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# GEOMETRIC QUANTIZATION OF THE LANDAU PROBLEM ON HYPERBOLIC RIEMANN SURFACES

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ABSTRACT. The techniques of symplectic reduction are applied to the study of the Landau problem on the Poincaré plane and compact Riemann surfaces of genus $g > 1$. This allows us to study the manifold of orbits of constant energy. Once this is done we proceed to study the geometric quantization of this problem showing its relationship, in the compact case, with Maass automorphic forms and operators.

## 1. Introduction

The quantum Hall effect deals with electrons moving in a two dimensional conductor sample under the influence of a magnetic field orthogonal to its surface [1, 2]. One uses the Hamiltonian for a single electron moving in this magnetic field, with a potential which represents the action of the periodic ionic lattice and usually one assumes this lattice to be an euclidean one (torus) [3, 4]. However, the first stage in the treatment of the Hall effect requires the study of the so called Landau problem; that is, one neglects the above mentioned potential term [5].

A generalization of this set-up involving a hyperbolic lattice ( a lattice generated under the action of a Fuchsian group on the hyperbolic plane or a Riemann surface of genus $g > 1$) has been considered recently and some quantum aspects of this problem have been analyzed in [6, 7, 8, 9, 10].

In this paper we study the symplectic aspects and the geometric quantization of the hyperbolic Landau problem. These techniques allow us to prove that the prequantization line bundles on the Riemann surfaces are precisely the line bundles of automorphic forms of a given weight which depends on the genus of the

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Riemann surface and the magnetic field. In particular, using the Blattner-Kostant-Sternberg quantization method we stablish the relationship between the quantum Hamiltonian and the Maass operator which is used in the Physics literature.

## 2. Symplectic aspects of the hyperbolic Landau problem

The Landau problem is the study of the motion of a charged particle on an oriented Riemannian surface $(X, T_2)$ under the action of a covariantly constant magnetic field $F = B\Omega_2$ where $B \in \mathbb{R}$ and $\Omega_2$ is the area element associated with the metric tensor $T_2$ (see [11]).

As is well known [12], a symplectic formulation of this problem can be given on the tangent bundle $TX \xrightarrow{\tau} X$ with the symplectic 2-form $\omega_2 = md\theta + e\tau^* F$ where $\theta$ is the pullback to $TX$ of the canonical 1-form of $T^*X$ under the isomorphism $TX \xrightarrow{\flat} T^*X$ determined by the metric, and the constants $m$ and $e$ are the mass and the charge of the particle respectively. The Hamiltonian is $H(D_x) = \frac{1}{2} m T_2(D_x, D_x)$ with $D_x \in TX$.

In this paper $(X, T_2)$ will be the hyperbolic plane or a compact Riemann surface.

**2.1. The Landau problem on the hyperbolic plane.** The configuration space is the hyperbolic plane $\mathbb{H} = \{z \in \mathbb{C}/Imz > 0\}$ with the metric $T_2 = \frac{R^2}{y^2}(dx^2 + dy^2)$ of Gaussian negative curvature $K = -\frac{1}{R^2}$. We have the following explicit expressions:

$$\omega_2 = \frac{mR^2}{y^2}(dx \wedge d\dot{x} + dy \wedge d\dot{y}) + \frac{eBR^2}{y^2} dx \wedge dy$$

$$eF = \frac{eBR^2}{y^2} dx \wedge dy$$

$$H = \frac{mR^2}{2y^2}(\dot{x}^2 + \dot{y}^2)$$

Let us note that $(\mathbb{H}, eF)$ is a symplectic manifold and that the group $\mathbb{P}SL(2, \mathbb{R})$ acts transitive and symplectically on $\mathbb{H}$ by isometries as follows

$$\begin{array}{cccccc} & \mathbb{P}SL(2,\mathbb{R}) & \times & \mathbb{H} & \longrightarrow & \mathbb{H} \\ (\gamma = & \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} & , & z) & \mapsto & \gamma \cdot z = R(\dfrac{a_\gamma z + b_\gamma R}{c_\gamma z + d_\gamma R}) \end{array}$$

This action on $(\mathbb{H}, eF)$ admits an equivariant moment map $J : \mathbb{H} \to \mathfrak{sl}(2, \mathbb{R})^*$ with respect to the coadjoint action, which is given by

$$J_1 = J^* A_1 = -\tfrac{eBR}{2y}(x^2 + y^2 - R^2)$$

$$J_2 = J^* A_2 = \tfrac{eBR^2}{y} x$$

$$J_3 = J^* A_3 = -\tfrac{eBR}{2y}(x^2 + y^2 + R^2)$$

where $< A_1, A_2, A_3 >$is a basis of $\mathfrak{sl}(2, \mathbb{R})$.

The previous action of $\mathbb{P}SL(2,\mathbb{R})$ on $\mathbb{H}$ induces a second symplectic action on $(T\mathbb{H},\omega_2)$ without fixed points that preserves the Hamiltonian $H$ and admits an equivariant moment map $\bar{J}\colon T\mathbb{H} \to \mathfrak{sl}(2,\mathbb{R})^*$, which is called hyperbolic angular momentum. One has the following expressions for the hyperbolic angular momenta [13]:

$$\bar{J}_1 = \bar{J}^* A_1 = -\tfrac{mR}{2y^2}\left((x^2-y^2-R^2)\dot{x} + 2xy\dot{y}\right) - \tfrac{eBR}{2y}(x^2+y^2-R^2)$$

$$\bar{J}_2 = \bar{J}^* A_2 = \tfrac{mR^2}{y^2}(x\dot{x}+y\dot{y}) + \tfrac{eBR^2}{y}x$$

$$\bar{J}_3 = \bar{J}^* A_3 = -\tfrac{mR}{2y^2}\left((x^2-y^2+R^2)\dot{x} + 2xy\dot{y}\right) - \tfrac{eBR}{2y}(x^2+y^2+R^2)$$

It is important to remark that the Hamiltonian can be expressed in terms of the hyperbolic angular momenta as

$$H = \frac{1}{2mR^2}(\bar{J}_1^2 + \bar{J}_2^2 - \bar{J}_3^2 + e^2B^2R^4)$$

In order to study the submanifold of constant energy of $T\mathbb{H}$, let us recall that

**Proposition.** *The coadjoint orbits of the group $\mathbb{P}SL(2,\mathbb{R})$ are:*

(1) $\mathcal{O}_0 = (0,0,0)$.
(2) $\mathcal{O}^{hyp}_{k^2,+} = \{\alpha \;/\; \alpha\cdot\alpha = -k^2 \;,\; \alpha_3 > 0\} \;, \forall\; k^2 \in \mathbb{R}^+$.
(3) $\mathcal{O}^{hyp}_{k^2,-} = \{\alpha \;/\; \alpha\cdot\alpha = -k^2 \;,\; \alpha_3 < 0\} \;, \forall\; k^2 \in \mathbb{R}^+$.
(4) $\mathcal{O}^{cone}_{+} = \{\alpha \neq (0,0,0) \;/\; \alpha\cdot\alpha = 0 \;,\; \alpha_3 > 0\}$.
(5) $\mathcal{O}^{cone}_{-} = \{\alpha \neq (0,0,0) \;/\; \alpha\cdot\alpha = 0 \;,\; \alpha_3 < 0\}$.
(6) $\mathcal{O}^{hyp\,1\,sheet}_{k^2} = \{\alpha \;/\; \alpha\cdot\alpha = k^2\} \;, \forall\; k^2 \in \mathbb{R}^+$.

*where $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathfrak{sl}(2,\mathbb{R})^*$ and $\alpha\cdot\alpha = \alpha_1^2+\alpha_2^2-\alpha_3^2$.*

**Proposition.** *If $G_\alpha$ denotes the isotropy subgroup of a point $\alpha \in \mathfrak{sl}(2,\mathbb{R})^*$ with respect to the coadjoint action of $\mathbb{P}SL(2,\mathbb{R})$, then one has that:*

(1) *If $\alpha \in \mathcal{O}_0$, then $G_\alpha = \mathbb{P}SL(2,\mathbb{R})$.*
(2) *If $\alpha \in \mathcal{O}^{hyp}_{k^2,\pm}$, then $G_\alpha \simeq SO(2)$.*
(3) *If $\alpha \in \mathcal{O}^{cone}_{\pm}$, then $G_\alpha \simeq E(1)$.*
(4) *If $\alpha \in \mathcal{O}^{hyp\,1\,sheet}_{k^2}$ then $G_\alpha \simeq SO(1,1)^o$.*

Let us denote by $T^\epsilon\mathbb{H} = \{D_x \in T\mathbb{H}/H(D_x) = \epsilon\}$ the submanifold of $T\mathbb{H}$ of constant energy $\epsilon$ and assume that $eB < 0$ (similar results hold for $eB > 0$).

**Theorem.** *Taking into account that $\epsilon = \frac{1}{2mR^2}(J\cdot J + e^2B^2R^4)$ with $J\cdot J = \bar{J}_1^2+\bar{J}_2^2-\bar{J}_3^2$, one has that*

(1) $T^0\mathbb{H} = \bar{J}^{-1}(\mathcal{O}^{hyp}_{k^2,+})$ *for* $\epsilon = 0$.
(2) $T^\epsilon\mathbb{H} = \bar{J}^{-1}(\mathcal{O}^{hyp}_{k^2,+})$ *for* $0 < \epsilon < \frac{e^2B^2R^2}{2m}$.
(3) $T^\epsilon\mathbb{H} = \bar{J}^{-1}(\mathcal{O}^{cone}_{+})$ *for* $\epsilon = \frac{e^2B^2R^2}{2m}$.
(4) $T^\epsilon\mathbb{H} = \bar{J}^{-1}(\mathcal{O}^{hyp\,1\,sheet}_{k^2})$ *for* $\epsilon > \frac{e^2B^2R^2}{2m}$.

**Corollary.** *Let $H = \epsilon$ be the energy of the particle, then:* (a) *If $\epsilon = 0$, then the trajectory on $\mathbb{H}$ is a point.* (b) *If $0 < \epsilon < \frac{e^2B^2R^2}{2m}$, then the trajectory on $\mathbb{H}$ is closed (periodic).* (c) *If $\epsilon \geq \frac{e^2B^2R^2}{2m}$, then the trajectory on $\mathbb{H}$ is open.*

For $\epsilon = 0$, the manifold of orbits of zero energy $T^0\mathbb{H}$ is identified with the zero section of the tangent bundle $T\mathbb{H}$, and $\omega_{2|T^0\mathbb{H}}$ is symplectic.

For $\epsilon \neq 0$, $T^\epsilon\mathbb{H}$ is a three dimensional submanifold of $T\mathbb{H}$. Let $\omega_2^\epsilon = \omega_{2|T^\epsilon\mathbb{H}}$ and let us denote by $ker\omega_2^\epsilon$ its null foliation. One has $ker\omega_2^\epsilon = < D_H^\epsilon >$ where $D_H^\epsilon$ is the restriction to $T^\epsilon\mathbb{H}$ of the Hamiltonian vector field $D_H$ associated to the Hamiltonian $H$ , that is $i_{D_H}\omega_2 = -dH$.

**Definition.** *The manifold of orbits of constant energy (when it exists) is the symplectic manifold $(\overline{\mathbb{H}}^\epsilon \equiv T^\epsilon\mathbb{H}/ker\omega_2^\epsilon, \overline{\omega}_2^\epsilon)$ and $\phi^*\overline{\omega}_2^\epsilon = \omega_2^\epsilon$ where $\phi : T^\epsilon\mathbb{H} \longrightarrow \overline{\mathbb{H}}^\epsilon$ is the natural projection.*

As a consequence of the Kazhdan-Kostant-Sternberg theorem [12] one has:

**Theorem.** [13] *The moment map $\bar{J}$ induces symplectic diffeomorphisms*

(1) $\hat{J} : \overline{\mathbb{H}}^0 \to \mathcal{O}^{hyp}_{e^2B^2R^2,+}$ *for* $\epsilon = 0$.

(2) $\hat{J} : \overline{\mathbb{H}}^\epsilon \to \mathcal{O}^{hyp}_{k^2,+}$ *for* $0 < \epsilon < \frac{e^2B^2R^2}{2m}$.

(3) $\hat{J} : \overline{\mathbb{H}}^\epsilon \to \mathcal{O}^{cone}_{+}$ *for* $\epsilon = \frac{e^2B^2R^2}{2m}$.

(4) $\hat{J} : \overline{\mathbb{H}}^\epsilon \to \mathcal{O}^{hyp\,1\,sheet}_{k^2}$ *for* $\epsilon > \frac{e^2B^2R^2}{2m}$

*where every coadjoint orbit is endowed with the Kostant-Kirillov-Souriau symplectic form.*

**2.2. The Landau problem on compact Riemann surfaces.** Let $S$ be a compact Riemann surface of genus $g > 1$ with a metric $T_2$ of constant curvature $K = -\frac{1}{R^2}$. Riemann's uniformization theorem tells us that $S \simeq \mathbb{H}/\Gamma$ for some Fuchsian group $\Gamma$ containing only hyperbolic elements. Moreover $\mathbb{H} \xrightarrow{p} \mathbb{H}/\Gamma$ is the universal Riemannian covering [14].

Let $\tilde{\omega}_2$ be the symplectic form and $\tilde{H}$ the Hamiltonian on $TS$. Since $p^*\tilde{F} = F$, $p^*\tilde{\omega}_2 = \omega_2$ and $p^*\tilde{H} = H$, the local expressions for $\tilde{\omega}_2$, $\tilde{F}$ and $\tilde{H}$ are the same as in $\mathbb{H}$. But we have

**Proposition.** [13] *$(TS, \tilde{\omega}_2, \tilde{H})$ does not admit Hamiltonian vector fields coming from $S$ and leaving $\tilde{H}$ invariant (that is, there are no infinitesimal symmetries).*

However, $\Gamma$ acts on $T\mathbb{H}$ and $TS = T(\mathbb{H}/\Gamma) \simeq T\mathbb{H}/\Gamma$. Let us denote by $T^\epsilon S$ the submanifold of $TS$ of constant energy. Then $T^\epsilon S \simeq T^\epsilon\mathbb{H}/\Gamma$. One also has that $ker\omega_2^\epsilon$ is $\Gamma$-invariant, therefore the manifold of orbits of constant energy $\overline{S}^\epsilon$ is identified with $\overline{\mathbb{H}}^\epsilon/\Gamma \simeq \mathcal{O}/\Gamma$. But the action $\Gamma \times \mathcal{O} \xrightarrow{\Phi} \mathcal{O}$ can have fixed points. In fact, given $\alpha \in \mathcal{O}$, its isotropy group for the action $\Phi$ is $\Gamma_\alpha = \Gamma \cap G_\alpha$.

Since all the elements of $\Gamma$ are hyperbolic, $G_\alpha$ is trivial except in the case $G_\alpha \simeq SO(1,1)^o$ and this happens for $\epsilon > \frac{e^2B^2R^2}{2m}$.

**Theorem.** *We have symplectic diffeomorphisms*

(1) $\tilde{J} : \overline{S}^0 \to \mathcal{O}^{hyp}_{e^2B^2R^2,+}/\Gamma$ for $\epsilon = 0$

(2) $\tilde{J} : \overline{S}^{\epsilon} \to \mathcal{O}^{hyp}_{k^2,+}/\Gamma$ for $0 < \epsilon < \frac{e^2B^2R^2}{2m}$,
(3) $\tilde{J} : \overline{S}^{\epsilon} \to \mathcal{O}^{cone}_{+}/\Gamma$ for $\epsilon = \frac{e^2B^2R^2}{2m}$

Thus we obtain another proof of the following result given in [15]

**Corollary.** (1) *For $0 < \epsilon < \frac{e^2B^2R^2}{2m}$ every trajectory on $S$ is closed and contractible.*
(2) *For $\epsilon = \frac{e^2B^2R^2}{2m}$ every trajectory on $S$ is open.*
(3) *For $\epsilon > \frac{e^2B^2R^2}{2m}$ there is a non contractible closed trajectory representing every homotopy class in $\pi_1(S) \simeq \Gamma$.*

## 3. Geometric Quantization of the Landau problem

Our configuration space $X$ is an oriented Riemannian surface and the phase space is $(TX,\ \omega_2 = md\theta + e\tau^*F)$. We assume that $[\frac{1}{\hbar}eF] \in H^2(X,\mathbb{Z})$. Then the prequantization condition $[\frac{1}{\hbar}\omega_2] \in H^2(TX,\mathbb{Z})$ is fulfilled.

Let $\bar{\pi} : (\bar{L}, \overline{\nabla}, \langle\ ,\ \rangle_{TX}) \to (TX, \omega_2)$ be a prequantization line bundle [16, 17] for $(TX, \omega_2)$; that is, a hermitian line bundle with a connection such that:

a) $iCurv(\overline{\nabla}) = \frac{1}{\hbar}\omega_2$
b) $D\langle s_1, s_2\rangle_{TX} = \langle \overline{\nabla}_D s_1, s_2\rangle_{TX} + \langle s_1, \overline{\nabla}_D s_2\rangle_{TX}$

Therefore the prequantization operator $\bar{Q}(\bar{f}) : \Gamma(TX,\bar{L}) \longrightarrow \Gamma(TX,\bar{L})$ corresponding to $\bar{f} \in C^\infty(TX)$ is given by

$$\bar{Q}(\bar{f}) = -i\hbar\overline{\nabla}_{D_{\bar{f}}} + \bar{f}$$

As we have pointed out, the magnetic field defines a symplectic structure $eF$ on the configuration space $X$. Since $[\frac{1}{\hbar}\omega_2] \in H^2(TX,\mathbb{Z})$ if and only if $[\frac{1}{\hbar}eF] \in H^2(X,\mathbb{Z})$ one has the following

**Proposition.** *For every prequantization line bundle $(\bar{L}, \overline{\nabla}, \langle\,,\,\rangle_{TX}) \xrightarrow{\bar{\pi}} (TX, \omega_2)$ there exists a unique (up to equivalence) prequantization line bundle $(L, \nabla, \langle\ ,\ \rangle_X) \xrightarrow{\pi} (X, eF)$ such that:*

(1) *$\tau^*L = \bar{L}$. Then, if $V$ is the vertical polarization of the tangent bundle and $\Gamma(TX,\bar{L})^V$ denotes the set of vertical polarized sections, there exists an isomorphism*

$$\begin{array}{ccc} \Gamma(X,L) & \xrightarrow{\tau^*} & \Gamma(TX,\bar{L})^V \\ s & \longmapsto & s^v \end{array}$$

(2) $\forall\, s_1\, s_2 \in \Gamma(X,L)$ , $\langle s_1, s_2\rangle_X = \langle s_1^v, s_2^v\rangle_{TX}$
(3) $\forall\, s \in \Gamma(X,L), \forall D \in \mathfrak{X}(X)$ , $(\nabla_D s)^v = \overline{\nabla}_{\bar{D}} s^v + \frac{im}{\hbar}\theta(\bar{D})s^v$ *where $\bar{D}$ is the canonical lifting of $D \in \mathfrak{X}(X)$ to $TX$.*

For every $f \in C^\infty(X)$ , let $D_f$ be the Hamiltonian vector field associated to $f$ with respect to the symplectic structure $eF$ (that is $i_{D_f}eF = -df$). Then the prequantization operator $Q(f) : \Gamma(X,L) \longrightarrow \Gamma(X,L)$ is given by

$$Q(f) = -i\hbar\nabla_{D_f} + f$$

If we consider the following map

$$\begin{array}{rcl} C^\infty(X) & \xrightarrow{\wedge} & C^\infty(TX) \\ f & \longmapsto & \widehat{f} = \tau^*(f) + m\theta(\bar{D}_f) \end{array}$$

then we have

**Proposition.** *$\forall f \in C^\infty(X)$, there is a commutative diagram*

$$\begin{array}{ccc} \Gamma(X,L) & \xrightarrow{\tau^*} & \Gamma(TX,\bar{L})^V \\ {\scriptstyle Q(f)}\big\downarrow & & \big\downarrow{\scriptstyle \bar{Q}(\hat{f})} \\ \Gamma(X,L) & \xrightarrow{\tau^*} & \Gamma(TX,\bar{L})^V \end{array}$$

**Remark.** *In the case $X = \mathbb{H}$, we can see that $\widehat{J}_i$, $\forall 1 \le i \le 3$, are precisely the hyperbolic angular momenta $\bar{J}_i$. The prequantization of the hyperbolic angular momenta $\bar{Q}(\bar{J}_i)$ acting on $\Gamma(TX,\bar{L})^V$ coincides, via $\tau^*$, with the prequantization operators $Q(J_i)$ acting on $\Gamma(X,L)$.*

Since $X$ is oriented, its orientation induces a metaplectic structure on $(TX,\omega_2)$ which leads to a trivial half-form bundle for the vertical polarization. Therefore

**Proposition.** [16, 18] *The Blattner-Kostant-Sternberg quantization method for the vertical polarization allows us to quantize the Hamiltonian $H$ giving*

$$Q_H^{BKS} = \frac{\hbar^2}{2m}(\nabla^*\nabla + \frac{s}{6})$$

*where $\nabla^*\nabla$ is the Bochner Laplacian of $(L,\nabla)$ and $s$ is the scalar curvature of the metric. Therefore $Q_H^{BKS}$ acts on $L^2(X,L)$.*

**3.1. The Hyperbolic plane $\mathbb{H}$.** On the hyperbolic plane $\mathbb{H}$ the prequantization line bundle $L \xrightarrow{\pi} \mathbb{H}$ is trivial. We can take a global section $s_0$ such that $\nabla s_0 = -\frac{i}{\hbar}A \otimes s_0$ where $A = eBR^2\frac{dx}{y}$ and therefore we can take $\langle s_0, s_0\rangle_{\mathbb{H}} = 1$.

For every $f \in C^\infty(\mathbb{H})$ we have

$$Q(f) = -\frac{i\hbar y^2}{eBR^2}\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right) + y\frac{\partial f}{\partial y} + f$$

In particular we have for $J_1$, $J_2$, $J_3$

$$Q(J_1) = -\tfrac{i\hbar}{2R}\left((R^2 - x^2 + y^2)\tfrac{\partial}{\partial x} - 2xy\tfrac{\partial}{\partial y}\right) - eBRy$$

$$Q(J_2) = -i\hbar\left(x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y}\right)$$

$$Q(J_3) = \tfrac{i\hbar}{2R}\left((R^2 + x^2 - y^2)\tfrac{\partial}{\partial x} + 2xy\tfrac{\partial}{\partial y}\right) - eBRy$$

**Proposition.** *The Casimir operator of the action of $\mathbb{P}SL(2,\mathbb{R})$ on $\Gamma(X,L)$ is*

$$Q(J_1)^2 + Q(J_2)^2 - Q(J_3)^2 = -\hbar^2\left(y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2i\frac{eBR^2}{\hbar}y\frac{\partial}{\partial x}\right)$$

**Proposition.**

$$\nabla^*\nabla = -\frac{1}{R^2}\left(y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - 2i\frac{eBR^2}{\hbar}y\frac{\partial}{\partial x} - \frac{e^2B^2R^4}{\hbar^2}\right)$$

**Corollary.** *The Hamiltonian of the quantum Landau problem on the hyperbolic plane is*

$$Q_H^{BKS} = \frac{1}{2mR^2}(Q(J_1)^2 + Q(J_2)^2 - Q(J_3)^2 + e^2B^2R^4) - \frac{\hbar^2}{6mR^2}$$

**Remark.** *This expression is equal to the classical one except for the term* $-\frac{\hbar^2}{6mR^2}$

**3.2. Compact Riemann surfaces $S$ of genus $g > 1$.** Let $S = \mathbb{H}/\Gamma$ and $\mathbb{H} \xrightarrow{p} S = \mathbb{H}/\Gamma$ be the canonical projection. The symplectic form on $S$ is given by $e\tilde{F} = eB\tilde{\Omega}_2$ where $\tilde{\Omega}_2$ is the area element with $p^*\tilde{\Omega}_2 = \Omega_2$.

**Proposition.** *There exists a prequantization line bundle* $(\tilde{L}, \tilde{\nabla}, \langle\ ,\ \rangle_S) \xrightarrow{\tilde{\pi}} (S, e\tilde{F})$ *if and only if*

$$\frac{eB}{\hbar}\tilde{\Omega}_2 \in H^2(S,\mathbb{Z}) \ \Leftrightarrow -\frac{eBR^2}{\hbar}\chi(S) = c_1(\tilde{L}) \in \mathbb{Z}$$

**Theorem.** *Let* $(L, \nabla, \langle\ ,\ \rangle_{\mathbb{H}}) \xrightarrow{\pi} (\mathbb{H}, eF)$ *be a prequantization line bundle. If* $-\frac{eBR^2}{\hbar} \in \frac{4\pi}{s\mu(\mathcal{F})}\mathbb{Z} = \frac{1}{\chi(S)}\mathbb{Z}$ *then there exists an action of* $\Gamma$ *on* $L \xrightarrow{\pi} \mathbb{H}$ *covering the action of* $\Gamma$ *on* $\mathbb{H}$ *such that:*

1. It leaves $\nabla$ and $\langle\ ,\ \rangle_{\mathbb{H}}$ invariant.
2. $(L/\Gamma, \nabla/\Gamma, \langle\ ,\ \rangle_{\mathbb{H}}/\Gamma) \xrightarrow{\tilde{\pi}} (S, e\tilde{F})$ is a prequantization line bundle.

**Proof.** Let $\bar{\Gamma}$ be the group in $SL(2,\mathbb{R})$ covering $\Gamma$. Taking a global section $s_0$ of $L \xrightarrow{\pi} \mathbb{H}$ one has that $L \simeq \mathbb{H}\times\mathbb{C}$ and the action is given by an automorphy factor

$$\begin{array}{ccccc} \bar{\Gamma} & \times & L & \longrightarrow & L \\ (\gamma & , & (z,\alpha)) & \mapsto & (\gamma(z), e^{i\xi(\gamma,z)}\alpha) \end{array}$$

where

$$\begin{array}{cccccc} \xi: & \bar{\Gamma} & \times & \mathbb{H} & \longrightarrow & \mathbb{R} \\ & (\gamma & , & z) & \mapsto & \xi(\gamma,z) \end{array}$$

verifies $\xi(\gamma_1\gamma_2, z) - \xi(\gamma_1, \gamma_2(z)) - \xi(\gamma_2, z) \in 2\pi\mathbb{Z}$. The connection 1-form of $\nabla$ is $\omega = -\frac{eBR^2}{\hbar}\frac{dx}{y} + d\sigma$ with $d\sigma$ being the Maurer-Cartan 1-form of $U(1)$ and the condition $\gamma^*\omega = \omega\ \forall\gamma\in\bar{\Gamma}$, implies that

$$\xi(\gamma,z) = 2\frac{eBR^2}{\hbar}\arg(c_\gamma z + d_\gamma R) + \mu_B(\gamma)$$

where $\bar{\Gamma} \xrightarrow{\mu_B} \mathbb{R}$ verifies $\mu_B(\gamma_1\gamma_2) - \mu_B(\gamma_1) - \mu_B(\gamma_2) - 4\pi\frac{eBR^2}{\hbar}w_B(\gamma_1,\gamma_2) \in 2\pi\mathbb{Z}$ with

$$w_B(\gamma_1,\gamma_2) = \tfrac{1}{2\pi}\{\arg(c_{\gamma_1}\gamma_2(z) + d_{\gamma_1}R) + \arg(c_{\gamma_2}z + d_{\gamma_2}R) - \arg(c_{\gamma_3}z + d_{\gamma_3}R)\}$$

Finally the map

$$\begin{array}{rcccl} \sigma_B: & \bar{\Gamma} & \times & \bar{\Gamma} & \longrightarrow U(1) \\ & (\gamma_1 & , & \gamma_2) & \mapsto \sigma_B(\gamma_1,\gamma_2) = e^{i4\pi\frac{eBR^2}{\hbar}w_B(\gamma_1,\gamma_2)} \end{array}$$

must define a trivial cohomology class $[\sigma_B] \in H^2(\bar{\Gamma}, U(1))$ and [19, 20, 21] this is the case if and only if $-\frac{eBR^2}{\hbar} \in \frac{4\pi}{s\mu(\mathcal{F})}\mathbb{Z} = \frac{1}{\chi(S)}\mathbb{Z}$. □

**Corollary.** *The sections of the prequantization line bundle $\tilde{L} \xrightarrow{\tilde{\pi}} S$ are identified with the automorphic forms $\mathcal{A}^k(\Gamma)$ on $\mathbb{H}$ of weight $k = \frac{eBR^2}{\hbar}$ and multiplier $\nu_B = e^{i\mu_B}$, that is functions $\psi$ on $\mathbb{H}$ such that*

$$\psi(\gamma(z)) = e^{i\mu_B(\gamma)}e^{2i\frac{eBR^2}{\hbar}\arg(c_\gamma z + d_\gamma R)}\psi(z) \quad \forall \gamma \in \bar{\Gamma}$$

Let $\mathcal{D}_B = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2i\frac{eBR^2}{\hbar}y\frac{\partial}{\partial x}$ be the Maass Laplacian acting on automorphic forms $\mathcal{A}^k(\Gamma)$ [22].

**Proposition.** *The Hamiltonian $Q_H^{BKS} = \frac{\hbar^2}{2m}(\nabla^*\nabla + \frac{s}{6})$ acting on $L^2(S,\tilde{L})$ is identified with the operator*

$$Q_H^{BKS} = -\frac{\hbar^2}{2mR^2}\mathcal{D}_B + \frac{e^2B^2R^2}{2m} - \frac{\hbar^2}{6mR^2}$$

**Theorem.** *The discrete series of the spectrum of $Q_H^{BKS}$ acting on $\mathcal{A}^k(\Gamma)$ is*

$$E_l^{(k)} = \frac{\hbar^2}{2mR^2}[(2l+1)|k| - l(l+1)] - \frac{\hbar^2}{6mR^2}$$

*with $l \in \mathbb{Z}$ and $0 \leq l < |k| - \frac{1}{2}$*

The degeneration of each eigenvalue can be calculated using the Riemann-Roch theorem

**Theorem.** dim $H_l^{(k)} = \frac{1}{2}\chi(S)(2l - 2|k| + 1)$.

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