} \pi x_n) e_n||$ is equal to 1, for $-1 \leq x \leq 1$. This implies

$$\int_{-\mu_n^{d/2}}^{\mu_n^{d/2}} ||\cos(m_n \mu_n^{-d/2} \pi x_n) + \sin(m_n \mu_n^{-d/2} \pi x_n) e_n||^2 dx_n = 2\mu_n^{d/2}.$$

Hence we may regularize the integral of $f = \prod f_n$ on Q^∞ as follows.

$$\int_{Q^\infty} ||f||^2 dx_1 dx_2 \cdots = 2\ \nu (detG)^{d/2}.$$

Therefore, we define the L^2 norm of f as an element of $L^2(Q^\infty)$ by $||f||^2 = 2^{\nu/2}(detG)^{d/4}$, that is we define

$$||2^{-\nu/2}(detG)^{-d/4} f|| = 1.$$

We set $\mathbf{m} = (m_1, m_2, \ldots) \in \mathbf{Z}^\infty$ and denote

$$\prod (\cos(2m_n \mu^{-d/2} \pi x_n) + \sin(2m_n \mu^{-d/2} \pi x_n) e_n) = f(\mathbf{m}; x).$$

Then the set

$$\{2^{-\nu/2}(detG)^{-d/4} f(\mathbf{m}; x) | \mathbf{m} \in \mathbf{Z}^\infty\},$$

forms an ortho-normal system in $L^2(Q^\infty)$. It is complete as an ortho-normal system of $L^2(T^\infty)$. Therefore above proper values and functions exhaust the proper values and functions of $: D :$ with the periodic boundary condition.

Note 1. Strictly saying, $L^2(Q^\infty)$ and $L^2(T^\infty)$ are not defined. The above discussion claims the validity to define $L^2(T^\infty)$ to be the Hilbert space spanned by $f(\mathbf{m}; x), (m) \in \mathbf{Z}^\infty$.

Note 2. T^∞ is the product space of $T^\infty(0)$ and S^1. Since this last S^1 corresponds to the proper value $\zeta(G, -d/2)$, the diameter of S^1 is $|\zeta(G, -d/2)/2\pi|$ and the orientation of S^1 is plus if $\zeta(G, -d/2)$ is positive and minus if $\zeta(G, -d/2)$ is negative.

In [1] (cf.[2]), we constructed the determinant bundle of a Hilbert manifold modelled by $\{H, G\}$ by using the determinant of G. Hence above calculation of the norm of $f(\mathbf{m}; x)$ shows it is better to consider proper functions of : D : to be the sections of the determinant bundle of H. More precisely, $H^-(finite))$ is not a subspace of H^-. It is the total space of the determinant bundle of $H^-(0)$. Related to this remark, we note that $f(\mathbf{m}; x)$ vanishes on $H^-(0)$, unless $\mathbf{m}$ belongs to $\mathbf{Z}^\infty(0)$:

$$f(\mathbf{m}; x) = 0, x \in Q^\infty(0), if$$
$$\mathbf{m} \notin \mathbf{Z}^\infty(0).$$

That is unless $f(\mathbf{m}; x)$ comes from finite dimensional cases. In other word, without using determinant bundle, we can not find out the effect of zeta-regularization.

6. Geometric meanings of zeta-regularization

By the results in Section 5, we can give the following geometric meanings of zeta-regularization.

1. *Zeta-regularization and the lattice generated by the periodic boundary condition.*

The lattice in $H^-(0)$ generated by the periodic boundary condition is $\mathbf{Z}^\infty(0)$. As showed in Section 5, zeta-regularization provides no new proper values and functions other than come from finite dimensional case on this space. To get the effects of zeta-regularization, we need to use $H^(finite)$, that is the lattice $\mathbf{Z}^\infty$. $\mathbf{Z}^\infty$ is the free abelian group generated by $\mathbf{e}_n, n \in \mathbf{N}$, $\mathbf{e}_n = (m_1, m_2, \ldots), m_n = 1, m_k = 0, k \neq n$, where $\mathbf{N}$ is the set of natural numbers. While $\mathbf{Z}^\infty$ is generated by $(\mathbf{e}_n, 0), n \in \mathbf{N}$ and $(0, 0, \ldots, 1)$. We may regard the set $\{(\mathbf{e}_n, 0) | n \in \mathbf{N}\} \cup \{(0, 0, \ldots, 1)\}$ to be the one point compactification of the set $\{(\mathbf{e}_n, 0) | n \in \mathbf{N}\}$. Therefore we may say

Zeta-regularization can be interpreted as the one point compactification of the set of generators of the free abelian group $\mathbf{Z}^\infty(0)$.

2. *Zeta-regularization and the infinite spinor.*

Proper functions of : D : comes from the finite dimensional case take the values in $Cl(H^-(finite))$. In fact, they take the values in $Cl(H)$. While proper functions of : D : appeared after zeta-regularization take the values in $Cl(H^-(finite))e_\infty$ at some points of Q^∞, where e_∞ is the infinite spinor (γ_5) adjoined to $Cl(H^-(finite))$. Without e_∞, zeta-regularization has no effects. Moreover, we need only one infinite spinor e_∞ thought to be $\prod e_n = e_1 e_2 \cdots$. That is we need not to adjoin half infinite spinor and so on to $Cl(H^-(finite))$.Therefore we may say

Zeta-regularization can be interpreted as the infinite spinor (γ_5), adjoined to $Cl(H^-(finite))$.

3. *Zeta-regularization and the determinant bundle.*

It is known the infinite spinor is a non-trivial section of the determinant bundle and generate (the fibre of) the determinant bundle ([12]). So we can interpret zeta-regularization as the determinant bundle. This interpretation is seen more directly by the calculation of the norm of proper functions. As we have shown in Section 5, $||f(\mathbf{m};x)||$ is regularized to be $2^{\nu/2}(detG)^{-d/4}$. Since $(detG)^{d/2}$ is a section of the determinant bundle of H^- and $H^-(0)$ is a subspace of H^-, it is better to regard $f(\mathbf{m};x)$ to be a section of the root of the dual bundle of the determinant bundle of $H^-(0)$. In this interpretation, we need to regard $H^-(finite)$ to be the total space of the (root of the dual bundle of the) determinant bundle of $H^-(0)$. Therefore we may say

Zeta-regularization can be interpreted as the (root of the dual bundle of the) determinant bundle of $H^-(0)$.

This third interpretation is the most geometric in the above three interpretations. Snce $H^-(0)$ is a flat space, so its determinant bundle is trivial and determinant bundle interpretation seems to have no meaning. But we can define zeta-regularization of differential operators on a mapping space $Map(X,M)$, X is a compact Riemannian manifold, in the following way: We fix a (positive) non-degenerate self-adjoint elliptic operator D acting on the $\mathbf{R}^d$-valued functions on X, where n is the dimension of X. Then we can regard $Map(X,M)$ to be a Sobolev manifold modelled by $\{W^k \otimes \mathbf{R}^d, G\}$, G is the Green operator of D.Here W^k means the k-th Sobolev space on X.

Note. Our calculation in this paper shows W^k is not appropriate for the study of regularization of differential operators on $Map(X,M)$. We need to take $W^{k-0}(0)$ and $W^{k-0}(finite)$ as the models. Precisely saying, original $Map(X,M)$ should be the manifold modelled by $W^{k-0}(0)$ (cf. [13]), while the manifold modelled by $W^{k-0}(finite)$ is the total space of the determinant bundle of $Map(X,M)$ modelled by $W^{k-0}(0)$.

But to select $W^{k-0}(0)$ and $W^{k-0}(finite)$ as models of $Map(X,M)$, there might exist obstructions related to the topology of M. Study on the symmetry of $\{H,G\}$ is also required. These must be the next problem.

In general, D is not defined on the (co)tangent space of $Map(X,M)$. So we add the connection $\{A_U\}$ to D and denote the Green operator of $\mathrm{D}+A_U$ by G_U (Note that if D is positive, there exists connection $\{A_U\}$ such that $\mathrm{D}+A_U$ is non-degenerate). Then for a differential operator D on $Map(X,M)$, we can define the operator $D(s)$ by $G_U^{-s}DG_U^s$, and the regularization $:D:$ of D by

$$:D:f = D(s)f|_{s=0}.$$

In this case, the determinant bundle of $Map(X, M)$ is trivial, and remained problem is the possibility of global definition of root of the determinant bundle (cf.[2]). On the other hand, if X is a spin manifold and take the Dirac operator to be D, $\mathrm{D} + A_U$ must degenerate at some point of $Map(X, M)$ and the determinant bundle of $Map(X, M)$ should be non-trivial in general ([2]). So we need further geometric studies when D is the Dirac operator. For example, taking $X = S^1$, that is $Map(X, M) = \Omega M$, $\mathrm{D} = -d^2/dt^2 + m$, where t is the circle variable and $m > 0$ *and* $\notin \{2n\pi | n \in \mathbf{N}\}$, the determinant bundle of ωM is trivial if M is a spin manifold. While considering complexification of ΩM and take $\mathrm{D} = -id/dt + m$, the determinant bundle is non-trivial unless the 1-st Pontrjagin class of M vanishes([14], cf [2]).

We conclude this paper by the following note.

Note. The polar coordinate of H has only latitude and lacks longitude. In the study of regularized spherical Laplacian of H, we need to add longitude to H to get the effect of zeta-regularization. This suggests the possibility to interpret the longitude of H to be the (fibre of the) determinant bundle of H.

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