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QUANTISING A RIGID BODY

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Abstract. This paper concerns the quantisation of a rigid body in the framework of the "covariant quantisation" on a curved spacetime with absolute time proposed by A. Jadczyk and M. Modugno.

We start with a spacetime for a pattern one–body mechanics, which is constituted by a 4–dimensional affine space fibred over time and equipped with a vertical Euclidean metric and an electromagnetic field. Then, we obtain the multi–spacetime for n–body mechanics by taking the n–fold fibred product of the above structure. Eventually, we obtain the spacetime for a rigid body by considering the fibred subbundle of the multi–spacetime defined by a rigid constraint.

We show that the general scheme of the "covariant quantisation" can be easily applied to the rigid body. In particular, we are concerned with the existence and classification of the inequivalent quantum structures.

Introduction

A covariant formulation of classical and quantum mechanics on a curved spacetime with absolute time based on fibred manifolds, jets, non linear connections, cosymplectic forms and Frölicher smooth spaces has been proposed by A. Jadczyk and M. Modugno [6, 7] and further developed in [1, 5, 10, 14, 15] and references therein. We shall briefly call this approach "covariant quantisation". It presents analogies with geometric quantisation ([3, 4, 8, 13, 12, 16] and references therein), but important novelties as well. In fact, it overcomes typical difficulties of geometric quantisation such as the problem of polarisations; moreover, in the flat case it reproduces the standard quantum mechanics (hence all standard examples).

Then, it would be interesting to treat new physically relevant examples. Here, we discuss an original geometric formulation of classical and quantum mechanics for a rigid body according to the general scheme of "covariant quantisation". The present analysis is more geometrical than physical, and has to be considered as a preliminar research, especially for what concerns the quantum theory. But we hope

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that this geometric setting could help understanding concrete physical objects (for examples, moleculae).

We start the paper with a sketch of the essential features of the general "covariant quantisation" following [5, 7, 10, 14, 15]. The classical theory is based on a fibred manifold ("spacetime") over time equipped with a vertical Riemannian metric ("space–like metric"), a time preserving linear connection ("gravitational connection") and a 2–form ("electromagnetic field"). The above objects yield a 2–form on the first jet space of spacetime ("phase space"), which is assumed to be closed. This form controls the classical dynamics. The quantum theory is based on a line bundle over spacetime equipped with a Hermitian and universal connection, whose curvature is proportional to the above classical 2–form. This quantum structure yields in a natural way a Lagrangian (hence the dynamics) and quantum operators.

In view of the formulation of classical and quantum mechanics of the rigid body in the framework of the above theory, we proceed in three steps ([11]). Namely, we start with a flat spacetime for a pattern one–body mechanics. Then, we consider an n–fold fibred product of the pattern structure as multi–spacetime for the n–body mechanics. Eventually, we consider the subbundle of the multi–spacetime induced by a rigid constraint as spacetime for the rigid body mechanics. Clearly, this spacetime fits the requirements of the "covariant quantisation"; hence, the general machinery for classical and quantum mechanics can be easily applied to the rigid body. In particular, we discuss the existence and classification of the inequivalent quantum structures. In forthcoming work we shall pursue the physical analysis.

We assume manifolds and maps to be C^∞.

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1. Covariant quantisation

We start with a brief sketch of the the covariant formulation of classical and quantum mechanics on a curved spacetime with absolute time proposed by A. Jadczyk and M. Modugno [6, 7] and further developed in [1, 5, 10, 14, 15] and in several other papers. We shall briefly call this approach *covariant quantisation*. For further details and discussions the reader will refer to the above literature and references therein.

In order to make the independence of classical and quantum mechanics from scales explicit, we introduce the "spaces of units of measurement" [7]. Roughly speaking, a unit space has the algebraic structure of $\mathbb{R}^+$ but has no distinguished 'basis'. The basic objects of our theory (metric, electromagnetic field, etc.) will be valued into *scaled* vector bundles, that is into vector bundles multiplied tensorially with unit spaces. In this way, each tensor field carries explicit information on its physical dimensions.

Actually, we assume the following basic unit spaces:
– $\mathbb{T}$, the space of *time intervals*;
– $\mathbb{L}$, the space of *lengths*;
– $\mathbb{M}$, the space of *masses*.
We set $\mathbb{T}^{-1} := \mathbb{T}^*$. We shall use rational tensor powers of unit spaces. We assume the *Planck's constant* $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$. Moreover, a *particle* will be assumed to have a *mass* $m \in \mathbb{M}$ and a *charge* $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$.

1.1. Covariant classical mechanics.

Assumption G.1. [5, 7] We assume an oriented manifold $\boldsymbol{E}$ of dimension $1+n$ equipped with
- a fibring $t : \boldsymbol{E} \to \boldsymbol{T}$ over an oriented 1-dimensional affine space $\boldsymbol{T}$ (associated with the vector space $\mathbb{T} \otimes I\!\!R$),
- a scaled vertical Riemannian metric $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes S^2 V^* \boldsymbol{E}$,
- a dt–preserving linear connection $K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} TT\boldsymbol{E}$,
- a scaled 2–form $f : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* \boldsymbol{E}$. □

We interpret $\boldsymbol{E}$ as *spacetime*, t as *absolute time map*, $\boldsymbol{T}$ as *absolute time*, g as *spacelike metric*, $K^\natural$ as *gravitational field* and f as *electromagnetic field*.

With reference to a given particle with mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$, it is convenient to *normalise* the metric and electromagnetic fields by setting $G := \frac{m}{\hbar} g : \boldsymbol{E} \to \mathbb{T} \otimes S^2 V^* \boldsymbol{E}$ and $F := \frac{q}{\hbar} f : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E}$.

We shall refer to fibred chart (x^0, x^i) on $\boldsymbol{E}$ adapted to a time scale $u_0 \in \mathbb{T}$. Latin indices $i, j, \ldots$ and Greek indices $\lambda, \mu, \ldots$ will denote space–like and spacetime coordinates, respectively. For short, we shall denote the induced dual bases of vector fields and forms by ∂_λ and d^λ. The vertical restriction will be denoted by the Check $\check{}$.

We have the coordinate expression $G = G_{ij} u_0 \otimes \check{d}^i \otimes \check{d}^j$. The fact that $K^\natural$ is dt–preserving reads in coordinates as $K^\natural{}_\lambda{}^0{}_\mu = 0$.

A section $s : \boldsymbol{T} \to \boldsymbol{E}$ is said to be a *motion*.

We assume [5, 7] as *phase space* for classical mechanics the first jet space [9] of motions $J_1\boldsymbol{E}$; the first jet prolongation of a motion s is said to be its *velocity*. We denote by (x^λ, x^i_0) the chart induced on $J_1\boldsymbol{E}$. We shall be involved with the natural complemetary maps $д : J_1\boldsymbol{E} \times \mathbb{T} \to T\boldsymbol{E}$ and $\theta : J_1\boldsymbol{E} \times_{\boldsymbol{E}} T\boldsymbol{E} \to V\boldsymbol{E}$, with expressions $д = u^0 \otimes (\partial_0 + x^i_0 \partial_i)$ and $\theta = (d^i - x^i_0 d^0) \otimes \partial_i$. We set $\theta^i \equiv d^i - x^i_0 d^0$.

An *observer* is defined to be a section $o : \boldsymbol{E} \to J_1\boldsymbol{E}$. Let o be an observer. Then, we obtain the map $\nabla[o] : J_1\boldsymbol{E} \to \mathbb{T}^{-1} \otimes \boldsymbol{E} : j_e - o(e)$. If s is a motion, then we define the *observed velocity* to be the section $\nabla[o] \circ j_1 s$. We define *observed kinetic energy* and *momentum*, respectively, as the maps $\mathcal{K}[o] := \frac{1}{2} G(\nabla[o], \nabla[o]) : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ and $\mathcal{Q}[o] := \theta^* \circ G^\flat(\nabla[o]) : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, with coordinate expressions $\mathcal{K}[o] = \frac{1}{2} G_{ij} x^i_0 x^j_0 \, d^0$ and $\mathcal{Q}[o] = G_{ij} x^j_0 \, \theta^i$.

The linear connection $K^\natural$ yields [5, 7] an affine connection $\Gamma^\natural$ on the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, with coordinate expressions $\Gamma^\natural{}_\lambda{}^i_0{}^0_\mu = K^\natural{}_\lambda{}^i{}_\mu$, and the non linear connection $\gamma^\natural := д \lrcorner \Gamma^\natural : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$ on the fibred manifold $J_1\boldsymbol{E} \to \boldsymbol{T}$, with coordinate expression $\gamma^\natural = u^0 \otimes (\partial_0 + x_0^i \partial_i + \gamma_0{}_0^i \partial_i^0)$, where $\gamma_0{}_0^i := K^\natural{}_h{}^i{}_k x_0^h x_0^k + 2K^\natural{}_h{}^i{}_0 x_0^h + K^\natural{}_0{}^i{}_0$. Moreover, $\Gamma^\natural$ yields [5, 7] the 2–form $\Omega^\natural := \nu[\Gamma^\natural] \bar{\wedge} \theta : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1 \boldsymbol{E}$, where $\nu[\Gamma^\natural]$ is the vertical projection complementary to $\Gamma^\natural$ and $\bar{\wedge}$ is the wedge product followed by a contraction with G. We have the coordinate expression $\Omega^\natural = G_{ij}\,(d_0^i - \gamma_0{}_0^i d^0 + \Gamma_h{}^i{}_0{}^0{}_h \theta^h) \wedge \theta^j$. The 2–form Ω is *non degenerate* as $dt \wedge \Omega \wedge \ldots \wedge \Omega$ is a scaled volume form of $J_1\boldsymbol{E}$.

There is a natural geometric way [5, 7] to couple the gravitational and electromagnetic objects into *global* objects, in such a way that all mutual relations holding for gravitational objects are preserved for total objects. Later on we shall refer to such total objects and we can forget about the two component fields. In particular, we deal with the total 2–form $\Omega := \Omega^\natural + \frac{1}{2}F$ and the total connection $\gamma = \gamma^\natural + \gamma^e$, where γ^e turns out to be the Lorentz force $\gamma^e = -G^\sharp(\breve{д} \lrcorner F)$.

Assumption G.2. We assume $d\Omega^\natural = 0$ and $dF = 0$. □

The first equation has several important consequences [5, 7]. In particular, it yields $\nabla g = 0$. The second equation is the first Maxwell equation.

The closed form Ω controls the classical dynamics [5, 10, 10]. It admits local potentials, called *Poincaré–Cartan forms*, of the type $\Theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, with coordinate expression $\Theta = -(\frac{1}{2} G_{ij} x_0^i x_0^j + A_i x_0^i) d^0 + (G_{ij} x_0^j + A_i) d^i$, where $A : \boldsymbol{E} \to T^*\boldsymbol{E}$ is a local potential of the closed 2–form $2o^*\Omega : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$ and the pullback is taken with respect to the observer o associated with the chosen chart. We define the local *Lagrangian* associated with a Poincaré–Cartan form Θ to be the form $\mathcal{L} := д \lrcorner \Theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, with coordinate expression $\mathcal{L} = (\frac{1}{2} G_{ij} x_0^i x_0^j + A_i x_0^i + A_0) u^0$. The Euler–Lagrange equation, in the unknown motion s, associated with this Lagrangian turns out to be the global equation $\nabla[\gamma] j_1 s = 0$, that is $\nabla[\gamma^\natural] j_1 s = -\gamma^e \circ j_1 s$. This equation is just the generalised Newton equation of motion for a charged particle in a given gravitational and electromagnetic field; we assume it to be our classical equation of motion. Moreover, given an observer o, we define the *Hamiltonian* and the *momentum* as the maps $\mathcal{H}[o] := -o \lrcorner \Theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ and $\mathcal{P}[o] := -\nu[o]^* \circ V_{\boldsymbol{E}} \mathcal{L} : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ (where $\nu[o]$ is the induced projection $\nu[o] : T\boldsymbol{E} \to V\boldsymbol{E}$), with coordinate expressions $\mathcal{H}[o] = (\frac{1}{2} G_{ij}\, x_0^i x_0^j - A_0) d^0$ and $\mathcal{P}[o] = (G_{ij} x_0^j + A_i) d^i$.

1.2. Covariant quantum mechanics. A *quantum bundle (over spacetime)* is defined [5, 7] to be a complex line bundle $\boldsymbol{Q} \to \boldsymbol{E}$, equipped with a Hermitian metric h with values in $\mathbb{C} \otimes \Lambda^3 V^*\boldsymbol{E}$. The choice of such a Hermitian metric allows us to avoid half–densities. A *quantum section* $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ describes a quantum particle.

Let η denote the vertical volume form induced by g. A local section $b : \boldsymbol{R} \to \mathbb{L}^{3/2} \otimes \boldsymbol{Q}$, such that $h(b, b) = \eta$, is a local base, said to be *normal*. We denote the local complex dual base of b by $z : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{C}$. If Ψ is a quantum section, then we write locally $\Psi = \psi b$.

The *Liouville vector field* is the vector field $и : \boldsymbol{Q} \to V\boldsymbol{Q} : q \mapsto (q, q)$, with coordinate expression $и = z\partial_z$.

A family of Hermitian connections on $\boldsymbol{Q} \to \boldsymbol{E}$ parametrised by sections of $J_1\boldsymbol{E} \to \boldsymbol{E}$ determines a Hermitian connection on $\boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \times_{\boldsymbol{E}} \boldsymbol{Q} \to J_1\boldsymbol{E}$, called *universal*.

We define [5, 7] a *quantum connection* to be a connection ч on $\boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$, which is Hermitian, universal, and whose curvature is given by $R[\text{ч}] = i\Omega \otimes \text{и}$.

We remark that the equation $d\Omega = 0$ turns out to be just the Bianchi identity for a quantum connection ч.

A pair $(\boldsymbol{Q}, \text{ч})$ is said to be a *quantum structure*.

Two complex line bundles $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$ on $\boldsymbol{E}$ are said to be *equivalent* if there exists an isomorphism of Hermitian line bundles $f : \boldsymbol{Q}_1 \to \boldsymbol{Q}_2$ on $\boldsymbol{E}$ (the existence of such an f is equivalent to the existence of an isomorphism of line bundles). Two quantum structures $(\boldsymbol{Q}_1, \text{ч}_1)$, $(\boldsymbol{Q}_2, \text{ч}_2)$, are said to be *equivalent* if there exists an equivalence $f : \boldsymbol{Q}_1 \to \boldsymbol{Q}_2$ which maps ч_1 into ч_2.

A quantum bundle is said to be *admissible* if it admits a quantum connection. Actually, the following theorem holds.

Let us consider the Čech cohomology $H^*(\boldsymbol{E}, X)$ with values in $X = \mathbb{R}$ or $X = \mathbb{Z}$, the inclusion morphism $i : \mathbb{Z} \to \mathbb{R}$ and the induced group morphism $i^* : H^*(\boldsymbol{E}, \mathbb{Z}) \to H^*(\boldsymbol{E}, \mathbb{R})$.

Proposition 1.1. [14, 15] *The following conditions are equivalent:*
i. there exists a quantum structure on $\boldsymbol{E}$*,*
ii. the class $[\Omega] \in H^2(\boldsymbol{E}, \mathbb{R})$ *lies in the subgroup* $i^2\left(H^2(\boldsymbol{E}, \mathbb{Z})\right) \subset H^2(\boldsymbol{E}, \mathbb{R})$.
Moreover, inequivalent quantum structures are in bijection with the set

$$(i^2)^{-1}([\Omega]) \times H^1(\boldsymbol{R}, \mathbb{R})/H^1(\boldsymbol{R}, \mathbb{Z})\,.$$

More precisely, the first factor parametrises admissible quantum bundles and the second factor parametrises quantum connections. □

The quantum theory is based on the only assumption of a quantum structure, supposing that the background spacetime admits one.

Assumption G.3. [5, 7] We assume a quantum bundle $\boldsymbol{Q}$ equipped with a quantum connection ч to be given. □

The quantum connection ч can be locally expressed [5, 7] as $\text{ч} = \text{ч}^{||} + i\Theta \otimes \text{и}$, where $\text{ч}^{||}$ is the flat connection induced by a local trivialisation of $\boldsymbol{Q}^{\uparrow}$ and Θ is a Poincaré–Cartan form.

All further quantum objects will be derived from the above quantum structure by natural procedures [5, 7].

We have been forced to assume ч on the pullback bundle $\boldsymbol{Q}^{\uparrow}$ because of the required link with the 2–form Ω. On the other hand, we wish to derive from ч new quantum objects, which are observer independent, hence living on the quantum bundle. For this purpose we follow a successful projectability criterion.

So, by means of a projectability criterion we can exibit [5, 7] a natural Lagrangian on $\boldsymbol{Q}$, which yields, by a standard procedure, the momentum, the Euler–Lagrange equation (generalised *Schrödinger equation*) and a conserved form (*probability current*). The coordinate expression of the Schrödinger equation is

$$i\,\big((\partial_0 - iA_0 + \frac{\partial_0\sqrt{|G|}}{\sqrt{|G|}}) + \frac{1}{2}\overset{o}{\Delta}_0\big)\psi = 0\,,$$

where $\overset{o}{\Delta}_0 := G^{hk}(\partial_h - iA_h)(\partial_k - iA_k) + \frac{\partial_h(G^{hk}\sqrt{|G|})}{\sqrt{|G|}}(\partial_k - iA_k)$.

Next, we sketch the formulation of quantum operators.

By means of a projectability criterion we can exhibit [5, 7] on the classical phase space a distinguished Lie algebra $\mathcal{Q}(J_1\boldsymbol{E}) \subset C^\infty(J_1\boldsymbol{E})$ of functions (*observables*), which are polynomial of second degree with respect to the affine fibring $J_1\boldsymbol{E} \to \boldsymbol{E}$ and whose second fibre derivative is proportional to the metric G. Functions of the above type are said to be *quantisable* and have coordinate expression of the type $f = \frac{1}{2}f^0\,G_{ij}\,x^i_0x^j_0 + f_i\,x^i_0 + f_0$, where f^0, f_i, f_0 are functions of $\boldsymbol{E}$. The bracket of this algebra is defined in terms of the Poisson bracket and the connection γ.

Then, by classifying [5, 7] the vector fields on $\boldsymbol{Q}^\uparrow$, which preserve the quantum structure and are projectable on $\boldsymbol{Q}$, we see that their projections on the quantum bundle constitute a Lie algebra of vectors fields, which is isomorphic to the Lie algebra of quantisable functions. These fields can be regarded as *pre–quantum operators* $Z[f]$ acting on quantum sections.

The *quantum bundle over time* is defined [5, 7] to be the bundle $\boldsymbol{S} \to \boldsymbol{T}$, whose fibres $\boldsymbol{S}_\tau$ are constituted by smooth quantum sections at the time τ with compact support. This infinite dimensional complex vector bundle turns out to be F–smooth in the sense of Frölicher [2] and inherits a pre–Hilbert structure via integration on the fibres. A Hilbert bundle can be obtained by completion.

We can prove [5, 7] that there is a unique linear connection χ on $\boldsymbol{S} \to \boldsymbol{T}$, such that $\nabla[\chi]$ is proportional to the Schrödinger operator.

Eventually, a natural procedure yields [5, 7] the symmetric *quantum operator* $\hat{f}$ on the pre–Hilbert bundle associated with every quantisable function f, as a linear combination of the corresponding pre–quantum operator $Z[f]$ and $\nabla[\chi]$. We have the coordinate expression

$$\hat{f}(\Psi) = \big(f_0 - i\frac{1}{2}\partial_h f^h - if^h(\partial_h - i\,A_h) - \frac{1}{2}\,f^0\overset{o}{\Delta}_0\big)\psi b\,.$$

For example, we have

$$\widehat{x^\alpha}(\Psi) = x^\alpha\Psi\,,\ \ \widehat{\mathcal{P}_j}(\Psi) = -i(\partial_j + \frac{1}{2}\frac{\partial_j\sqrt{|G|}}{\sqrt{|G|}})\psi b\,,\ \ \widehat{\mathcal{H}_0}(\Psi) = (\frac{1}{2}\overset{o}{\Delta}_0 - A_0)\psi b\,.$$

2. Rigid body

Now, we consider a rigid body and show how it can be quantised according to the scheme of the obove general theory. The procedure consists in three steps:

- we start with a flat “pattern spacetime” for the formulation of one–body classical and quantum mechanics;

- then, we consider the n–fold fibred product of the flat pattern spacetime and related structures as the framework for n–body classical and quantum mechanics;

- eventually, we consider the rigid constrained fibred submanifold of the above n–fold fibred product along with the induced structures as the framework for rigid–body classical and quantum mechanics.

Each of the above steps fits the general setting of the “covariant quantisation” sketched in the above section. So, that scheme can be applied to these specific cases.

2.1. Rigid body classical mechanics.

Pattern one–body mechanics. Following the general scheme, we start by assuming [11] as spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ for one–body mechanics a flat spacetime, which is called *pattern spacetime*. All objects related to this pattern spacetime are called *pattern objects* and labelled by the subscript p.

Assumption R.1. We assume as *pattern spacetime* a 4–dimensional affine space $\boldsymbol{E}_p$, equipped with an affine map t_p as time map, the connection $K_p^\natural$ induced by the affine structure as gravitational connection, a space–like metric g_p and an electromagnetic field f_p. Moreover, we assume $d\Omega_p^\natural = 0, df_p = 0$. □

A motion $s_p : \boldsymbol{T} \to \boldsymbol{E}_p$ and an observer $o : \boldsymbol{E} \to J_1\boldsymbol{E}$ are said to be *inertial* if they are affine maps.

The affine structure induces some trivialisations. We denote by $\bar{\boldsymbol{E}}_p$ the vector space associated with $\boldsymbol{E}_p$. Let us consider the maps $Dt_p : \bar{\boldsymbol{E}}_p \to \mathbb{T} \otimes I\!R$ and $\mathrm{id}_{\mathbb{T}^{-1}} \otimes Dt_p : \bar{\boldsymbol{E}}_p \to (\mathbb{T}^{-1} \otimes \mathbb{T}) \otimes I\!R$ and the induced vector and affine subspaces $\boldsymbol{S}_p := Dt_p^{-1}(0) \subset \bar{\boldsymbol{E}}_p$ and $\boldsymbol{U}_p := (\mathrm{id} \otimes Dt_p)^{-1}(1) \subset \mathbb{T}^{-1} \otimes \bar{\boldsymbol{E}}_p$.

Then, $t_p : \boldsymbol{E}_p \to \boldsymbol{T}$ turns out to be a principal bundle associated with the abelian group $\boldsymbol{S}_p$. Moreover, we have the natural isomorphisms $V\boldsymbol{E}_p \simeq \boldsymbol{E}_p \times \boldsymbol{S}_p$ and $J_1\boldsymbol{E}_p \simeq \boldsymbol{E}_p \times \boldsymbol{U}_p$.

The axiom $d\Omega_p^\natural = 0$ implies $\nabla g_p = 0$; hence, g_p can be regarded as a Euclidean metric on $\boldsymbol{S}_p$. Conversely, starting with a Euclidean metric on $\boldsymbol{S}_p$, we would get $d\Omega_p^\natural = 0$.

We stress, that, because of the affine structure of spacetime, Ω is globally exact; in particular, for each inertial observer o, $\Theta_p^\natural[o] := -\mathcal{K}[o] + \mathcal{Q}[o]$ is a distinguished global potential of $\Omega_p^\natural$.

Multi-body mechanics. Then, following the general scheme, we continue by assuming [11] as spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ for n–body mechanics the n–fold fibred product of the pattern spacetime, which is called *multi–spacetime* and denoted by $t_m : \boldsymbol{E}_m \to \boldsymbol{T}$. All objects related to this multi–spacetime are called *multi–objects* and labelled by the subscript m.

Thus, we consider a system on n particles with masses $m_1, \dots, m_n \in \mathbb{M}$ and charges $q_1, \dots, q_n \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$.

We define $m_0 := \sum_i m_i \in \mathbb{M}$ to be the *total mass* and $\mu_i := m_i/m_0 \in I\!R^+$ the *i-th weights*. For $i = 1, \dots, n$, we introduce n identical copies of the pattern objects $\boldsymbol{E}_i := \boldsymbol{E}_p$, $\boldsymbol{S}_i := \boldsymbol{S}_p$, $\boldsymbol{U}_i := \boldsymbol{U}_p$, $g_i := g_p$, $f_i := f_p$.

Assumption R.2. We assume as *multi–spacetime* the affine fibred product

$$\boldsymbol{E}_m := \boldsymbol{E}_1 \times_T \cdots \times_T \boldsymbol{E}_n ,$$

equipped with the associated projection t_m as time map, the connection $K^\natural_m$ induced by the affine structure as gravitational connection, $g_m := (\mu_1 g_1, \dots, \mu_n g_n)$ as space–like metric and $F_m := (\frac{q_1}{\hbar} f_1, \dots, \frac{q_n}{\hbar} f_n)$ as normalised electromagnetic field. □

We define the normalised metric as $G_m = \frac{m_0}{\hbar} g_m$.
Of course, we obtain $d\Omega^\natural_m = 0$ and $dF_m = 0$; then, Ω_r is globally exact.

Thus, the structure of multi–spacetime is analogous to that of pattern spacetime. The different dimension of the fibre in the two cases has no importance in many respects; hence, most concepts and results can be straightfowardly translated from the first case to the second one. So, we can formulate the classical and quantum mechanics of an n-body analogously to that of a one–body equipped with the total mass and effected by the given multi–metric, multi–gravitational field and normalised multi–electromagnetic field.

On the other hand, the multi–spacetime is equipped with the projections of the fibred product, which provide the information concerning each particle.

Moreover, due to the affine structure, the multi–spacetime is equipped [11] with another important splitting, which is related to the center of mass. Namely, the multi–spacetime splits naturally into the 3–dimensional affine subspace of center of mass and the $(3n-3)$–dimensional space of distances relative to the center of mass. This splitting will effect all geometrical, kinematical and dynamical structures, including the equation of motion.

In view of the definition of the center of mass and in order to emphasise its role, we consider a spacetime structure constituted by the copies $\boldsymbol{E}_0 := \boldsymbol{E}_p$ of the pattern spacetime, $g_0 := g_p$ of the pattern metric and $K^\natural_0 := K^\natural_p$ of the pattern gravitational connection. Moreover, we refer to the total mass $m_0 := \sum_i m_i$, the induced normalised metric $G_0 = \frac{m_0}{\hbar} g_0$ and the *total electromagnetic field* $F_0 := \sum_i F_i$. All objects related to this spacetime structure are labelled by the subscript 0.

For each $\tau \in \boldsymbol{T}$, we define the *center of mass* of $\boldsymbol{e} \in \boldsymbol{E}_{0\tau}$ to be the unique element $e_0 \in \boldsymbol{E}_{0\tau}$ such that $\sum_i \mu_i(e_i - e_0) = 0$. We say that $\boldsymbol{E}_0$ is the *space of centers of mass* and denote the associated projection by $\pi_0 : \boldsymbol{E} \to \boldsymbol{E}_0$.

Moreover, we call $\boldsymbol{S}_d := \{\boldsymbol{v} \in \boldsymbol{S}_m \mid \sum_i \mu_i v_i = 0\} \subset \boldsymbol{S}_m$ the *difference space*. Then, we have the affine splitting $\boldsymbol{E} \to \boldsymbol{E}_0 \times \boldsymbol{S}_d : \boldsymbol{e} \mapsto (e_0, v_d) := (\pi_0(\boldsymbol{e}), \boldsymbol{e} - e_0)$, which is orthogonal with respect to the multi–metric G_m.

This splitting is reflected on most multi–objects, whose components are accordingly labelled by the subscripts 0 and d.

Rigid body mechanics. Eventually, we achieve the scheme for a rigid body in the framework of the "covariant quantisation" by considering a rigid constraint on the multi–spacetime and assuming as spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ for the rigid body the constrained subbundle of the multi–spacetime, which is called *rigid–body spacetime* and denoted by $t : \boldsymbol{R} \to \boldsymbol{T}$. All objects related to this rigid–spacetime are called *rigid–body objects* and labelled by the subscript r.

In order to avoid trivialities, we suppose throughout the rest of the paper that $n \geq 2$.

We consider a set $\{l_{ij} \in \mathbb{L}^2 \,|\, i, j = 1, \dots, n,\, i \neq j, l_{ij} = l_{ji}, l_{ik} \leq l_{ij} + l_{jk}\}$ and define the subsets

$$i_r : \boldsymbol{R} := \{\boldsymbol{e} \in \boldsymbol{E} \mid \|e_i - e_j\| = l_{ij},\ 1 \leq i < j \leq n\} \hookrightarrow \boldsymbol{E}\,,$$
$$i_a : \boldsymbol{R}_a := \{\boldsymbol{v} \in \boldsymbol{S}_d \,|\, \|v_i - v_j\| = l_{ij}\,, 1 \leq i < j \leq n\} \hookrightarrow \boldsymbol{S}_d\,.$$

The affine splitting $\boldsymbol{E}_m \to \boldsymbol{E}_0 \times \boldsymbol{S}_d$ restricts to a splitting $\boldsymbol{R} \to \boldsymbol{E}_0 \times \boldsymbol{R}_a$. Thus, we obtain a curved fibred manifold $t : \boldsymbol{R} \to \boldsymbol{T}$ consisting of the cartesian product of the affine bundle $t : \boldsymbol{E}_0 \to \boldsymbol{T}$ with the (spacelike) submanifold $\boldsymbol{R}_a \subset \boldsymbol{S}_d$.

The metric g_m yields an orthogonal projection $\pi_a : T\boldsymbol{S}_d \to T\boldsymbol{R}_a$. The connection $K^\natural_d$ yields a connection $K^\natural_a$ on the bundle $\boldsymbol{T} \times \boldsymbol{R}_a \to \boldsymbol{T}$ via pullback i^*_r and projection π_a.

Assumption R.3. We assume as *rigid–body spacetime* the fibred manifold $t : \boldsymbol{R} \to \boldsymbol{T}$ equipped with the induced *spacelike metric* $g_r := i^*_r g_m$, the *gravitational connection* $K^\natural_r := K^\natural_0 \times K^\natural_a$ and the normalised *electromagnetic field* $F_r := i^*_r F_m$. □

Regarding the rigid body as a one–body with mass m_0, we refer to the normalised metric $G_r = \frac{m_0}{\hbar} g_r$.

The connection $\gamma^\natural_d$ yields a connection $K^\natural_a$ on the bundle $\boldsymbol{R}_a \to \boldsymbol{T}$ via pullback i^*_a and projection π_a. The connection $\gamma^\natural_r$ on the bundle $J_1\boldsymbol{R} \to \boldsymbol{T}$ induced by $K^\natural_r$ turns out to be $\gamma^\natural_r = \gamma^\natural_0 \times \gamma^\natural_a$.

The tensor γ^e_d yields a tensor γ^e_a on the bundle $J_1\boldsymbol{R}_a$ via pullback i^*_a and projection π_a. The tensor γ^e_r on $J_1\boldsymbol{R}$ induced by F_r and G_r coincides with γ^e_r.

We obtain $\Omega_r = i^*_r \Omega_m$. Hence, $d\Omega_r = 0$ and Ω_r is globally exact.

The geometry of $\boldsymbol{R}_a$ depends on the initial mutual positions of particles and is time independent. In particular, particles can either lie on a straight line, or lie on a plane, or "fill" the whole space. We say $\boldsymbol{R}$ to be *degenerate* in the first case, *weakly non degenerate* in the second case, *non degenerate* in the third case. Of course, if $n = 2$, then $\boldsymbol{R}$ is degenerate; if $n = 3$, then $\boldsymbol{R}$ can be degenerate or weakly non degenerate. It can be easily proved [11] that the choice of a point $\boldsymbol{r} \in \boldsymbol{R}$ yields the following diffeomorphisms:

– if $\boldsymbol{R}$ is degenerate, then $\boldsymbol{R}_a \simeq S^2$, where S^2 is the unit sphere in $\mathbb{R}^2$;

– if $\boldsymbol{R}$ is weakly non degenerate, then $\boldsymbol{R}_a \simeq SO(3, \boldsymbol{S}_p)$;

– if $\boldsymbol{R}$ is non degenerate, then $\boldsymbol{R}_a \simeq O(3, \boldsymbol{S}_p)$.

Henceforth, in this section, we shall refer only to the most interesting (weakly) non degenerate case. Moreover, we shall refer only to one of the two connected components of $\boldsymbol{R}_a$ (which is diffeomorphic to $SO(3, \boldsymbol{S}_p)$), for continuity reasons.

The velocity space of $\boldsymbol{R}$ splits as $J_1\boldsymbol{R} \simeq (\boldsymbol{E}_0 \times \boldsymbol{U}_0) \times (\mathbb{T}^{-1}{\otimes}T\boldsymbol{R}_a)$. So, to understand the geometry of $J_1\boldsymbol{R}$ we shall concentrate on the space $T\boldsymbol{R}_a$.

According to the classical formula for the velocity of a rigid body [11], for each $(d_1, \dots, d_n;\, v_1, \dots, v_n) \in T\boldsymbol{R}_a \subset T\boldsymbol{S}_d$, there is a unique $\omega \in \mathbb{L}^{-1}{\otimes}\boldsymbol{S}_p$ such that $v_i = \omega \times d_i$, where $\times$ is the cross product induced by the metric g_p. Hence, we obtain the fibred linear isomorphism over $\boldsymbol{R}_a$

$$T\boldsymbol{R}_a \to \boldsymbol{R}_a \times (\mathbb{L}^{-1}{\otimes}\boldsymbol{S}_p) : (d_1, \dots, d_n;\; v_1, \dots, v_n) \to (d_1, \dots, d_n;\; \omega)\,.$$

This result can also been interpreted in terms of the classical parallelisation of the tangent space of a Lie group, by recalling the Lie algebra $so(3, \boldsymbol{S}_p) \simeq \mathbb{L}^2{\otimes}\Lambda^2\boldsymbol{S}_p^*$ and the Hodge isomorphism $* : \mathbb{L}^2{\otimes}\Lambda^2\boldsymbol{S}_p^* \to \mathbb{L}^{-1}{\otimes}\boldsymbol{S}_p$.

In virtue of the above parallelisation, the metric G_a can be regarded as the *inertia tensor* $G_a : \boldsymbol{R}_a \times ((\mathbb{L}^{-1}{\otimes}\boldsymbol{S}_a) \times (\mathbb{L}^{-1}{\otimes}\boldsymbol{S}_a)) \to \mathbb{T}{\otimes}I\!R$ given by $(\Omega, \Omega') \mapsto \sum_i (G_i(d_i, d_i) g_a(\omega, \omega') - G_i(d_i, \omega) G_i(d_i, \omega'))$.

Just to compare the above scheme to the standard dynamics of a rigid body, we can prove [11] that the the equation of motion becomes the Euler system of equations

$$\nabla[\gamma_0^\natural] j_1 s_0 = \gamma_0^e \circ j_1 s_r\,, \qquad \nabla[\gamma_a^\natural] j_1 s_a = \gamma_a^e \circ j_1 s_a\,,$$

where $\nabla[\gamma_0^\natural] j_1 s_0$ is the acceleration of the center of mass, $\nabla[\gamma_a^\natural] j_1 s_a$ is the angular acceleration, $\gamma_0^e \circ j_1 s_r$ turns out to be the total force and $\gamma_a^e \circ j_1 s_a$ turns out to be the total momentum of forces.

2.2. Rigid body quantum mechanics. The setting of the above section shows that we can apply straightfowardly the "covariant quantisation" scheme to the rigid body.

First, we analyse the existence and classification of quantum structures according to Proposition 1.1.

The form Ω_r is exact, hence the existence condition is fulfilled. So, we have just to compute all possible inequivalent quantum structures. We recall that $[\Omega_r] = 0 \in H^2(\boldsymbol{R}, I\!R)$ due to the exactness of Ω_r.

If $\boldsymbol{R}$ is (weakly) non degenerate, then the universal covering $\pi : SU(2) \to \boldsymbol{R}_a$ is a principal bundle whose group is $\pi_1(\boldsymbol{R}_a) = \mathbb{Z}_2$. This group acts on the right on $SU(2)$ via deck transformations, which, in this case, turn out to be multiplication by the identity matrix I or by $-I$. It also acts on the left on $\mathbb{C}$ with two representations ρ_f and ρ_s, respectively the trivial representation and the representation via multiplication by $+1$ or -1. Accordingly, we have two bundles associated with π, namely

$$\boldsymbol{Q}_f = SU(2) \underset{\rho_0}{\times} \mathbb{C} \qquad \text{and} \qquad \boldsymbol{Q}_s = SU(2) \underset{\rho_1}{\times} \mathbb{C}\,.$$

Of course, $\boldsymbol{Q}_f = \boldsymbol{R}_a \times \mathbb{C} \to \boldsymbol{R}_a$ is the trivial bundle and $\boldsymbol{Q}_s \to \boldsymbol{R}_a$ is a non trivial bundle whose cocycle is the same as the cocycle of π. We believe that the second bundle might be related to spin.

Moreover, both $\boldsymbol{Q}_f$ and $\boldsymbol{Q}_s$ have a natural flat connection, which we denote by $\text{ч}^{\|}_f$, $\text{ч}^{\|}_s$, respectively.

Theorem 2.1. (Classification I) *Let $\boldsymbol{R}$ be (weakly) non degenerate. Then,*

i. the only inequivalent admissible quantum bundles on $\boldsymbol{R}$ are $\boldsymbol{Q}_f$ and $\boldsymbol{Q}_s$;

ii. the only inequivalent quantum structures are $(\boldsymbol{Q}_f, ч_f)$ and $(\boldsymbol{Q}_s, ч_s)$, with

$$ч_f = ч^{||}_f + i\Theta_f \otimes и\,, \qquad ч_s = ч^{||}_s + i\Theta_s \otimes и\,,$$

where Θ_f and Θ_s are two global potentials of Ω.

Proof. In fact, we have

$$(i^2)^{-1}([\Omega_r]) = \ker i^2 \simeq H^2(\boldsymbol{R}, \mathbb{Z}) \simeq H^2(SO(3), \mathbb{Z}) = \mathbb{Z}_2\,;$$

hence we have only the trivial bundle and a non trivial bundle as representatives of the two equivalence classes of admissible quantum bundles. These bundles coincide with $\boldsymbol{Q}_f$ and $\boldsymbol{Q}_s$.

The second statement comes from the fact that there is exactly one equivalence class of quantum connections on $\boldsymbol{Q}_f$ and $\boldsymbol{Q}_s$ (see Proposition 1.1). □

Theorem 2.2. (Classification II) *Let $\boldsymbol{R}$ be degenerate. Then,*

i. the only (inequivalent) admissible quantum bundle on $\boldsymbol{R}$ is the trivial bundle $\boldsymbol{Q} := \boldsymbol{R} \times \mathbb{C}$;

ii. the only (inequivalent) quantum structure on $\boldsymbol{R}$ is $(\boldsymbol{Q}, ч)$, with

$$ч = ч^{||} + i\Theta \otimes и\,,$$

where $ч^{||}$ is the natural flat connection on $\boldsymbol{Q} \to \boldsymbol{R}$ and Θ is a global potential of Ω_r.

Proof. It comes from $H^2(\boldsymbol{R}, X) = H^2(S^2, X) = X$, with $X = \mathbb{R}$ or $X = \mathbb{Z}$, $H^1(S^2, \mathbb{R}) = H^1(S^2, \mathbb{Z}) = 0$, and $i^2 : \mathbb{Z} \to \mathbb{R}$ the inclusion map. □

Then, we can choose each one of the above quantum structures and apply the machinery of the "covariant quantisation". In particular, we can easily write the Schrödinger equation and the Hamiltonian operator.

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