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ON HOLLAND'S FRAME FOR RANDERS SPACE AND ITS APPLICATIONS IN PHYSICS

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ABSTRACT. From a rigorous Finsler geometric perspective, we re-examine Holland's Randers space theory of motion of an electron in flat Minkowski space-time permeated with an electromagnetic field. Holland's theory was motivated by analogy with plastic deformation and dislocation in Bravais crystals, through work of D. Bohm on Quantum Mechanics.

We develop the anholonomic geometry of a Randers space using Holland's frame and determine two Finsler connections, one of them the crystallographic connection, the other just Cartan's connection expressed in terms of anholonomy. Corresponding to these, we give two fully covariant versions of Holland's theory each of which covers the case of a curved Minkowski space-time. The crystallographic theory with extra matter is most promising.

0. INTRODUCTION

The well-known approach of D. BOHM (see [BH]) to quantum mechanics casts quantum theory into a classical form so that classical and quantum physics can be more easily compared. As a result, the so-called "quantum potential" arises as a "guide" to motion of particles in space-time as in de Broglie's pilot-wave theory [C]. P. HOLLAND [H_1], [H_2] noted that a draw-back of this quantum potential method is that it must derive from ad hoc use of Schrödinger's equation and further asked whether the converse procedure, "that of reformulating classical theory in a manner which is more in accord with the spirit of quantum theory (particularly in a way which emphasizes the unity of field and particle), might not suggest an alternative starting point for the theoretical treatment of quantum effects", [H_1]. His differential geometric method is based on fundamental work of S. AMARI on a Finsler approach to crystal dislocation theory, [A].

Holland studies a unified formalism which uses a *anholonomic frame* (non-integrable 1-form) on *space-time*, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space-time viewed as a strained medium. In fact, Ingarden [I] was first

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to point out that the Lorentz force law, in this case, can be written as a geodesic equation on a Finsler space called Randers space [R]. This results in geometrical entities which depend on the electromagnetic field (vector potential), particle (velocity) and background space-time parameters, [H_1]. The Finsler structure implies the existence of a global anholonomic (Holland) frame which in turn yields a connection with torsion and vanishing Finsler curvatures. Holland shows that the usual electromagnetic theory can be recovered from an averaging process applied to the Holland frame and corresponding flat connection, [H_1].

Unfortunately, Holland's theory is not truly Finslerian as he presented it. Technically, his notion of covariant differential in Randers space (equation (19) in [H_1] and his local expression for the flat connection (equation (21) in [H_1] appear to be incorrect. In the present paper we use Finsler geometry to derive Holland's frame for an arbitrary Randers space; see [AIM], [M], [MR] and [R] for further references on these Finsler spaces.

Holland's idea has led us to find Holland type frames for Kropina spaces, as well, and these results will be reported on elsewhere. The upshot of our work so far is that it allows us to define intrinsic frames for C-reducible Finsler spaces of dimension exceeding three. Previous workers have not succeeded in finding parallel translation invariant frames because of the failure of the so-called *strongly non-Riemannian* condition, [M], [MI]. We have found these for the crystallographic connection, defined herein. They are *conformally invariant*.

The final remarks in this paper show how to obtain two "fully covariant" curved space-time versions of Holland's Theory via Theorem 3.3 or Theorem 3.4. Especially significant for the latter is the analogy with extra matter defects in crystal lattices [GZ], [K_1], [K_2]. These may be considered to be solutions to problems posed in [H_1], [H_2]. After preliminary material on Finsler geometry in Section 2 we introduce anholonomic frames, and the associated crystallographic connection. In Section 3 we introduce Holland's frame for Randers spaces and prove the main Theorems 3.3 and 3.4 of our paper. In Section 4 we consider the original set-up of Holland and rederive his results taking care to explain them in our notation.

1. Finsler Spaces and Finsler Connections

Let M be a real n-dimensional connected manifold of C^∞-class and (TM, π, M) its tangent bundle with zero section removed. Every local chart $\big(U, \varphi = (x^i)\big)$ on M induces a local chart $\big(\pi^{-1}(U), \phi = (x^i, y^i)\big)$ on TM. The kernel of the linear mapping $\pi_* = TTM \to TM$ is called the *vertical distribution* and is denoted by VTM. For every $u \in TM$, $\operatorname{Ker} \pi_{*,u} = V_uTM$ is spanned by $\big\{\frac{\partial}{\partial y^i}|_u\big\}$. By a *nonlinear connection* on TM we mean a regular n-dimensional distribution $H : u \in TM \mapsto H_uTM$ which is supplementary to the vertical distribution i.e.

$$T_uTM = H_uTM \oplus V_uTM, \quad \forall\, u \in TM. \tag{1.1}$$

A basis for T_uTM adapted to the direct sum (1.1) has the form $\big(\frac{\delta}{\delta x^i}|_u = \frac{\partial}{\partial x^i}|_u - N_i^j(u)\frac{\partial}{\partial y^j}|_u, \frac{\partial}{\partial y^i}|_u\big)$. The dual basis of this is $(dx^i, \delta y^i = dy^i + N^i_j dx^j)$. These are the *Berwald bases.* The vector field mapping $J : \mathfrak{X}(TM) \to \mathfrak{X}(TM)$, defined by:

$$J = \frac{\partial}{\partial y^i} \otimes dx^i, \tag{1.2}$$

is globally defined on TM and is called the *almost tangent structure.* It has the properties:

1°. $J^2 = 0$;

2°. $\operatorname{Im} J = \operatorname{Ker} J = VTM$. See [MA] or [AIM] for more discussion.

A *linear connection* (Koszul connection) on TM is a map $\Delta : (X, Y) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto \Delta_X Y \in \mathfrak{X}(TM)$ for which we have:

1°. $\Delta_{fX+gY} Z = f\Delta_X Z + g\Delta_Y Z$;

2°. $\Delta_X fY = X(f) + f\Delta_X Y$; $\quad \Delta_X(Y + Z) = \Delta_X Y + \Delta_X Z$.

Definition 1.1. A linear connection Δ on TM is called a *Finsler connection* (or an *N-linear connection*) if:

1° Δ preserve by parallelism the horizontal distribution HTM;

2° The almost tangent structure J is absolute parallel with respect to Δ.

For a Finsler connection Δ it is immediate that Δ preserves also the vertical distribution. In the Berwald basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ adapted to the decomposition (1.1), a Finsler connection can be expressed as:

$$\begin{cases} \Delta_{\frac{\delta}{\delta x^i}} \dfrac{\delta}{\delta x^j} = F^k_{ji} \dfrac{\delta}{\delta x^k}; & \Delta_{\frac{\delta}{\delta x^i}} \dfrac{\partial}{\partial y^j} = F^k_{ji} \dfrac{\partial}{\partial y^k} \\ \Delta_{\frac{\partial}{\partial y^i}} \dfrac{\delta}{\delta x^j} = C^k_{ji} \dfrac{\delta}{\delta x^k}; & \Delta_{\frac{\partial}{\partial y^i}} \dfrac{\partial}{\partial y^j} = C^k_{ji} \dfrac{\partial}{\partial y^k}. \end{cases} \tag{1.3}$$

Observe that under a change of induced coordinates on TM the functions F^k_{ji} transform like the coefficients of a linear connection on M and C^k_{ji} as the components of a $(1, 2)$ tensor field on M. We will say that C^k_{ji} is a $(1, 2)$-type Finsler tensor field. In general, a tensor field of (r, s)-type on TM is called *Finsler tensor field* (or *d-tensor field*) if under a change of induced coordinates on TM its components transform like the components of a (r, s) type tensor on the base manifold M.

For $X = X^i \frac{\delta}{\delta x^i}$, a *horizontal vector field*, the absolute differential with respect to the Finsler connection Δ is given by:

$$\Delta X^i = dX^i + F^i_{jk} X^j dx^k + C^i_{jk} X^j \delta y^k = dX^i + \omega^i_j X^j, \tag{1.4}$$

where $\omega^i_j = F^i_{jk} dx^k + C^i_{jk} \delta y^k$ is the connection 1-form of Δ. The formula (1.4) can be written in an equivalent form:

$$\Delta X^i = X^i_{\ |k} dx^k + X^i|_k \delta y^k. \tag{1.4$'$}$$

Here, $X^i_{|k}$ and $X^i|_k$ are the *horizontal* and the *vertical covariant derivatives* of X^i, respectively. These must satisfy

$$\Delta_{\frac{\delta}{\delta x^k}} X^i \frac{\delta}{\delta x^i} = X^i_{|k} \frac{\delta}{\delta x^i}; \qquad \Delta_{\frac{\partial}{\partial y^k}} X^i \frac{\delta}{\delta x^i} = X^i|_k \frac{\delta}{\delta x^i}, \qquad \text{i.e.} \tag{1.5}$$

$$X^i_{|k} = \frac{\delta X^i}{\delta x^k} + F^i_{rk} X^r; \qquad X^i|_k = \frac{\partial X^i}{\partial y^k} + C^i_{rk} X^r. \tag{1.5$'$}$$

Of course, the horizontal and vertical covariant derivatives with respect to a Finsler connection Δ, can be defined in general for a Finsler tensor field. For example, if T^i_j are the components of an $(1,1)$ Finsler tensor field then h and v-covariant derivatives are given by:

$$\begin{cases} T^i_{j|k} = \dfrac{\delta T^i_j}{\delta x^k} + F^i_{rk} T^r_j - F^r_{jk} T^i_r \quad \text{and} \\ T^i_j|_k = \dfrac{\partial T^i_j}{\partial y^k} + C^i_{rk} T^r_j - C^r_{jk} T^i_r. \end{cases} \tag{1.6}$$

For a Finsler connection Δ one considers typically

$$T(X,Y) = \Delta_X Y - \Delta_Y X - [X,Y] \qquad \text{and}$$

$$R(X,Y)Z = \Delta_X \Delta_Y Z - \Delta_Y \Delta_X Z - \Delta_{[X,Y]} Z$$

the *torsion* and the *curvature*. It is well known [AIM], [MA] that in the basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ there are only five components of torsion and three components of curvature. The five components of torsion are:

$$\begin{cases} hT\left(\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta x^j}\right) =: T^k_{ij} \dfrac{\delta}{\delta x^k} = (F^k_{ji} - F^k_{ij}) \dfrac{\delta}{\delta x^k}; & (h)\ h\text{-torsion} \\ vT\left(\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta x^j}\right) =: R^k_{ij} \dfrac{\partial}{\partial y^k} = \left(\dfrac{\delta N^k_i}{\delta x^j} - \dfrac{\delta N^k_j}{\delta x^i}\right) \dfrac{\partial}{\partial y^k}; & (v)\ h\text{-torsion} \\ hT\left(\dfrac{\partial}{\partial y^i}, \dfrac{\delta}{\delta x^j}\right) = C^k_{ji} \dfrac{\delta}{\delta x^k}; & (h)\ hv\text{-torsion} \\ vT\left(\dfrac{\partial}{\partial y^i}, \dfrac{\delta}{\delta x^j}\right) =: P^k_{ij} \dfrac{\partial}{\partial y^k} = \left(\dfrac{\partial N^k_j}{\partial y^i} - F^k_{ij}\right) \dfrac{\partial}{\partial y^k}; & (v)\ hv\text{-torsion} \\ vT\left(\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j}\right) =: S^k_{ij} \dfrac{\partial}{\partial y^k} = \left(C^k_{ji} - C^k_{ij}\right) \dfrac{\partial}{\partial y^k} & (v)\ v\text{-torsion.} \end{cases} \tag{1.7}$$

The three components of curvature are given by:

$$
\begin{cases}
R^i_{jkh} = \dfrac{\delta F^i_{jk}}{\delta x^h} - \dfrac{\delta F^i_{jh}}{\delta x^k} + F^m_{jk}F^i_{mh} - F^m_{jh}F^i_{mk} + C^i_{jm}R^m_{kh} \\
P^i_{jkh} = \dfrac{\partial F^i_{jk}}{\partial y^h} - C^i_{{}_{jk|h}} + C^i_{jm}P^m_{kh} \\
S^i_{jkr} = \dfrac{\partial C^i_{jk}}{\partial y^r} - \dfrac{\partial C^i_{jr}}{\partial y^k} + C^m_{jk}C^i_{mr} - C^m_{jr}C^i_{mk}
\end{cases}
\tag{1.8}
$$

For a Finsler connection Δ we have the following Ricci identity

$$
\begin{cases}
X^i_{{}_{|k|r}} - X^i_{{}_{|r|k}} = X^m R^i_{mkr} - X^i_{{}_{|m}} T^m_{kr} - X^i|_m R^m_{kr} \\
X^i_{{}_{|k}}|_r - X^i|_{{}_{r|k}} = X^m P^i_{mkr} - X^i_{{}_{|m}} C^m_{kr} - X^i|_m P^m_{kr} \\
X^i|_k|_r - X^i|_r|_k = X^m S^i_{mkr} - X^i|_m S^m_{kr}.
\end{cases}
\tag{1.9}
$$

2. The Parallel Displacement Determined by an Anholonomic Finsler Frame

Let V be an open subset on TM and

$$Y_\alpha : u \in V \to Y_\alpha(u) \subset V_u TM, \quad \alpha = \overline{1,n} \tag{2.1}$$

be a vertical frame over V. Assume that $V = TM$. If we consider $Y_\alpha(u) = Y^i_\alpha(u)\frac{\partial}{\partial y^i}|_u$, then $\left(Y^i_\alpha(u)\right)$ are the entries of a nonsingular matrix. Denote by (Y^α_j) the components of the inverse of these matrix. This means that:

$$Y^i_\alpha Y^\alpha_j = \delta^i_j, \qquad Y^i_\alpha Y^\beta_i = \delta^\beta_\alpha. \tag{2.2}$$

We call (Y^i_α) an *anholonomic Finsler frame*. Naturally, every geometric object field on TM can be expressed in anholonomic Finsler frames. For example, if T^i_j are the components of a $(1,1)$ Finsler field, then the anholonomic components are given by:

$$T^\alpha_\beta = Y^\alpha_i T^i_j Y^j_\beta. \tag{2.3}$$

Consider the anholonomic Berwald basis:

$$\frac{\delta}{\delta x^\alpha} = Y^i_\alpha \frac{\delta}{\delta x^i} \quad \text{and} \quad \frac{\partial}{\partial y^\alpha} = Y^i_\alpha \frac{\partial}{\partial y^i}. \tag{2.4}$$

Let Δ be a Finsler connection with the holonomic coefficients $(F^k_{ij},\, C^k_{ij})$ given by (1.3). As we have seen Δ preserves, by parallelism, the horizontal and vertical distribution, so in the anholonomic Berwald basis $(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^\alpha})$ the Finsler connection

Δ can be expressed as:

$$\left\{\begin{array}{l}\Delta_{\frac{\delta}{\delta x^\alpha}}\dfrac{\delta}{\delta x^\beta}=F^\gamma_{\beta\alpha}\dfrac{\delta}{\delta x^\gamma},\Delta_{\frac{\delta}{\delta x^\alpha}}\dfrac{\partial}{\partial y^\beta}=F^\gamma_{\beta\alpha}\dfrac{\partial}{\partial y^\gamma}\\ \Delta_{\frac{\partial}{\partial y^\alpha}}\dfrac{\delta}{\delta x^\beta}=C^\gamma_{\beta\alpha}\dfrac{\delta}{\delta x^\gamma},\Delta_{\frac{\partial}{\partial y^\alpha}}\dfrac{\partial}{\partial y^\beta}=C^\gamma_{\beta\alpha}\dfrac{\partial}{\partial y^\gamma}.\end{array}\right. \tag{2.5}$$

The functions $(F^\gamma_{\beta\alpha}, C^\gamma_{\beta\alpha})$ are called the *anholonomic coefficients of the Finsler connections*, Δ.

Proposition 2.1. *Let Δ be a Finsler connection with the holonomic coefficients $(F^k_{ij},\ C^k_{ij})$ and (Y^i_α) be an anholonomic Finsler frame. Then, the anholonomic coefficients of the Finsler connection Δ are given by:*

$$\left\{\begin{array}{l}F^\gamma_{\alpha\beta}=\left(\dfrac{\delta Y^k_\alpha}{\delta x^i}+Y^j_\alpha F^k_{ji}\right)Y^i_\beta Y^\gamma_k=Y^j_\alpha\left(-\dfrac{\delta Y^\gamma_j}{\delta x^i}+F^k_{ji}Y^\gamma_k\right)Y^i_\beta\\ \qquad =Y^k_{\alpha|i}Y^i_\beta Y^\gamma_k=-Y^\gamma_{j|i}Y^j_\alpha Y^i_\beta\\ C^\gamma_{\alpha\beta}=\left(\dfrac{\partial Y^k_\alpha}{\partial y^i}+Y^j_\alpha C^k_{ji}\right)Y^i_\beta Y^\gamma_k=Y^j_\alpha\left(-\dfrac{\partial Y^\gamma_j}{\partial y^i}+C^k_{ji}Y^\gamma_k\right)Y^i_\beta\\ \qquad =Y^k_\alpha|_i Y^i_\beta Y^\gamma_k=-Y^\gamma_j|_i Y^j_\alpha Y^i_\beta.\end{array}\right. \tag{2.6}$$

Conversely, if for a Finsler connection Δ we know the anholonomic coefficients $(F^\alpha_{\beta\gamma},\ C^\alpha_{\beta\gamma})$ then the holonomic coefficients are given by:

$$\left\{\begin{array}{l}F^k_{ji}=Y^k_\alpha Y^\alpha_{j|\beta}Y^\beta_i=Y^k_\alpha\left(\dfrac{\delta Y^\alpha_j}{\delta x^\beta}+Y^\delta_j F^\alpha_{\delta\beta}\right)Y^\beta_i\\ \qquad =-Y^k_{\alpha|\beta}Y^\alpha_j Y^\beta_i=\left(-\dfrac{\delta Y^k_\alpha}{\delta x^\beta}+Y^k_\delta F^\delta_{\alpha\beta}\right)Y^\alpha_j Y^\beta_i\\ C^k_{ji}=Y^k_\alpha Y^\alpha_j|_\beta Y^\beta_i=Y^k_\alpha\left(\dfrac{\partial Y^\alpha_j}{\partial Y^\beta}+Y^\delta_j C^\alpha_{\delta\beta}\right)Y^\beta_i\\ \qquad =-Y^k_\alpha|_\beta Y^\alpha_j Y^\beta_i=\left(-\dfrac{\partial Y^k_\alpha}{\partial Y^\beta}+Y^k_\delta C^\delta_{\alpha\beta}\right)Y^\alpha_j Y^\beta_i.\end{array}\right. \tag{2.6$'$}$$

Proof. The first formulae for $F^\gamma_{\alpha\beta}$ and $C^\gamma_{\alpha\beta}$ can be obtained if we compare (1.3) and (2.5). For the second we have to take into account

$$Y^i_\alpha\frac{\delta Y^\alpha_j}{\delta x^\beta}=-Y^\alpha_j\frac{\delta Y^i_\alpha}{\delta x^\beta};\qquad Y^i_\alpha\frac{\delta Y^\beta_i}{\delta x^j}=-Y^\beta_i\frac{\delta Y^i_\alpha}{\delta x^j}$$

and

$$Y^i_\alpha\frac{\partial Y^\alpha_j}{\partial y^\beta}=-Y^\alpha_j\frac{\partial Y^i_\alpha}{\partial y^\beta};\qquad Y^i_\alpha\frac{\partial Y^\beta_i}{\partial y^j}=-Y^\beta_i\frac{\partial Y^i_\alpha}{\partial y^j}$$

which follows from (2.2) by differentiation. $\square$

We then find that for any Finsler connection (1.3)

$$\text{(2.7)}\quad \begin{aligned} &\text{(a)} && \left[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right] = \Omega^\gamma_{\alpha\beta}\frac{\delta}{\delta x^\gamma} + \mathbb{R}^\gamma_{\alpha\beta}\frac{\partial}{\partial y^\gamma} \\ &\text{(b)} && \left[\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^\beta}\right] = (\mathbb{P}^\gamma_{\alpha\beta} + F^\gamma_{\beta\alpha})\frac{\partial}{\partial y^\gamma} + \Omega^\gamma_{\alpha(\beta)}\frac{\delta}{\delta x^\gamma} \\ &\text{(c)} && \left[\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right] = \eta^\gamma_{\alpha\beta}\frac{\partial}{\partial y^\gamma}. \end{aligned}$$

See [MI] for more discussion. Here, $\mathbb{R}^\gamma_{\alpha\beta}$ are the anholonomic components of the $(1,2)$ type tensor field $\mathbb{R}^i_{jk}$ given by (1.7), $\Omega^\gamma_{\alpha\beta}$ and $\Omega^\gamma_{\alpha(\beta)}$ are the *anholonomic objects* defined by

$$\text{(2.8)}\quad \begin{aligned} &\text{(a)} && \Omega^\gamma_{\alpha\beta} = Y^\gamma_i\left(\frac{\delta Y^i_\beta}{\delta x^\alpha} - \frac{\delta Y^i_\alpha}{\delta x^\beta}\right) \qquad \text{and} \\ &\text{(b)} && \Omega^\gamma_{\alpha(\beta)} = Y^i_\alpha\left(\frac{\partial}{\partial y^j}Y^\gamma_i\right)Y^j_\beta. \end{aligned}$$

Also, the (v)-hv-torsion has components

$$\text{(2.8)}\qquad \text{(c)}\qquad \mathbb{P}^\gamma_{\alpha\beta} = \left(\frac{\partial N^i_k}{\partial y^j} - F^i_{jk}\right)Y^\gamma_i Y^j_\alpha Y^k_\beta$$

while the (v)-v-*torsion* is

$$\text{(2.8)}\qquad \text{(d)}\qquad \eta^\gamma_{\alpha\beta} = Y^\gamma_i\left(\frac{\partial Y^i_\beta}{\partial y^\alpha} - \frac{\partial Y^i_\alpha}{\partial y^\beta}\right).$$

The frame Y^i_α is *holonomic* if and only if there exists n functions φ^α on M for which $Y^\alpha_i = \frac{\partial\varphi^\alpha}{\partial x^i}$, i.e. the 1-form $Y^\alpha_i v dx^i$ is *exact.*

Proposition 2.2. *The frame Y^i_α is holonomic if and only if $\Omega^\gamma_{\alpha\beta} = 0 = \Omega^\gamma_{\alpha(\beta)}$.*

Proof. $\Omega^\gamma_{\alpha(\beta)} = 0$ implies Y^i_α is independent of y, then $\Omega^\gamma_{\alpha\beta} = 0$ implies Y^i_α are given by gradients of n functions on M. □

Remark. Y^i_α is holonomic if and only if $\left[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right]$ and $\left[\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^\beta}\right]$ are in VTM, i.e. iff these Lie brackets are vertical. Of course, 2.7(c) expresses the fact that the fibre of TM is integrable.

Let X^α be the anholonomic components of a Finsler vector field X. Then the absolute differential of X with respect to the Finsler connection Δ can be expressed as:

$$\text{(2.9)}\qquad \Delta X^\alpha = dX^\alpha + F^\alpha_{\beta\gamma}X^\beta dx^\gamma + C^\alpha_{\beta\gamma}X^\beta\delta y^\gamma.$$

(Here, the reader may note the difference with (19) in [H$_1$].

Proposition 2.3. *Given an anholonomic Finsler frame* (Y^i_α), *there exists a unique Finsler connection* Δ *for which the given frame is* h*- and* v*-covariant constant. We call this the Crystallographic connection.*

Proof. From (2.6) we have that $Y^i_{\alpha|j} = 0$ is equivalent with $F^\gamma_{\alpha\beta} = 0 \Longleftrightarrow$

$$F^k_{ji} = -Y^\alpha_j \frac{\delta Y^k_\alpha}{\delta x^i} = \frac{\delta Y^\gamma_j}{\delta x^i} Y^k_\gamma. \tag{2.10}$$

Similarly, $Y^i_\alpha|_j = 0 \Longleftrightarrow C^\gamma_{\alpha\beta} = 0$ which is also equivalent with:

$$C^k_{ji} = -Y^\alpha_j \frac{\partial Y^k_\alpha}{\partial y^i} = \frac{\partial Y^\gamma_j}{\partial y^i} Y^k_\gamma. \tag{2.10$'$}$$

(The reader may note the difference between (2.10) and (21) in [H$_1$]). □

Proposition 2.4. *For the Finsler connection given by Proposition* (2.3), *all three components of curvature vanish.*

Proof. According to $Y^i_{\alpha|k} = 0$, $Y^i_\alpha|_k = 0$ and the Ricci identities (1.9) we obtain $R^i_{jk\ell} = P^i_{jk\ell} = S^i_{jk\ell} = 0$. □

3. The Anholonomic Finsler Frame Determined by a Randers Metric (The Holland Frame)

Let us consider a Finsler space $F^n = (M, \alpha)$ on an n-dimensional manifold M. This means that:

1° $\alpha : TM \to \mathbb{R}$ is of C^∞-class and continuous on the zero section;

2° α is positively homogeneous with respect to y;

3° The matrix with the entries:

$$a_{ij} = \frac{1}{2}\frac{\partial^2 \alpha^2}{\partial y^i \partial y^j} \quad \text{has a constant rank } n \text{ on } TM. \tag{3.1}$$

It's known that a Finsler space has a canonical nonlinear connection HTM, with the local coefficients:

$$N^i_j = \frac{1}{2}\frac{\partial}{\partial y^j}(\gamma^i_{k\ell} y^k y^\ell), \tag{3.2}$$

where $\gamma^i_{k\ell}$ are the Christoffel symbols of second kind for the metric tensor a_{ij}. There is also a Finsler connection (*Cartan connection*) with the holonomic coefficients given by:

$$\begin{cases} F^k_{ij} = \dfrac{1}{2} a^{kr}\left(\dfrac{\delta a_{ri}}{\delta x^j} + \dfrac{\delta a_{rj}}{\delta x^i} - \dfrac{\delta a_{ij}}{\delta x^r}\right) \\[2ex] C^k_{ij} = \dfrac{1}{2} a^{kr}\left(\dfrac{\partial a_{ri}}{\partial y^j} + \dfrac{\partial a_{rj}}{\partial y^i} - \dfrac{\partial a_{ij}}{\partial y^r}\right). \end{cases} \tag{3.3}$$

As is well-known, this connection is metrical ($a_{ij|k} = 0$ and $a_{ij}|_k = 0$) and h and v symmetric ($T^k_{ij} = 0$ and $S^k_{ij} = 0$) [AIM], [MA].

Together with the Finsler space $F^n = (M, \alpha)$ we shall consider a covector field $b_i(x)dx^i$ on M (or an open set V of M). Then $\beta(x,y) = b_i(x)y^i$ is a scalar function on TM (or on $\pi^{-1}(V)$).

The function $L : TM \to \mathbb{R}$, defined by:

$$L(x,y) = \alpha(x,y) + \beta(x,y) \tag{3.4}$$

is also the fundamental function of a Finsler space [AIM], [MR]. The pair (M, L) is called a *Randers space*. Denote by:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j} \tag{3.5}$$

the fundamental tensor of the Randers space (M, L). Taking into account the homogeneity of α and L we have:

$$\begin{cases} p^i := \dfrac{1}{\alpha}y^i = a^{ij}\dfrac{\partial \alpha}{\partial y^j}; & p_i := a_{ij}p^j = \dfrac{\partial \alpha}{\partial y^i} \\ \ell^i := \dfrac{1}{L}y^i = g^{ij}\dfrac{\partial L}{\partial y^j}; & \ell_i := g_{ij}\ell^j = \dfrac{\partial L}{\partial y^i} = p_i + b_i \\ \ell^i = \dfrac{\alpha}{L}p^i; & \ell^i\ell_i = p^ip_i = 1; \quad \ell^ip_i = \frac{\alpha}{L}; \quad p^i\ell_i = \dfrac{L}{\alpha} \\ b_ip^i = \dfrac{\beta}{\alpha}, & b_i\ell^i = \dfrac{\beta}{L}. \end{cases} \tag{3.6}$$

The metric tensors (a_{ij}) and (g_{ij}) are related by:

$$\begin{aligned} g_{ij} &= \frac{L}{\alpha}a_{ij} + b_ip_j + p_ib_j + b_ib_j - \frac{\beta}{\alpha}p_ip_j \\ &= \frac{L}{\alpha}(a_{ij} - p_ip_j) + \ell_i\ell_j. \end{aligned} \tag{3.7}$$

Let us consider now $\left(X^i_\alpha(x,y)\right)$ an arbitrary, but fixed anholonomic frame (but it could be also holonomic). Denote by

$$a_{\alpha\beta}(x,y) = X^i_\alpha(x,y)X^j_\beta(x,y)a_{ij}(x,y),$$

the components of the metric (a_{ij}) with respect to (X^i_α). If (X^α_i) are the components of the inverse matrix of (X^i_α), denote:

$$\ell^\alpha = X^\alpha_i\ell^i, \quad p^\alpha = X^\alpha_ip^i, \quad \ell_\alpha = X^i_\alpha\ell_i \quad \text{and} \quad p_\alpha = X^i_\alpha p_i.$$

Consider (Y^i_j) the matrix with the entries:

$$Y^i_j(x,y) = \sqrt{\frac{\alpha}{L}}\left(\delta^i_j - \ell^i\ell_j + \sqrt{\frac{\alpha}{L}}p^ip_j\right).$$

Then this matrix is invertible and the components of its inverse are:

$$(Y^{-1})^i_j = \sqrt{\frac{L}{\alpha}}\left(\delta^i_j + \sqrt{\frac{L}{\alpha}}\ell^i\ell_j - p^ip_j\right).$$

Theorem 3.1. *For a Randers space (M, L) consider in the case $\frac{L}{\alpha} > 0$:*

$$Y^i_\gamma = \sqrt{\frac{\alpha}{L}}\left(\delta^i_\gamma - \ell^i \ell_\gamma + \sqrt{\frac{\alpha}{L}}\, p^i p_\gamma\right) \tag{3.8}$$

defined on open set V in TM where $\frac{L}{\alpha} > 0$. Then $(Y_\gamma = Y^i_\gamma \frac{\partial}{\partial y^i})_{\gamma=\overline{1,n}}$ is an anholonomic Finsler frame.

We call it, *Holland's frame of Randers space* [H$_1$], [H$_2$]. It is p-homogeneous of degree zero in y and a conformal invariant in the sense that $L \mapsto e^{\phi(x)} \cdot L$ leaves Y^i_γ fixed.

Proof. Consider also:

$$Y^\gamma_j = \sqrt{\frac{L}{\alpha}}\left(\delta^\gamma_j + \sqrt{\frac{L}{\alpha}}\ell^\gamma \ell_j - p^\gamma p_j\right). \tag{3.9}$$

We have to check that $Y^i_\gamma Y^\gamma_j = \delta^i_j$ and $Y^i_\gamma Y^\beta_i = \delta^\beta_\gamma$. Let us verify the former:

$$\begin{aligned} Y^i_\gamma Y^\gamma_j &= \left(\delta^i_\gamma + \sqrt{\frac{L}{\alpha}}\ell^i \ell_\gamma - p^i p_\gamma\right)\left(\delta^\gamma_j - \ell^\gamma \ell_j + \sqrt{\frac{\alpha}{L}} p^\gamma p_j\right) \\ &= \delta^i_j - \ell^i \ell_j + \sqrt{\frac{\alpha}{L}}\, p^i p_j + \sqrt{\frac{L}{\alpha}}\ell^i \ell_j - \sqrt{\frac{L}{\alpha}}\ell^i \ell_j + \ell^i \ell_\gamma p^\gamma p_j \\ &\quad - p^i p_j + p^i p_\gamma \ell^\gamma \ell_j - \sqrt{\frac{\alpha}{L}} p^i p_j = \delta^i_j. \end{aligned}$$

□

Theorem 3.2. *With respect to Holland's frame the holonomic components of the Finsler metric tensor $(a_{\alpha\beta})$ is the Randers metric (g_{ij}), that is:*

$$g_{ij} = Y^\alpha_i Y^\beta_j a_{\alpha\beta}. \tag{3.10}$$

Proof. We have

$$\begin{aligned} Y^\beta_j a_{\alpha\beta} &= \sqrt{\frac{L}{\alpha}}\left(\delta^\beta_j + \sqrt{\frac{L}{\alpha}}\ell^\beta \ell_j - p^\beta p_j\right) a_{\alpha\beta} \\ &= \sqrt{\frac{L}{\alpha}} a_{\alpha j} + p_\alpha \ell_j - \sqrt{\frac{L}{\alpha}} p_\alpha p_j \end{aligned}$$

$$\begin{aligned} Y^\alpha_i Y^\beta_j a_{\alpha\beta} &= \sqrt{\frac{L}{\alpha}}\left(\delta^\alpha_i + \sqrt{\frac{L}{\alpha}}\ell^\alpha \ell_i - p^\alpha p_i\right) \cdot \left(\sqrt{\frac{L}{\alpha}} a_{\alpha j} + p_\alpha \ell_j - \sqrt{\frac{L}{\alpha}} p_\alpha p_j\right) \\ &= \frac{L}{\alpha} a_{ij} + \sqrt{\frac{L}{\alpha}} p_i \ell_j - \frac{L}{\alpha} p_i p_j + \frac{L}{\alpha} \cdot \frac{\alpha}{L}\sqrt{\frac{L}{\alpha}} p_j \ell_i \\ &\quad + \frac{L}{\alpha} \cdot \frac{\alpha}{L}\ell_i \ell_j - \frac{L}{\alpha} \cdot \sqrt{\frac{L}{\alpha}}\frac{\alpha}{L}\ell_i p_j - \frac{L}{\alpha} p_j p_i - \sqrt{\frac{L}{\alpha}} p_i \ell_j + \frac{L}{\alpha} p_i p_j \end{aligned}$$

$$= \frac{L}{\alpha}(a_{ij} - p_i p_j) + \ell_i \ell_j = g_{ij}.$$

$\square$

Corollary. *We have*

$$Y_i^\alpha p_\alpha = \ell_i, \quad Y_\alpha^i p^\alpha = \ell^i.$$

Theorem 3.3. *Consider the Randers space (M, L) and Holland's Frame (Y_α^i). Then there exists a unique Finsler connection Δ (Cartan connection) with local coefficients $(F_{\beta\gamma}^\alpha, C_{\beta\gamma}^\alpha)$ for which*

(1) $a_{\alpha\beta|\gamma} = 0$ *and* $a_{\alpha\beta}|_\gamma = 0$

(2) *the anholonomic components of the (h) h-torsion (v) v-torsion are*

$$\tau_{\beta\gamma}^\alpha := F_{\gamma\beta}^\alpha - F_{\beta\gamma}^\alpha = -\Omega_{\beta\gamma}^\alpha \quad \textit{and}$$

$$\Sigma_{\beta\gamma}^\alpha := C_{\gamma\beta}^\alpha - C_{\beta\gamma}^\alpha = -\eta_{\beta\gamma}^\alpha.$$

Proof. Let us consider Δ the Cartan connection of the Randers space (M, L). Denote by (F_{jk}^i, C_{jk}^i) it's holonomic coefficients. This is the unique Finsler connection which satisfies Matsumoto's Axioms:

(a) $g_{ij|k} = 0$ and $g_{ij}|_k = 0$;

(b) $T_{jk}^i = 0$ and $S_{jk}^i = 0$.

Consider $(F_{\beta\gamma}^\alpha, C_{\beta\gamma}^\alpha)$ the anholonomic coefficients of this Cartan connection (2.6). All we have to prove is that (1) and (a) are equivalent and also (2) and (b). First we prove that

$$g_{ij|k} = Y_k^\alpha Y_j^\beta a_{\alpha\beta|\gamma} Y_k^\gamma \quad \text{and} \tag{3.12}$$

$$g_{ij}|_k = Y_i^\alpha Y_j^\beta a_{\alpha\beta}|_\gamma Y_k^\gamma.$$

Let us start with the (RHS) of $(3.12)_1$,

$$\begin{aligned}
a_{\alpha\beta|\gamma} Y_i^\alpha Y_j^\beta Y_k^\gamma &= \left(\frac{\delta a_{\alpha\beta}}{\delta x^\gamma} - a_{\delta\beta} F_{\alpha\gamma}^\delta - a_{\delta\alpha} F_{\beta\gamma}^\delta \right) Y_i^\alpha Y_j^\beta Y_k^\gamma \\
&= Y_k^\gamma \, \frac{\delta a_{\alpha\beta}}{\delta x^\gamma} Y_i^\alpha Y_j^\beta - a_{\delta\beta} Y_j^\beta F_{\alpha\gamma}^\delta Y_k^\gamma Y_i^\alpha - a_{\delta\alpha} Y_i^\alpha F_{\beta\gamma}^\delta Y_k^\gamma Y_j^\beta \\
&= \frac{\delta a_{\alpha\beta}}{\delta x^k} Y_i^\alpha Y_j^\beta - g_{j\ell} Y_\delta^\ell F_{\alpha\gamma}^\delta Y_k^\gamma Y_i^\alpha - g_{i\ell} Y_\delta^\ell F_{\beta\gamma}^\delta Y_k^\gamma Y_j^\beta \\
&= \frac{\delta}{\delta x^k} (g_{\ell m} Y_\alpha^\ell Y_\beta^m) Y_i^\alpha Y_j^\beta - g_{j\ell} \left(\frac{\delta Y_\alpha^\ell}{\delta x^k} + Y_\alpha^m F_{mk}^\ell \right) Y_i^\alpha \\
&\quad - g_{i\ell} \left(\frac{\delta Y_\beta^\ell}{\delta x^k} + Y_\beta^m F_{mk}^\ell \right) Y_j^\beta \\
&= \frac{\delta g_{ij}}{\delta x^k} - g_{j\ell} F_{ik}^\ell - g_{i\ell} F_{jk}^\ell + g_{\ell j} \, \frac{\delta Y_\alpha^\ell}{\delta x^k} Y_i^\alpha + g_{im} \, \frac{\delta Y_\beta^m}{\delta x^k} Y_j^\beta
\end{aligned}$$

$$- g_{j\ell} \frac{\delta Y^\ell_\alpha}{\delta x^k} Y^\alpha_i - g_{i\ell} \frac{\delta Y^\ell_\beta}{\delta x^k} Y^\beta_j = g_{ij|k}.$$

We have used: $a_{\delta\beta} Y^\beta_j = g_{j\ell} Y^\ell_\delta$ and $a_{\delta\alpha} Y^\alpha_i = g_{i\ell} Y^\ell_\delta$ according to (3.10) and

$$F^\delta_{\alpha\beta} Y^\ell_\delta Y^\beta_k = \frac{\delta Y^\ell_\alpha}{\delta x^k} + Y^m_\alpha F^\ell_{mk} \quad \text{according to (2.6).}$$

A similar formula works for v-covariant derivative. We also have, as in [MI],

$$\begin{cases} \tau^\alpha_{\beta\gamma} + \Omega^\alpha_{\beta\gamma} = T^i_{jk} Y^\alpha_i Y^j_\beta Y^k_\gamma \\ C^\alpha_{\beta\gamma} + \Omega^\alpha_{\beta(\gamma)} = C^i_{jk} Y^\alpha_i Y^j_\beta Y^k_\gamma \qquad \text{(equivalent to } (2.6)_2) \\ \sum^\alpha_{\beta\gamma} + \eta^\alpha_{\beta\gamma} = S^i_{jk} Y^\alpha_i Y^j_\beta Y^k_\gamma . \end{cases}$$

Using these formulae we have that (2) and (b) are equivalent. □

Remark. From the second formula of (3.13) we can determine the vertical anholonomic coefficients of the Cartan connection as:

$$C^\alpha_{\beta\gamma} = -\Omega^\alpha_{\beta(\gamma)} + C^i_{jk} Y^\alpha_i Y^j_\beta Y^k_\gamma, \quad \text{where} \quad C^i_{jk} = \frac{1}{4} g^{i\ell} \frac{\partial^3 L^2}{\partial y^\ell \partial y^j \partial y^k}.$$

Theorem 3.4. *Consider the Randers space (M, L) with Holland's Frame. The crystallographic connection satisfies:*

1) $g_{ij|k} = Y^\alpha_i Y^\beta_j Y^\gamma_k \dfrac{\delta a_{\alpha\beta}}{\delta x^\gamma}$
   $g_{ij}|_k = Y^\alpha_i Y^\beta_j Y^\gamma_k \dfrac{\partial a_{\alpha\beta}}{\partial y^\gamma}.$
2) *The holonomic components of the* (h) *h-torsion,* (v) *v-torsion and* (h) *hv-torsion are:*

$$T^i_{jk} = \Omega^\alpha_{\beta\gamma} Y^i_\alpha Y^\beta_j Y^\gamma_k; \quad S^i_{jk} = \eta^\alpha_{\beta\gamma} Y^i_\alpha Y^\beta_j Y^\gamma_k \quad \textit{and}$$

$$C^i_{jk} = \Omega^\alpha_{\beta(\gamma)} Y^i_\alpha Y^\beta_j Y^\gamma_k, \quad \textit{respectively.}$$

3) *This connection is flat and $Y^i_{\alpha|j} = 0$, $Y^i_\alpha\big|_j = 0$.*

Proof. According to Propositions 2.3 and 2.4, the anholonomic components for the crystallographic connection are $F^\alpha_{\beta\gamma} = C^\alpha_{\beta\gamma} = 0$. Then $a_{\alpha\beta|\gamma} = \frac{\delta a_{\alpha\beta}}{\delta x^\gamma}$ and $a_{\alpha\beta}|_\gamma = \frac{\partial a_{\alpha\beta}}{\partial y^\gamma}$. As (3.12) holds also for this connection we have that (1) is verified.

Since $F^\alpha_{\beta\gamma} = C^\alpha_{\beta\gamma} = 0$ then, $\sum^\alpha_{\beta\gamma} = \tau^\alpha_{\beta\gamma} = 0$ and (3.13) gives (2). Proposition 2.4 gives the condition (3). □

Remark. Since the Holland frame is a conformal invariant it follows from $(2.8)_b$ and $(2.8)_d$ that the anholonomic objects $\Omega^\alpha_{\beta(\gamma)}$ and $\eta^\alpha_{\beta\gamma}$ are conformal invariants.

Also from Theorem 3.4, 2) we have that the (v) v-torsion S^i_{jk} and (h) hv-torsion C^i_{jk} of the crystallographic connection are conformal invariant.

4. Anholonomic Geometry of Flat Riemannian Space Versus Randers Space

Let (a_{ij}) be a regular Riemannian metric possibly with non-positive-definite signature. Suppose (M, a_{ij}) is a flat space. Denote by (γ^i_{jk}) the Christoffel symbols of second kind of the Riemannian metric.

Since the Riemannian space (M, a_{ij}) is flat there exists a frame $X^i_\alpha(x)$ (holonomic or not) on the base manifold such that $a_{\alpha\beta} = X^i_\alpha(x)X^j_\beta(x)a_{ij}(x)$ are constants. With respect to this frame the coefficients of Levi-Civita connection are $\gamma^\alpha_{\beta\gamma} = 0$.

Then $\alpha(x,y) = \sqrt{a_{ij}(x)y^iy^j} = \sqrt{a_{\alpha\beta}y^\alpha y^\beta}$ is the fundamental function of a Finsler metric.

If we assume that the manifold M is endowed with a covector field $b = b_i(x)dx^i$, then $L : TM \to \mathbb{R}$, given by $L(x,y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i$ is the fundamental function of a Randers space.

Theorem 4.1. *For the Randers space (M, L) above with Holland's Frame, the Cartan connection is the unique connection which in anholonomic coordinates satisfies:*

1) $a_{\alpha\beta|\gamma} = 0$ *and* $a_{\alpha\beta}|\gamma = 0$*;*
2) $\tau^\alpha_{\beta\gamma} = -\Omega^\alpha_{\beta\gamma}$ *and* $\sum^\alpha_{\beta\gamma} = -\eta^\alpha_{\beta\gamma}$.

The anholonomic coefficients of the Cartan connection are given by

$$F^\alpha_{\beta\gamma} = \frac{1}{2}a^{\alpha\delta}(a_{\beta\varepsilon}\Omega^\varepsilon_{\delta\gamma} + a_{\gamma\varepsilon}\Omega^\varepsilon_{\delta\beta} - a_{\delta\varepsilon}\Omega^\varepsilon_{\beta\gamma}) \tag{4.1}$$

$$C^\alpha_{\beta\gamma} = \frac{1}{2}a^{\alpha\delta}(a_{\beta\varepsilon}\eta^\varepsilon_{\delta\gamma} + a_{\gamma\varepsilon}\eta^\varepsilon_{\delta\beta} - a_{\delta\varepsilon}\eta^\varepsilon_{\beta\gamma}).$$

Proof. According to Theorem 3.3 the Cartan connection is the unique Finsler connection of the Randers space (M, L) for which (1) and (2) hold.

¿From (1) and (2) by some standard computation we can get:

$$F^\alpha_{\beta\gamma} = \frac{1}{2}a^{\alpha\delta}\left(\frac{\delta a_{\delta\beta}}{\delta x^\gamma} + \frac{\delta \alpha_{\delta\gamma}}{\delta x^\beta} - \frac{\delta a_{\beta\gamma}}{\delta x^\delta}\right) + \frac{1}{2}a^{\alpha\delta}(a_{\beta\varepsilon}\Omega^\varepsilon_{\delta\gamma} + \alpha_{\gamma\varepsilon}\Omega^\varepsilon_{\delta\beta} - a_{\delta\varepsilon}\Omega^\varepsilon_{\beta\gamma})$$

and

$$C^\alpha_{\beta\gamma} = \frac{1}{2}a^{\alpha\delta}\left(\frac{\partial a_{\delta\beta}}{\partial y^\gamma} + \frac{\partial a_{\delta\gamma}}{\partial y^\beta} - \frac{\partial a_{\beta\gamma}}{\partial y^\delta}\right) + \frac{1}{2}a^{\alpha\delta}(a_{\beta\varepsilon}\eta^\varepsilon_{\delta\gamma} + a_{\gamma\varepsilon}\eta^\varepsilon_{\delta\beta} - a_{\delta\varepsilon}\eta^\varepsilon_{\beta\gamma}).$$

Taking into account that $(a_{\alpha\beta})$ are constants we get (4.1).

Denote now by "s" the arc length with respect the Riemannian metric, i.e. $ds = (a_{ij}(x)dx^idx^j)^{1/2} = (a_{\alpha\beta}dx^\alpha dx^\beta)^{1/2}$ and by "S" the arc length with respect to Finsler metric, i.e.

$$dS = (a_{ij}(x)dx^i dx^j)^{1/2} + b_i(x)dx^i = (g_{ij}(x,y)dx^i dx^j)^{1/2}. \tag{4.2}$$

It is known that if we perform a variation of (4.2) we get the Lorentz force equation:

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = F^i_j \frac{dx^j}{ds}, \tag{4.3}$$

where γ^i_{jk} are the Christoffel symbols of the Riemannian metric (a_{ij}) and $F^i_j = a^{ik}(\frac{\partial b_k}{\partial x^j} - \frac{\partial b_j}{\partial x^k})$ is the electro-magnetic tensor field. But the equation (4.3) has an equivalent form:

$$\frac{d^2x^i}{dS^2} + F^i_{jk}\frac{dx^j}{dS}\frac{dx^k}{dS} = 0, \tag{4.3$'$}$$

where F^i_{jk} are coefficients of the Cartan connection of the Randers space (M, L). In anholonomic coordinates given by Holland's Frame, the equation $(4.3)'$ becomes:

$$\frac{d^2x^\alpha}{dS^2} + F^\alpha_{\beta\gamma}\frac{dx^\beta}{dS}\frac{dx^\gamma}{dS} = 0. \tag{4.4}$$

If we take into account the Theorem (4.1), then the Lorentz equation (4.3) can be expressed anholonomically as:

$$\frac{d^2x^\alpha}{dS^2} + a^{\alpha\delta}(a_{\beta\varepsilon}\Omega^\varepsilon_{\delta\gamma})\frac{dx^\beta}{dS}\,\frac{dx^\gamma}{dS} = 0. \tag{4.5}$$

□

Theorem 4.2. *Let (M, L) be a Randers space as above with Holland's Frame and crystallographic connection. Then $\tau^\alpha_{\beta\gamma} = 0$, $\sum^\alpha_{\beta\gamma} = 0$ and $C^\alpha_{\beta\gamma} = 0$. Moreover, this Finsler connection is flat and metric and $Y^i_{\alpha|j} = 0$, $Y^i_\alpha\big|_j = 0$.*

Proof. As $(a_{\alpha\beta})$ are constant we have by Theorem 3.4(1) that the crystallographic connection is metric. Flatness follows from Proposition 2.4 and the parallel translation invariance of Y^i_α from Proposition 2.3. □

Remark. As for the crystallographic connection, we have the anholonomic coefficients $F^\alpha_{\beta\gamma} = C^\alpha_{\beta\gamma} = 0$, and the induced absolute differential is:

$$\Delta X^\alpha = dX^\alpha. \tag{4.6}$$

Consequently, the geodesic equations for this connection are $\frac{d^2x^\alpha}{dS^2} = 0$.

If the Riemannian metric is positive definite we can choose the frame $\big(X^i_\alpha(x)\big)$ such that

$$X^i_\alpha(x)X^j_\beta(x)a_{ij}(x) = \delta_{\alpha\beta}.$$

In this case according with (3.10) we have:

$$g_{ij}Y^i_\alpha Y^j_\beta = \delta_{\alpha\beta}$$

and this means exactly that $\big(Y_\alpha = Y^i_\alpha\frac{\partial}{\partial y^i}\big)_{\alpha=\overline{1,n}}$ is an orthogonal frame.

Final Remark. Consider the absolute differential with holonomic coefficients (2.9), (2.10) and $(2.10)'$ in the present paper and compare them to (19) and (21) in [H_1]. For the case of Theorem 4.2 these two sets are identical. However, introduction of matter curves the flat Minkowski's 4-space of special relativity. Holland claims that a fully covariant theory (i.e. a curved space version of Theorem 4.2) is still possible. Theorem 3.3 is one way to complete his program. In this case the Cartan torsion tensor plays a crucial role and the three curvature tensors will not vanish. Unfortunately, the Holland frame is not invariant under parallel translation.

Theorem 3.4 also completes Holland's program. Indeed, not only is Holland's frame invariant, but equation (1) in the statement of Theorem 3.4 has the character of "extra matter" caused by point defects [K_2]. Point defects in Bravais crystals are accounted for by non-metricity. Thus, matter in Minkowski 4-space is expressed according to defect theory via (1). It is obvious that this description is in the spirit of Holland's original idea, based on the geometry of defects in crystal lattices, [H_1], [H_2].

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