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ON THE GRADED LIE ALGEBRA OF QUANTISABLE FORMS

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Abstract. A quantisable form is defined to be a form on the 1-jet space of spacetime whose Hamiltonian lift is a tangent valued form projectable on spacetime. The quantisable forms constitute a graded Lie algebra with respect to a natural bracket. A natural homomorphism from the algebra of quantisable forms to the Frölicher–Nijenhuis algebra of tangent valued forms on the spacetime is exhibited.

Introduction

This paper is devoted to some new results on the covariant formulation of quantum mechanics originally proposed by A. Jadczyk and M. Modugno ([15, 17]) and further developed in cooperation with D. Canarutto, J. Janyška, R. Vitolo ([2, 3, 14, 19, 20, 37, 45]).

The original theory has been formulated in the framework of a curved *Galilei spacetime*, that is a curved spacetime fibred over absolute time. The key techniques are based on fibred manifolds, jets, non linear connections, cosymplectic forms, graded Lie algebra of tangent valued forms, universal connections and Frölicher "smooth spaces" of sections.

This theory yields a generalisation of the Schrödinger and Pauli equations and related Hilbert stuff on a curved spacetime with absolute time.

The theory is manifestly generally covariant. That is, its formulation is coordinate free and independent from the choice of any accelerated frames. Of course, due to the fibring of spacetime over absolute time, the underlying classical theory is unable to fully account for the effects of movement of masses and charges on the gravitational and electromagnetic fields. So, in a sense, this theory can be

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regarded as a rigorous mathematical theory, which physically stands between the standard non relativistic theory and a fully general relativistic theory.

The theory has several analogies with respect to other geometric approaches to quantum mechanics, such as "geometric quantisation". On the other hand it overcomes several typical problems of geometric quantisation. For instance, the theory has no problems of polarisations and half-densities, in virtue of the use of universal connections and Hermitian products valued in the bundle of volume forms.

Moreover, in the flat case, the theory recovers exactly the standard Schrödinger and Pauli equations and related Hilbert stuff. Hence, the theory allows us to recover directly all known standard examples of quantum systems.

One of the main topics of this theory is a geometric characterisation of the quantisable functions ([14, 15, 17]). They are functions on the jet bundle of space-time characterised by the projectability of their Hamiltonian lifts; they constitute a Lie algebra by means of a new bracket (different from the Poisson or the Jacobi brackets). A geometrical theorem yields an isomorphism between this Lie algebra of quantisable functions and a distinguished Lie algebra of projectable vector fields on the quantum bundle, as the core of pre–quantisation procedure. The above requirements of projectability are aimed at getting rid of the problem of polarisations in the quantum theory.

The goal of this paper is to extend the above procedure to forms of higher degree. In fact, we obtain a graded Lie algebra of quantisable forms (with respect to a new bracket) and a natural homomorphism with the graded Lie algebra of tangent valued forms (with respect to the Frölicher–Nijenhuis bracket). For instance, the classical Poincaré–Cartan form turns out to be quantisable. In a forthcoming paper we shall exhibit a natural isomorphism between the graded Lie algebra of quantisable forms and a distinguished graded Lie algebra of tangent valued forms on the quantum bundle. We expect that these results could be useful for a covariant formulation of the quantum measurement theory.

The reader can find some partial analogies with our approach in [1, 35], where the standard Hamiltonian formalism of a Poisson or a symplectic manifold is extended to forms. On the other hand, we stress that we are dealing with a co–symplectic manifold and with a distinguished type of forms.

We assume the following fundamental unit spaces, which are positive 1–dimensional "semi–vector spaces" over $\mathbb{R}^+$:

(1) the space $\mathbb{T}$ of *time intervals*,

(2) the space $\mathbb{L}$ of *lengths*,

(3) the space $\mathbb{M}$ of *masses*.

A *time unit of measurement* is defined to be an element $u_0 \in \mathbb{T}$, or its dual $u^0 \in \mathbb{T}^*$.

Moreover, we refer to the *Planck constant* $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

We shall be concerned with smooth manifolds and maps.

Let $\boldsymbol{M}$ be a manifold and $\boldsymbol{F} \to \boldsymbol{B}$ a fibred manifold. Then, $\pi_{\boldsymbol{M}} : T\boldsymbol{M} \to \boldsymbol{M}$ denotes the tangent bundle of $\boldsymbol{M}$ and $V\boldsymbol{F} \subset T\boldsymbol{F}$ the vertical subbundle. Moreover,

$\mathcal{M}(\boldsymbol{M}, \mathbb{S})$ denotes the sheaf of local maps $f : \boldsymbol{M} \to \mathbb{S}$ with values in a set $\mathbb{S}$. Furthermore, $\mathcal{S}(\boldsymbol{B}, \boldsymbol{F})$ denotes the sheaf of local sections $s : \boldsymbol{B} \to \boldsymbol{F}$; in particular, $\mathcal{T}(\boldsymbol{M}) \equiv \mathcal{S}(\boldsymbol{M}, T\boldsymbol{M})$ is the sheaf of local vector fields $X : \boldsymbol{M} \to T\boldsymbol{M}$, $\mathcal{T}^*(\boldsymbol{M}) \equiv \mathcal{S}(\boldsymbol{M}, T^*\boldsymbol{M})$ is the sheaf of local 1–forms $\alpha : \boldsymbol{M} \to T^*\boldsymbol{M}$ and $\mathcal{V}(\boldsymbol{F}) \equiv \mathcal{S}(\boldsymbol{F}, V\boldsymbol{F})$ is the sheaf of local vertical vector fields $X : \boldsymbol{F} \to V\boldsymbol{F}$.

1. Geometry of spacetime and phase space

First, we formulate a covariant scheme for the classical spacetime, the gravitational and electromagnetic fields and the mechanics of a classical particle [14, 15, 16, 17, 18, 19, 21, 37, 44, 46].

1.1. Spacetime. We start with the fibred and metric structure of spacetime. Later we shall add the connection structure.

We assume the classical *spacetime* to be a 4-dimensional orientable manifold $\boldsymbol{E}$, the *absolute time* to be a 1-dimensional oriented affine space $\boldsymbol{T}$ associated with the vector space $\mathbb{T} \otimes I\!\!R$ and the *time map* to be a surjective map $t : \boldsymbol{E} \to \boldsymbol{T}$ of rank 1. Moreover, we assume the fibres of spacetime to be equipped with a "scaled" family of Riemannian metrics $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V^*\boldsymbol{E})$.

Thus, we have the *time-form* $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$. Given a mass $m \in \mathbb{M}$, it is convenient to introduce the "normalised" metric $G := \frac{m}{\hbar} g : \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V^*\boldsymbol{E})$. We stress that the normalised metric and all objects which will be derived from it incorporate the chosen mass and the Planck constant.

We choose an orientation of spacetime. We shall refer to *spacetime charts* (x^λ), which are adapted to the fibring t and to the chosen orientation of $\boldsymbol{E}$, and such that x^0 is a Cartesian chart of $\boldsymbol{T}$ associated with a time unit of measurement u_0. The index 0 will refer to the base space, Latin indices $i, j, \dots = 1, 2, 3$ will refer to the fibres, while Greek indices $\lambda, \mu, \dots = 0, 1, 2, 3$ will refer both to the base space and the fibres. For short, we denote the local bases of vector fields and forms of $\boldsymbol{E}$ induced by a spacetime chart by (∂_λ) and (d^λ). The chart on the tangent space $T\boldsymbol{E}$ induced by a spacetime chart (x^λ) will be denoted by $(x^\lambda, \dot{x}^\lambda)$; accordingly, the induced local bases of vector fields and forms of $T\boldsymbol{E}$ will be denoted by $(\partial_\lambda, \dot{\partial}_\lambda)$ and $(d^\lambda, \dot{d}^\lambda)$. In general, the check symbol "ˇ" will indicate vertical restriction.

We have the coordinate expressions $dt = u_0 \otimes d^0$ and $G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j$, where $G^0_{ij} \equiv \frac{m}{\hbar} g_{ij} u^0 \in \mathcal{M}(\boldsymbol{E}, I\!\!R)$.

A *motion* is defined to be a section $s : \boldsymbol{T} \to \boldsymbol{E}$.

1.2. Phase space. The classical *phase space* is defined to be the first jet space $t^1_0 : J\boldsymbol{E} \equiv J_1\boldsymbol{E} \to \boldsymbol{E}$ of motions. The *velocity* of a motion s is defined to be its first jet prolongation $j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E}$. A spacetime chart (x^λ) induces on the phase space the chart (x^0, x^i, x^i_0), characterised by $(x^0, x^i, x^i_0) \circ j_1 s \equiv (x^0, s^i, \partial_0 s^i)$. We can naturally identify the phase space with the affine subbundle $J\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$ characterised by $x^i_0 = u_0 \otimes \dot{x}^i$. Thus, we obtain the well defined *jet contact maps* $д : J\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ and $\theta : J\boldsymbol{E} \to T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V\boldsymbol{E}$, with coordinate expressions $д \equiv u^0 \otimes д_0 = u^0 \otimes (\partial_0 + x^i_0\, \partial_i)$ and $\theta \equiv \theta^i \otimes \partial_i = (d^i - x^i_0\, d^0) \otimes \partial_i$.

We have the natural fibred isomorphism $V_{\boldsymbol{E}}J\boldsymbol{E} \simeq J\boldsymbol{E} \times_{\boldsymbol{E}} (\mathbb{T}^*\otimes V\boldsymbol{E})$ over $J\boldsymbol{E}$; it can be regarded as a scaled tangent valued form $\nu : J\boldsymbol{E} \to \mathbb{T}\otimes(V^*\boldsymbol{E}\otimes_{J\boldsymbol{E}}VJ\boldsymbol{E})$, with coordinate expression $\nu = u_0\otimes\check{d}^i\otimes\partial_i^0$.

An *observer* is defined to be a section $o : \boldsymbol{E} \to J\boldsymbol{E}$, that is a field of velocities. A chart is said to be *adapted* to an observer o if $o_0^i = 0$. There are many spacetime charts adapted to an observer; conversely, each spacetime chart yields one observer.

According to the principle of general relativity, we have not assumed spacetime to be split into the product of time and space (we have only the fibring over time). Analogously, we have no distinguished splitting of tangent vectors of spacetime into their time–like and space–like components (we have only the time–like component).

On the other hand, an observer o can be regarded as a non linear connection on the fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$. Hence, it allows us to split a tangent vector field X into its time–like component $\underline{X} := dt(X) : \boldsymbol{E} \to \mathbb{T} \otimes I\!R$ and space–like component $\nu[o](X) \equiv X - \langle o, \underline{X}\rangle$. Analogously, it allows us to split the velocity $j_1 s$ of a motion s into its time–like component $1 : \boldsymbol{T} \to \mathbb{T}^* \otimes \mathbb{T}$ and *space–like observed velocity* $\nabla[o]j_1 s \equiv j_1 s - o \circ s : \boldsymbol{T} \to \mathbb{T}^* \otimes V\boldsymbol{E}$.

1.3. Connections. We shall be concerned with several connections, which are mutually related and partially related to the metric.

First of all, the vertical Riemannian metric g yields the family of Riemannian connections $\varkappa : V\boldsymbol{E} \to V^*\boldsymbol{E}\otimes_{V\boldsymbol{E}}TV\boldsymbol{E}$ of the fibres of spacetime, whose coordinate expression is $\varkappa = \check{d}^i\otimes(\partial_i + \varkappa_i{}^j{}_h\, \dot{x}^h\, \dot{\partial}_j)$, where $\varkappa_h{}^i{}_k = -\frac{1}{2}g^{ij}(\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk})$.

Next, we analyse some further distinguished types of connections; later we shall make some assumptions on them.

A *spacetime connection* is defined to be a torsion free linear connection $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes_{T\boldsymbol{E}} TT\boldsymbol{E}$ of the bundle $T\boldsymbol{E} \to \boldsymbol{E}$ such that $\nabla dt = 0$. Its coordinate expression is of the type $K = d^\mu \otimes (\partial_\mu + K_\mu{}^i{}_\nu\, \dot{x}^\nu\, \dot{\partial}_i =)$, with $K_\mu{}^i{}_\nu = K_\nu{}^i{}_\mu \in \mathcal{M}(\boldsymbol{E}, I\!R)$.

A spacetime connection K yields, by restriction, a linear connection $K' : V\boldsymbol{E} \to T^*\boldsymbol{E} \otimes_{V\boldsymbol{E}} TV\boldsymbol{E}$ on the vertical bundle $V\boldsymbol{E} \to \boldsymbol{E}$, which is characterised in coordinates by $K'{}_\mu{}^i{}_j = K_\mu{}^i{}_j =$. Moreover, a spacetime connection K yields, by further restriction, a family of linear connections $\check{K} : V\boldsymbol{E} \to V^*\boldsymbol{E} \otimes_{V\boldsymbol{E}} VV\boldsymbol{E}$ of the fibres of spacetime, which is characterised in coordinates by $\check{K}_h{}^i{}_k = K_h{}^i{}_k$.

A spacetime connection K is said to be *metric* if $\nabla g = 0$.

Given an observer o, a spacetime connection K can be split into its vertical restriction $\check{K}$, a symmetric space–like tensor $\check{\Sigma}[o] : \boldsymbol{E} \to S^2V^*\boldsymbol{E}$ and an antisymmetric tensor $\Phi[o] : \boldsymbol{E} \to \Lambda^2T^*\boldsymbol{E}$, with coordinate expressions (in a chart adapted to o) $\Sigma_{ij} = -(G^0_{jh}K_i{}^h{}_0 + G^0_{ih}K_j{}^h{}_0)$, $\quad \Phi_{0j} = -G^0_{jh}K_0{}^h{}_0$, $\quad \Phi_{ij} = -(G^0_{jh}K_i{}^h{}_0 - G^0_{ih}K_j{}^h{}_0)$.

A spacetime connection K is metric if and only if $\check{K} = \varkappa$ and $\check{\Sigma}[o] = G^\flat L_o\bar{G}$, where $L_o\bar{G}$ is the Lie derivative, with respect to o, of the contravariant metric $\bar{G}$.

A *phase connection* is defined to be a torsion free affine connection $\Gamma : J\boldsymbol{E} \to T^*\boldsymbol{E} \otimes_{J\boldsymbol{E}} TJ\boldsymbol{E}$ of the bundle $t_0^1 : J\boldsymbol{E} \to \boldsymbol{E}$. Its coordinate expression is of the type $\Gamma = d^\mu \otimes (\partial_\mu + \Gamma_\mu{}^i_0\, \partial^0_i)$, where $\Gamma_\mu{}^i_0 = \Gamma_\mu{}^i_0{}^0_0 + \Gamma_\mu{}^i_0{}^0_j\, x^j_0$ with $\Gamma_\mu{}^i_0{}^0_\nu \in \mathcal{M}(\boldsymbol{E}, I\!R)$.

We can prove ([21]) that there is a natural bijection $K \mapsto \Gamma$ between spacetime connections and phase connections characterised in coordinates by $\Gamma_\mu{}^i_0{}^0_\nu = K_\mu{}^i{}_\nu$.

A (non linear) *second order connection* of $\boldsymbol{E}$ is defined as a section $\gamma : J_1\boldsymbol{E} \to J_2\boldsymbol{E}$. It can also be viewed as a (non linear first order) connection $\gamma : J\boldsymbol{E} \to \mathbb{T}^* \otimes TJ\boldsymbol{E}$ of the fibred manifold $J\boldsymbol{E} \to \boldsymbol{T}$, which projects on д. Its coordinate expression is of the type $\gamma = u^0 \otimes (\partial_0 + x^i_0\, \partial_i + \gamma_0{}^i_0\, \partial^0_i)$, where $\gamma_0{}^i_0 \in \mathcal{M}(J\boldsymbol{E}, I\!R)$. We say γ to be *homogeneous* if the coefficients $\gamma_0{}^i_0$ are second order polynomials in the coordinates x^i_0.

For each phase connection Γ, the section $\gamma = д \lrcorner \Gamma : J\boldsymbol{E} \to \mathbb{T}^* \otimes TJ\boldsymbol{E}$ turns out to be a homogeneous second order connection, with coordinate expression $\gamma_0{}^i_0 = \Gamma_h{}^i_0{}^0_k\, x^h_0 x^k_0 + 2\,\Gamma_h{}^i_0{}^0_0\, x^h_0 + \Gamma_0{}^i_0{}^0_0$. Even more, the map $\Gamma \mapsto \gamma \equiv д \lrcorner \Gamma$ is a bijection between phase connection and homogeneous second order connections.

Each aspect of the above objects in bijection (spacetime connections, phase connections and second order connections) will play a specific role: a spacetime connection will describe the gravitational and electromagnetic fields, the corresponding phase connection will equip the phase space and the corresponding second order connection will describe the dynamics of a classical particle.

1.4. Phase 2–form. A phase connection Γ on $J\boldsymbol{E}$ and the vertical metric G yield the *phase 2–form* $\Omega := \nu[\Gamma]\,\bar\wedge\,\theta : J\boldsymbol{E} \to \Lambda^2 T^* J\boldsymbol{E}$, whose coordinate expression is

$$\Omega = G^0_{ij}\,(d^i_0 - \Gamma_\mu{}^i{}_0 d^\mu) \wedge \theta^j\,, \tag{1.1}$$

where $\nu[\Gamma] : J\boldsymbol{E} \to \mathbb{T}^* \otimes (T^*J\boldsymbol{E} \otimes_{J\boldsymbol{E}} V\boldsymbol{E})$ is the vertical valued form associated with Γ and the contracted wedge product $\bar\wedge$ is taken with respect to G. In [18] it has been proved that the above form is the only non trivial natural 2–form which can be obtained from Γ and G.

The 2–form Ω is non–degenerate, in fact $dt \wedge \Omega \wedge \Omega \wedge \Omega : J\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^7 T^* J\boldsymbol{E}$ is non singular, hence it is a scaled volume form on $J\boldsymbol{E}$.

1.5. Phase 2-vector. A phase connection Γ on $J\boldsymbol{E}$ and the vertical metric G yield the *phase 2–vector* $\Lambda := \check\Gamma\,\bar\wedge\,\nu : J\boldsymbol{E} \to \Lambda^2 V J\boldsymbol{E}$, with coordinate expression

$$\Lambda = G^{ij}_0\,(\partial_i + \Gamma_i{}^h_0\,\partial^0_h) \wedge \partial^0_j\,, \tag{1.2}$$

where the contracted wedge product is taken with respect to $\bar G$.

The 2–vector Λ is non–degenerate, in fact $\gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda : J\boldsymbol{E} \to \mathbb{T}^* \otimes \Lambda^7 TJ\boldsymbol{E}$ is non singular, hence it is a scaled volume vector on $J\boldsymbol{E}$.

1.6. Connection structure. We assume spacetime to be equipped with a spacetime connection K.

Henceforth, we shall refer to this spacetime connection K and to the associated phase connection Γ, second order connection γ, phase 2–form Ω and phase 2–vector Λ.

Possibly, K can be regarded as the sum of a "gravitational" connection $K^\natural$ plus a tensor induced by an electromagnetic field F, according to a natural procedure described in detail in [14, 17].

1.7. Closure of the phase 2–form. We analyse the closure of the phase 2–form and the vanishing of related Schouten brackets.

Theorem 1.1. ([17]) *The following conditions are equivalent:*

$$d\Omega = 0\,, \tag{1.3}$$

$$\nabla G = 0\,, \qquad R^i{}_\lambda{}^j{}_\mu = R^j{}_\mu{}^i{}_\lambda\,, \tag{1.4}$$

$$\nabla G = 0\,, \qquad d\Phi[o] = 0. \tag{1.5}$$

Thus, in a coordinate chart adapted to the observer o, the phase 2–form Ω is closed if and only if

$$K_{ihj} = -\frac{1}{2}(\partial_i G^0_{hj} + \partial_j G^0_{hi} - \partial_h G^0_{ij})\,, \qquad K_{ij0} + K_{ji0} = -\partial_0 G^0_{ij}\,,$$
$$\partial_\nu \Phi_{\lambda\mu} + \partial_\mu \Phi_{\nu\lambda} + \partial_\lambda \Phi_{\mu\nu} = 0\,,$$

where the indices of K have been lowered by means of $G^\flat$. □

Now we analyse the Schouten brackets ([32]) $[\Lambda, \Lambda]$ and $[\gamma, \Lambda]$, and their relation with $d\Omega$.

Lemma 1.1. *We have the following coordinate expression*

$$\begin{aligned}
[\gamma, \Lambda] =& u^0 \otimes 2(\nabla_0 G^{ij}_0 + x^h_0 \nabla_h G^{ij}_0)\, \partial_i \wedge \partial^0_j \\
&+ u^0 \otimes 2\big(\Gamma_k{}^i_0 (\nabla_0 G^{jk}_0 + x^h_0 \nabla_h G^{jk}_0)\, \partial^0_i \wedge \partial^0_j \\
&+ u^0 \otimes 2\big(R^i{}_h{}^j{}_k\, x^h_0 x^k_0 + (R^i{}_0{}^j{}_h + R^i{}_h{}^j{}_0)\, x^h_0 + R^i{}_0{}^j{}_0\big)\, \partial^0_i \wedge \partial^0_j\,.
\end{aligned}$$

Proof. It follows from a computation in coordinates by recalling the formula

$$i([\gamma, \Lambda])\beta = i(\gamma) d i(\Lambda)\beta - i(\Lambda) d i(\Lambda)\beta\,,$$

which holds for every closed 3–form β. □

Lemma 1.2. *We have the following coordinate expression*

$$\begin{aligned}
[\Lambda, \Lambda] =& 12\, G^{kj}_0\, \nabla_k G^{hi}_0\, \partial_h \wedge \partial^0_i \wedge \partial^0_j \\
&- 2\left(6\, G^{li}_0\, \nabla_l G^{mj}_0 \Gamma_m{}^h_0 + 3\,(R^{ijh}{}_q x^q_0 + R^{ijh}{}_0)\right) \partial^0_i \wedge \partial^0_j \wedge \partial^0_h\,.
\end{aligned}$$ □

Theorem 1.2. *The following conditions are equivalent:*

$$d\Omega = 0\,, \tag{1.6}$$

$$[\gamma, \Lambda] = 0\,. \tag{1.7}$$

Moreover, the condition $d\Omega = 0$ implies $[\Lambda, \Lambda] = 0$. □

Remark 1.1. If $d\Omega = 0$, then it is easy to see that $[\Lambda, \Lambda] \neq 2\gamma \wedge \Lambda$, i.e. the vector field γ and the phase 2-vector Λ do not define a Jacobi structure on $J\boldsymbol{E}$ ([32]). □

Eventually, we make a fundamental assumption, which can be regarded as a field equation for the interaction between K and G. Namely, we assume:

$$d\Omega = 0\,.$$

2. Lagrangian and Hamiltonian formalism

2.1. Poincaré–Cartan forms. Next, we analyse some distinguished objects induced by the closed phase 2–form Ω.

A form $\alpha \in \Lambda^r \mathcal{T}^*(J\boldsymbol{E})$ is said to be $\boldsymbol{E}$*–horizontal* if $i(X)\alpha = 0$ for any $X \in \mathcal{V}_{\boldsymbol{E}}(J\boldsymbol{E})$. Thus, a form α is $\boldsymbol{E}$–horizontal, if and only if it can be regarded as a fibred morphism $\alpha : J\boldsymbol{E} \to \Lambda^r T^*\boldsymbol{E}$ over $\boldsymbol{E}$, i.e. if and only if its coordinate expression is of the type $\alpha = \alpha_{\lambda_1\ldots\lambda_r}\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r}$, with $\alpha_{\lambda_1\ldots\lambda_r} \in \mathcal{M}(J\boldsymbol{E}, I\!R)$.

The phase 2–form Ω can be locally derived from a potential $\Theta \in \mathcal{T}^*(J\boldsymbol{E})$. On the other hand, we can easily see that Ω admits $\boldsymbol{E}$–horizontal local potentials, i.e. local potentials of the type $\Theta : J\boldsymbol{E} \to T^*\boldsymbol{E}$. Each potential of the above type will be called a *Poincaré–Cartan form*. The coordinate expressions of Poincaré–Cartan forms Θ is of the type $\Theta = -(\frac{1}{2}G^0_{ij}\, x^i_0 x^j_0 - A_0)\, d^0 + (G^0_{ij} x^j_0 + A_i)\, d^i$, $A_\lambda \in \mathcal{M}(\boldsymbol{E}, I\!R)$. A Poincaré–Cartan form Θ is defined up to a closed local form $\alpha : \boldsymbol{E} \to T^*\boldsymbol{E}$.

We define the *Lagrangian* associated with a Poincaré–Cartan form Θ to be the local map $\mathcal{L} := д \lrcorner \Theta : J\boldsymbol{E} \to \mathbb{T}^* \otimes I\!R$, with coordinate expression $\mathcal{L} = (\frac{1}{2}G^0_{ij}\, x^i_0 x^j_0 + A_i\, x^i_0 + A_0)\, u^0$. By considering the gauge of Poincaré–Cartan forms, a Lagrangian $\mathcal{L}$ turns out to be defined up to a map of the type $д \lrcorner \alpha : J\boldsymbol{E} \to \mathbb{T}^* \otimes I\!R$, where $\alpha \in \mathcal{T}^*(\boldsymbol{E})$ is a closed form. We observe that, by pullback with respect to $t : \boldsymbol{E} \to \boldsymbol{T}$, a Lagrangian $\mathcal{L}$ can be also regarded as a fibred morphism $\mathcal{L} : J\boldsymbol{E} \to T^*\boldsymbol{E}$; accordingly, we shall write $\mathcal{L} \equiv \mathcal{L}_0\, u^0 \simeq \mathcal{L}_0\, d^0$.

We can say more about the relation between Lagrangians and Poincaré–Cartan forms.

We define, in the standard way, the *momentum* associated with a Lagrangian $\mathcal{L}$ as the map $\check{\mathcal{P}} := V_{\boldsymbol{E}}\mathcal{L} : J\boldsymbol{E} \to V^*\boldsymbol{E}$; the momentum turns out to be the vertical restriction of the 1–form $\mathcal{P} \equiv \theta \lrcorner \check{\mathcal{P}} : J\boldsymbol{E} \to T^*\boldsymbol{E}$. The coordinate expression of the momentum is given by $\check{\mathcal{P}} = (G^0_{ij}\, x^j_0 + A_i)\, \check{d}^i$, $\mathcal{P} = -(G^0_{ij}\, x^i_0\, x^j_0 + A_i\, x^i_0)\, d^0 + (G^0_{ij} x^j_0 + A_i)\, d^i$. Then, we can prove that each Poincaré–Cartan form Θ can be written as ([37]) $\Theta = \mathcal{L} + \mathcal{P} : J\boldsymbol{E} \to T^*\boldsymbol{E}$, where $\mathcal{L} \equiv д \lrcorner \Theta$ and $\mathcal{P} \equiv V_{\boldsymbol{E}}\mathcal{L}$.

Next, we describe the above objects in terms of observers. So, let us consider an observer o, the closed 2–form $\Phi[o] \equiv 2\, o^*\Omega : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$ and its local potential $A[o] : \boldsymbol{E} \to T^*\boldsymbol{E}$ (which is defined up to a closed form of $\boldsymbol{E}$).

We define the *Hamiltonian* associated with a Poincaré–Cartan form Θ and the observer o to be the local map $\mathcal{H}[o] := -\, o \lrcorner \Theta : J\boldsymbol{E} \to \mathbb{T}^* \otimes I\!R$; its coordinate expression in adapted coordinates is $\mathcal{H}[o] = (\frac{1}{2}G^0_{ij}\, x^i_0 x^j_0 - A_0)\, u^0$. We can express the Hamiltonian as $\mathcal{H}[o] = K[o] - o \lrcorner A[o]$. We observe that, by pullback with respect to $t : \boldsymbol{E} \to \boldsymbol{T}$, a Hamiltonian $\mathcal{H}[o]$ can be also regarded as a fibred morphism $\mathcal{H}[o] : J\boldsymbol{E} \to T^*\boldsymbol{E}$; accordingly, we shall write $\mathcal{H}[o] \equiv \mathcal{H}_0\, u^0 \simeq \mathcal{H}_0\, d^0$.

Moreover, we define the *observed momentum* associated with the Lagrangian $\mathcal{L}$ and the observer o to be the fibred morphism $\mathcal{P}[o] := \nu^*[o] \circ \check{\mathcal{P}} : J\boldsymbol{E} \to T^*\boldsymbol{E}$, whose coordinate expression in adapted coordinates is $\mathcal{P}[o] = (G^0_{ij} x^j_0 + A_i)\, d^i$.

Hence, we obtain the following expressions $\check{\mathcal{P}} = \check{\mathcal{Q}}[o] + \check{A}[o]$, $\mathcal{L} = \mathcal{K}[o] + д \lrcorner A[o]$, $\mathcal{L} = -\mathcal{H}[o] + д \lrcorner \mathcal{P}[o]$, $\Theta = -\mathcal{H}[o] + \mathcal{P}[o]$. Hence, we have the coordinate expressions $\Theta = -\mathcal{H}_0\, d^0 + \mathcal{P}_i\, d^i$, $\mathcal{L}_0 = (-\mathcal{H}_0 + \mathcal{P}_i\, x^i_0)\, d^0$, where $\mathcal{H}_0$ and $\mathcal{P}_i$ are the components of the Hamiltonian $\mathcal{H}[o]$ and of the observed momentum $\mathcal{P}[o]$ associated with the observer o attached to the chosen chart.

We stress that our definitions of a Poincaré–Cartan form, a Lagrangian and a momentum (defined locally up to a gauge) do not involve observers. On the other hand, each Poincaré–Cartan form, each Lagrangian and each momentum can be expressed as a combination of two terms related to an observer.

We recall that all formulas above incorporate the Planck constant because $\hbar$ has been used in the definition of the normalised metric G, hence in the definition of Ω.

2.2. Splitting of vector fields and forms. We start by analysing the splitting of vector fields and 1–forms induced by the time form dt and the second order connection γ.

Thus, let us consider the time form $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$ and the second order connection $\gamma : J\boldsymbol{E} \to \mathbb{T}^* \otimes TJ\boldsymbol{E}$. We have the identity $\langle dt, \gamma \rangle = 1$.

Then, we obtain the splittings

$$TJ\boldsymbol{E} = T_\gamma J\boldsymbol{E} \oplus T_{dt} J\boldsymbol{E} \equiv T_\gamma J\boldsymbol{E} \oplus VJ\boldsymbol{E}\,,$$
$$T^*J\boldsymbol{E} = T^*_{dt} J\boldsymbol{E} \oplus T^*_\gamma J\boldsymbol{E} \equiv HJ\boldsymbol{E} \oplus T^*_\gamma J\boldsymbol{E}\,,$$

where

- $T_\gamma J\boldsymbol{E} := \gamma(T\boldsymbol{E}) \subset TJ\boldsymbol{E}$ is the *the horizontal subbundle* with respect to the connection γ, i.e. the vector subbundle generated by γ through $\mathcal{M}(J\boldsymbol{E}, \mathbb{T} \otimes I\!\!R)$,
- $T^*_{dt} J\boldsymbol{E} \equiv HJ\boldsymbol{E} := t^{1*}_0(T^*\boldsymbol{T}) \subset T^*J\boldsymbol{E}$ is the *horizontal subbundle* with respect to the fibring t, i.e. the vector subbundle generated by dt through $\mathcal{M}(J\boldsymbol{E}, \mathbb{T}^* \otimes I\!\!R)$,
- $T_{dt} J\boldsymbol{E} \equiv VJ\boldsymbol{E} \subset TJ\boldsymbol{E}$ is the *vertical subbundle* with respect to the fibring t, i.e. the vector subbundle which vanishes on dt,
- $T^*_\gamma J\boldsymbol{E} := (\gamma(T\boldsymbol{E}))_\perp \subset T^*J\boldsymbol{E}$ is the *the orthogonal subbundle* with respect to the connection γ, i.e. the vector subbundle which vanishes on γ.

The corresponding projection maps are

$$\pi^{||} : TJ\boldsymbol{E} \to T_\gamma J\boldsymbol{E} : X \mapsto \langle i(X)dt, \gamma \rangle\,, \qquad \pi^\perp : TJ\boldsymbol{E} \to VJ\boldsymbol{E}\,,$$
$$\pi_{||} : T^*J\boldsymbol{E} \to HJ\boldsymbol{E} : \alpha \mapsto \langle i(\gamma)\alpha, dt \rangle\,, \qquad \pi_\perp : T^*J\boldsymbol{E} \to T^*_\gamma J\boldsymbol{E}\,.$$

Moreover, we have natural isomorphisms $(T_\gamma J\boldsymbol{E})^* \simeq HJ\boldsymbol{E}$, $(VJ\boldsymbol{E})^* \simeq T^*_\gamma J\boldsymbol{E}$.

2.3. Musical morphisms. Further, we analyse the morphisms and isomorphisms induced on the tangent and cotangent bundles of spacetime by Ω and Λ. We shall express these morphisms and isomorphisms in terms of their action on vector fields and forms.

We start with the main morphisms.

Proposition 2.1. *The phase 2–form Ω and the phase 2–vector Λ yield the $\mathcal{M}(J\boldsymbol{E}, I\!\!R)$–linear sheaf morphisms*

$$\Omega^\flat : \mathcal{T}(J\boldsymbol{E}) \to \mathcal{T}^*(J\boldsymbol{E}) : X \mapsto i(X)\Omega\,,$$
$$\Lambda^\sharp : \mathcal{T}^*(J\boldsymbol{E}) \to \mathcal{T}(J\boldsymbol{E}) : \alpha \mapsto i(\alpha)\Lambda\,. \qquad \square$$

Definition 2.1. The morphisms $\Omega^\flat$ and $\Lambda^\sharp$ are called *musical morphisms*. $\square$

Proposition 2.2. *We have*

$$\operatorname{im}(\Omega^\flat) = \mathcal{T}^*_\gamma(J\boldsymbol{E})\,, \qquad \operatorname{im}(\Lambda^\sharp) = \mathcal{T}_{dt}(J\boldsymbol{E}) \equiv \mathcal{V}(J\boldsymbol{E})\,,$$
$$\ker(\Omega^\flat) = \mathcal{T}_\gamma(J\boldsymbol{E})\,, \qquad \ker(\Lambda^\sharp) = \mathcal{T}^*_{dt}(J\boldsymbol{E}) \equiv \mathcal{H}(J\boldsymbol{E})\,.$$

Proof. It follows from the identities $i(\gamma)\Omega = 0$, $i(dt)\Lambda = 0$. $\square$

Proposition 2.3. *We obtain* $\Lambda^\sharp \circ \Omega^\flat = \pi^\perp : \mathcal{T}(J\boldsymbol{E}) \to \mathcal{T}_{dt}(J\boldsymbol{E})\,, \quad \Omega^\flat \circ \Lambda^\sharp = \pi_\perp : \mathcal{T}^*(J\boldsymbol{E}) \to \mathcal{T}^*_\gamma(J\boldsymbol{E})\,.$

In other words, for each $X \in \mathcal{T}(J\boldsymbol{E})$, $\alpha \in \mathcal{T}^*(J\boldsymbol{E})$, we have

$$(\Lambda^\sharp \circ \Omega^\flat)(X) = X - \langle i(X)dt, \gamma\rangle\,,$$
$$(\Omega^\flat \circ \Lambda^\sharp)(\alpha) = \alpha - \langle i(\gamma)\alpha, dt\rangle\,,$$

where $\langle\,,\,\rangle : \mathbb{T}^* \otimes \mathbb{T} \to I\!\!R$ denotes the natural contraction. $\square$

Theorem 2.1. *The restrictions of $\Omega^\flat$ and $\Lambda^\sharp$ to $\mathcal{T}_{dt}(J\boldsymbol{E})$ and $\mathcal{T}^*_\gamma(J\boldsymbol{E})$ yield inverse $\mathcal{M}(J\boldsymbol{E}, I\!\!R)$–linear sheaf isomorphisms*

$$\Omega^\flat_0 : \mathcal{V}(J\boldsymbol{E}) \to \mathcal{T}^*_\gamma(J\boldsymbol{E})\,, \qquad \Lambda^\sharp_0 : \mathcal{T}^*_\gamma(J\boldsymbol{E}) \to \mathcal{V}(J\boldsymbol{E})\,.$$

Proof. It follows from the coordinate expression by using (1.1) and (1.2). $\square$

In order to extend the above result, let us consider a given $(\mathbb{T}\otimes I\!\!R)$–valued *time function* $\tau : J\boldsymbol{E} \to \mathbb{T}\otimes I\!\!R$ and the induced vector field $\langle\tau, \gamma\rangle : J\boldsymbol{E} \to T_\gamma J\boldsymbol{E}$, with coordinate expressions $\tau = \tau^0 u_0$ and $\langle\tau,\gamma\rangle = \tau^0(\partial_0 + x^i_0\partial_i + \gamma_0{}^i_0\partial^0_i)$. Then, we introduce the affine subsheaf

$$\mathcal{T}_\tau(J\boldsymbol{E}) := \{X \in \mathcal{T}(J\boldsymbol{E}) \mid \langle dt, X\rangle = \tau\}$$
$$= \{X \in \mathcal{T}(J\boldsymbol{E}) \mid \pi^{||}(X) = \langle\tau,\gamma\rangle\} \subset \mathcal{T}(J\boldsymbol{E})\,.$$

Theorem 2.2. *The restriction to $\mathcal{T}_\tau(J\boldsymbol{E}) \subset \mathcal{T}(J\boldsymbol{E})$ of the $\mathcal{M}(J\boldsymbol{E}, I\!\!R)$–linear morphism $\Omega^\flat : \mathcal{T}(J\boldsymbol{E}) \to \mathcal{T}^*(J\boldsymbol{E})$ is an affine isomorphism $\Omega^\flat_\tau : \mathcal{T}_\tau(J\boldsymbol{E}) \to \mathcal{T}^*_\gamma(J\boldsymbol{E})$, whose inverse isomorphism is*

$$\Lambda^\sharp_\tau : \mathcal{T}^*_\gamma(J\boldsymbol{E}) \to \mathcal{T}_\tau(J\boldsymbol{E}) : X \mapsto \langle\tau,\gamma\rangle + \Lambda^\sharp(X)\,.$$

Proof. It follows from the coordinate expression. $\square$

Example 2.1. A computation in coordinates yields the following formulas

$$\Lambda^\sharp(dt) = 0\,, \qquad \Lambda^\sharp(\Theta) = G^\sharp(\check{\mathcal{P}})\,. \qquad \square$$

2.4. Extended musical morphisms. Next, we extend the above morphisms and isomorphisms to tangent valued forms and forms of higher degree.

Proposition 2.4. *Let $r \geq 1$. There are unique $\mathcal{M}(J\boldsymbol{E}, I\!\!R)$-linear sheaf morphisms*

$$\Omega^\flat : \Lambda^{r-1}\mathcal{T}^*(J\boldsymbol{E}) \otimes \mathcal{T}(J\boldsymbol{E}) \to \Lambda^r\mathcal{T}^*(J\boldsymbol{E})\,,$$
$$\Lambda^\sharp : \Lambda^r\mathcal{T}^*(J\boldsymbol{E}) \to \Lambda^{r-1}\mathcal{T}^*(J\boldsymbol{E})\,,$$

such that, respectively,

$$\Omega^\flat : \alpha_1 \wedge \ldots \wedge \alpha_{r-1} \otimes X \mapsto \alpha_1 \wedge \ldots \wedge \alpha_{r-1} \wedge \Omega^\flat(X)\,,$$
$$\Lambda^\sharp : \alpha_1 \wedge \ldots \wedge \alpha_r \mapsto \sum_{1\leq p\leq r} (-1)^{p-1} \alpha_1 \wedge \ldots \wedge \hat{\alpha}_p \wedge \ldots \wedge \alpha_r \otimes \Lambda^\sharp(\alpha_p)\,.$$

Proof. It follows from the universal property of the exterior product. $\square$

Remark 2.1. We have $\Omega^\flat(\beta) = i(\beta)\Omega$ for any tangent valued form β [22]. $\square$

Proposition 2.5. *Let $\beta, \beta' \in \Lambda^{r-1}\mathcal{T}^*(J\boldsymbol{E}) \otimes \mathcal{T}(J\boldsymbol{E})$, $\alpha, \alpha' \in \Lambda^r\mathcal{T}^*(J\boldsymbol{E})$, $\alpha'' \in \Lambda^s\mathcal{T}^*(J\boldsymbol{E})$, $f \in \mathcal{M}(J\boldsymbol{E}, I\!\!R)$.*

Then, we have

$$\Omega^\flat(\beta + \beta') = \Omega^\flat(\beta) + \Omega^\flat(\beta')\,, \qquad \Omega^\flat(f\beta) = f\Omega^\flat(\beta),$$
$$\Omega^\flat(\alpha'' \wedge \beta) = \alpha'' \wedge \Omega^\flat(\beta)\,,$$

and

$$\Lambda^\sharp(\alpha + \alpha') = \Lambda^\sharp(\alpha) + \Lambda^\sharp(\alpha')\,, \qquad \Lambda^\sharp(f\alpha) = f\Lambda^\sharp(\alpha)\,,$$
$$\Lambda^\sharp(\alpha'' \wedge \alpha) = \Lambda^\sharp(\alpha'') \wedge \alpha + (-1)^s\, \alpha'' \wedge \Lambda^\sharp(\alpha)\,. \quad \square$$

Proposition 2.6. *We have*

$$\mathrm{im}\,(\Omega^\flat) \subset \mathcal{T}^*_\gamma(J\boldsymbol{E}) \wedge \Lambda^{r-1}\mathcal{T}^*(J\boldsymbol{E})\,, \qquad \Lambda^{r-1}\mathcal{T}^*(J\boldsymbol{E}) \otimes \mathcal{T}_\gamma(J\boldsymbol{E}) \subset \ker\,(\Omega^\flat)\,,$$

and

$$\mathrm{im}\,(\Lambda^\sharp) \subset \Lambda^{r-1}\mathcal{T}^*(J\boldsymbol{E}) \otimes \mathcal{T}_{dt}(J\boldsymbol{E})\,, \qquad \Lambda^r\mathcal{T}^*_{dt}(J\boldsymbol{E}) \subset \ker\,(\Lambda^\sharp)\,.$$

Proof. It follows from Proposition 2.2. $\square$

Thus, for each $\alpha \in \Lambda^r\mathcal{T}^*(J\boldsymbol{E})$, $\Lambda^\sharp(\alpha)$ is a $\boldsymbol{T}$–vertical valued $(r-1)$–form.

Example 2.2. A computation in coordinates yields the following formulas

$$\Omega^\flat(\gamma) = 0\,, \qquad \Omega^\flat(\Gamma) = \Omega^\flat(\nu[\Gamma]) = \Omega\,, \qquad \Omega^\flat(\nu[\gamma]) = \Omega^\flat(1_{J\boldsymbol{E}}) = 2\Omega\,,$$

and

$$\Lambda^\sharp(\Omega) = -\nu[\gamma]\,. \qquad \square$$

Proposition 2.7. *For each* $\alpha \in \Lambda^r \mathcal{T}^*(J\boldsymbol{E})$, *we have*

$$(\Omega^\flat \circ \Lambda^\sharp)(\alpha) = (-1)^{r-1} \left(r\alpha - dt \mathbin{\bar{\wedge}} (i(\gamma)\alpha) \right),$$

where $\bar{\wedge}$ *denotes the wedge product along with the contraction* $\langle\, ,\, \rangle : \mathbb{T}^* \otimes \mathbb{T} \to I\!R$.

Proof. It is easy to prove it on decomposable forms. □

2.5. Hamiltonian lift of forms. Eventually, we introduce the Hamiltonian lift of forms.

Definition 2.2. For $r \geq 0$, we define the *Hamiltonian lift* of the form $\alpha \in \Lambda^r \mathcal{T}^*(J\boldsymbol{E})$ to be the $\boldsymbol{T}$–vertical valued form

$$H[\alpha] := \Lambda^\sharp(d\alpha). \qquad \square$$

Proposition 2.8. *If* $\alpha, \alpha' \in \Lambda^r \mathcal{T}^*(J\boldsymbol{E})$, $\beta \in \Lambda^s \mathcal{T}^*(J\boldsymbol{E})$, $k \in I\!R$, $f \in \mathcal{M}(J\boldsymbol{E}, I\!R)$, *then*

$$H[\alpha + \alpha'] = H[\alpha] + H[\alpha'], \qquad H[k\alpha] = kH[\alpha],$$

$$H[f\alpha] = fH[\alpha] + \alpha \otimes H[f] - df \wedge \Lambda^\sharp(\alpha),$$

$$H[\alpha \wedge \beta] = H[\alpha] \wedge \beta + \alpha \wedge H[\beta] - (-1)^r d\alpha \wedge \Lambda^\sharp(\beta) + (-1)^r \Lambda^\sharp(\alpha) \wedge d\beta. \qquad \square$$

2.6. τ-Hamiltonian lift of forms. More generally, in analogy to Theorem 2.2, let us consider a given $(\mathbb{T} \otimes I\!R)$–valued *time* $(r-1)$*–form* $\tau : J\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^{r-1} T^* J\boldsymbol{E}$, and the induced tangent valued $(r-1)$–form $\langle \gamma, \tau \rangle : J\boldsymbol{E} \to \Lambda^{r-1} T^* J\boldsymbol{E} \otimes_{J\boldsymbol{E}} TJ\boldsymbol{E}$.

Then, we obtain the affine sheaf morphism

$$\Lambda^\sharp_\tau : \Lambda^r \mathcal{T}^*(J\boldsymbol{E}) \to \Lambda^{r-1} \mathcal{T}^*(J\boldsymbol{E}) \otimes \mathcal{T}(J\boldsymbol{E}) : \alpha \mapsto \langle \gamma, \tau \rangle + \Lambda^\sharp(\alpha).$$

Definition 2.3. Let $\tau \in \mathbb{T} \otimes \Lambda^r \mathcal{T}^*(J\boldsymbol{E})$ be a time r–form. Then, we define the τ*–Hamiltonian lift* of the form $\alpha \in \Lambda^r \mathcal{T}^*(J\boldsymbol{E})$ to be the tangent valued form

$$H_\tau[\alpha] := \Lambda^\sharp_\tau(d\alpha) = \langle \gamma, \tau \rangle + H[\alpha] = \langle \gamma, \tau \rangle + \Lambda^\sharp(d\alpha). \qquad \square$$

In particular, we obtain $H_0[\alpha] = H[\alpha]$.

3. Quantisable forms

This section is devoted to the projectable Hamiltonian lift of forms and the induced graded Lie algebra of quantisable forms.

3.1. Projectable Hamiltonian lift of forms. Now, we specialise the results of the previous section, by considering projectable Hamiltonian lift of forms.

Let us start by recalling the notion of projectability for tangent valued forms.

A $T\boldsymbol{JE}$–valued r–form $\phi : \boldsymbol{JE} \to \Lambda^r T^* \boldsymbol{JE} \otimes_{\boldsymbol{JE}} T\boldsymbol{JE}$ is said to be *projectable* on $\boldsymbol{E}$ if there exists a $T\boldsymbol{E}$–valued r–form $\underline{\phi} : \boldsymbol{E} \to \Lambda^r T^* \boldsymbol{E} \otimes_{\boldsymbol{E}} T\boldsymbol{E}$ which makes the following diagram commutative

$$\begin{array}{ccc} \Lambda^r T\boldsymbol{JE} & \xrightarrow{\phi} & T\boldsymbol{JE} \\ {\scriptstyle \Lambda^r T t^1_0}\big\downarrow & & \big\downarrow{\scriptstyle T t^1_0} \\ \Lambda^r T\boldsymbol{E} & \xrightarrow{\underline{\phi}} & T\boldsymbol{E} \end{array}$$

Next, let us consider an r–form $\alpha \in \Lambda^r \mathcal{T}^*(\boldsymbol{JE})$.

The Hamiltonian lifts are not yet sufficient for our purposes (in view of developments for quantum theory) because they live on $\boldsymbol{JE}$, while we are looking for tangent valued forms on $\boldsymbol{E}$. Moreover, the construction of these tangent valued forms depends on an arbitrary time form τ, while we would like to get a natural construction. Both problems can be solved by means of a projectability method.

Lemma 3.1. *The tangent valued $(r-1)$-form $\Lambda^\sharp(\alpha)$ is projectable on $\boldsymbol{E}$ if and only if in local charts we have ${\alpha_{\lambda_1\dots\lambda_{r-1}}}^0_i \in \mathcal{M}(\boldsymbol{E}, I\!R)$.*

Proof. The tangent valued $(r-1)$-form $\Lambda^\sharp(\alpha)$ is projectable on $\boldsymbol{E}$ if and only if in local coordinates ${(\Lambda^\sharp(\alpha))_{\lambda_1\dots\lambda_{r-1}}}^\mu \in \mathcal{M}(\boldsymbol{E}, I\!R)$. But from (1.2) we have ${(\Lambda^\sharp(\alpha))_{\lambda_1\dots\lambda_{r-1}}}^0 = 0$ and ${(\Lambda^\sharp(\alpha))_{\lambda_1\dots\lambda_{r-1}}}^i = (-1)^r G^{ij}_0 {\alpha_{\lambda_1\dots\lambda_{r-1}}}^0_j$. □

Lemma 3.2. *Let $\tau \in \mathbb{T} \otimes \Lambda^{r-1}\mathcal{T}^*(\boldsymbol{JE})$ be a time form. Then, the tangent valued form $\Lambda^\sharp_\tau(\alpha)$ is projectable on $\boldsymbol{E}$ if and only if in local charts we have*

$${\alpha_{\lambda_1\dots\lambda_{r-1}}}^0_i = (-1)^r \left({\tau^0}_{\lambda_1\dots\lambda_{r-1}} G^0_{ij}\, x^j_0 + \alpha^0_{i|\lambda_1\dots\lambda_{r-1}} \right),$$

where ${\tau^0}_{\lambda_1\dots\lambda_{r-1}}, \alpha^0_{i|\lambda_1\dots\lambda_{r-1}} \in \mathcal{M}(\boldsymbol{E}, I\!R)$.

Proof. In local coordinates we have ${(\Lambda^\sharp_\tau(\alpha))_{\lambda_1\dots\lambda_{r-1}}}^0 = {\tau^0}_{\lambda_1\dots\lambda_{r-1}}$ which implies ${\tau^0}_{\lambda_1\dots\lambda_{r-1}} \in \mathcal{M}(\boldsymbol{E}, I\!R)$, and ${(\Lambda^\sharp_\tau(\alpha))_{\lambda_1\dots\lambda_{r-1}}}^i = x^i_0 {\tau^0}_{\lambda_1\dots\lambda_{r-1}} + (-1)^r G^{ij}_0 {\alpha_{\lambda_1\dots\lambda_{r-1}}}^0_j$, hence $\partial^0_i {\alpha_{\lambda_1\dots\lambda_{r-1}}}^0_j = (-1)^r G^0_{ij} {\tau^0}_{\lambda_1\dots\lambda_{r-1}}$; eventually, integrating this formula on the fibres of the affine bundle $t^1_0 : \boldsymbol{JE} \to \boldsymbol{E}$ we get the lemma. □

Lemma 3.3. *Let $\tau \in \mathbb{T} \otimes \Lambda^{r-1}\mathcal{T}^*(\boldsymbol{JE})$ be a time form. Then, the tangent valued form $H_\tau[\alpha]$ is projectable on $\boldsymbol{E}$ if and only if in local charts we have*

$$\partial_{\lambda_1} {\alpha_{\lambda_2\dots\lambda_r}}^0_i + (-1)^r \partial^0_i \alpha_{\lambda_1\dots\lambda_r} = (-1)^r \left({\tau^0}_{\lambda_1\dots\lambda_r} G^0_{ij}\, x^j_0 + \alpha^0_{i|\lambda_1\dots\lambda_r} \right),$$

where ${\tau^0}_{\lambda_1\dots\lambda_r}$, $\alpha^0_{i|\lambda_1\dots\lambda_r} \in \mathcal{M}(\boldsymbol{E}, I\!R)$.

Proof. It is a consequence of Lemma 3.2 and

$$(d\alpha)_{\lambda_1\dots\lambda_r}{}^0_i = \partial_{\lambda_1}\alpha_{\lambda_2\dots\lambda_r}{}^0_i + (-1)^r \partial^0_i \alpha_{\lambda_1\dots\lambda_r} . \qquad \square$$

In the following we shall consider only $\boldsymbol{E}$*–horizontal forms*, i.e. forms of the type $\alpha : J\boldsymbol{E} \to \Lambda^r T^*\boldsymbol{E}$, $\tau : J\boldsymbol{E} \to \mathbb{T}\otimes\Lambda^r T^*\boldsymbol{E}$.

Theorem 3.1. *The τ–Hamiltonian lift $H_\tau[\alpha]$ of the $\boldsymbol{E}$–horizontal r–form α is projectable on $\boldsymbol{E}$ if and only if α is, with respect to the fibres of $J\boldsymbol{E} \to \boldsymbol{E}$, a polynomial of degree 2, whose second fibre derivative is of the type $D^2\alpha = \alpha''\otimes G : \boldsymbol{E} \to \mathbb{T}^2\otimes(V^*\boldsymbol{E}\otimes_{\boldsymbol{E}} V^*\boldsymbol{E}\otimes_{\boldsymbol{E}}\Lambda^r T^*\boldsymbol{E})$, with $\alpha'' \in \mathbb{T}\otimes\Lambda^r\mathcal{T}^*(\boldsymbol{E})$, and $\tau = \alpha''$.*

Proof. It follows by integrating the formula of Lemma 3.3 on the fibres of the affine bundle $t^1_0 : J\boldsymbol{E} \to \boldsymbol{E}$. $\square$

Definition 3.1. Forms α of the above type are said to be *quantisable*. $\square$

3.2. Quantisable forms. Now, we analyse the structure of quantisable forms.

We can express the condition for an $\boldsymbol{E}$–horizontal form to be quantisable in coordinates or in terms of an observer.

Proposition 3.1. *An $\boldsymbol{E}$–horizontal r–form α is quantisable if and only if its coordinate expression is of the type*

$$\alpha = \Big(\frac{1}{2}\alpha^0{}_{\lambda_1\dots\lambda_r} G^0_{ij} x^i_0 x^j_0 + \alpha^0_{i|\lambda_1\dots\lambda_r} x^i_0 + \alpha_{o|\lambda_1\dots\lambda_r}\Big) d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r} ,$$

with $\alpha^0{}_{\lambda_1\dots\lambda_r}, \alpha^0_{i|\lambda_1\dots\lambda_r}, \alpha_{o|\lambda_1\dots\lambda_r} \in \mathcal{M}(\boldsymbol{E}, I\!R)$.

Actually, for a quantisable form α, we obtain $\alpha'' = \alpha^0{}_{\lambda_1\dots\lambda_r} u_0\otimes d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}$.

Moreover, let us consider an observer o and an adapted chart. An $\boldsymbol{E}$–horizontal r–form α is quantisable if and only if it is of the type $\alpha = \alpha''\otimes\mathcal{K}[o] + \langle\alpha'[o], \nabla[o]\rangle + \alpha[o]$, where

$$\alpha'' = \langle D^2\alpha, \bar{G}\rangle = \alpha^0{}_{\lambda_1\dots\lambda_r} u_0\otimes d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r} : \boldsymbol{E} \to \mathbb{T}\otimes\Lambda^r T^*\boldsymbol{E} ,$$

$$\alpha'[o] = D\alpha\circ o = \alpha^0_{i|\lambda_1\dots\lambda_r} u_0\otimes\check{d}^i\otimes d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r} : \boldsymbol{E} \to \mathbb{T}\otimes(\Lambda^r T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E})$$

$$\alpha[o] = \alpha\circ o = \alpha_{o|\lambda_1\dots\lambda_r} d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r} : \boldsymbol{E} \to \Lambda^r T^*\boldsymbol{E} . \qquad \square$$

Remark 3.1. We stress that in the coordinate expression of a quantisable form α there is no relation between the coefficients $\alpha^0{}_{\lambda_1\dots\lambda_r}$ and $\alpha_{o|\lambda_1\dots\lambda_r}$.

Moreover, the form α'' does not depend on the choice of the chart, while the forms $\alpha'[o], \alpha_0[o]$ depend on the chart through the associated observer o. $\square$

We set $\alpha^i{}_{\lambda_1\dots\lambda_r} := G^{ij}_0 \alpha^0_{j|\lambda_1\dots\lambda_r}$.

Remark 3.2. Each quantisable r–form can be locally written as a sum of terms of the type

$$\alpha = f\, d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r} ,$$

where f is a quantisable function. $\square$

3.3. Tangent lift of quantisable forms. Next, given a quantisable form, we can compute its tangent lift with respect to the related time form.

Thus, if α is a quantisable r–form, then we obtain its Hamiltonian lift with respect to the time-form α''

$$X^{\uparrow}[\alpha] := H_{\alpha''}[\alpha] : \boldsymbol{JE} \to \Lambda^r T^* \boldsymbol{JE} \underset{\boldsymbol{JE}}{\otimes} T\boldsymbol{JE}\,.$$

Proposition 3.2. *If α is a quantisable r–form, then we have the following coordinate expression*

$$\begin{aligned} X^{\uparrow}[\alpha] = {} & X_{\lambda_1\dots\lambda_r}{}^0\, d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes\partial_0 + X_{\lambda_1\dots\lambda_r}{}^i\, d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes\partial_i + \\ & + X_{\lambda_1\dots\lambda_r}{}^{i}_{0}\, d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes\partial_i^0 + X_{\lambda_1\dots\lambda_{r-1}}{}^{0\,i}_{h\,0}\, d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r-1}}\wedge d_0^h\otimes\partial_i^0, \end{aligned}$$

where

$$\begin{aligned} X_{\lambda_1\dots\lambda_r}{}^0 &= \alpha^0{}_{\lambda_1\dots\lambda_r} \\ X_{\lambda_1\dots\lambda_r}{}^i &= -G_0^{ij}\alpha^0_{j|\lambda_1\dots\lambda_r} \\ X_{\lambda_1\dots\lambda_r}{}^{i}_{0} &= \big(\alpha^0{}_{\lambda_1\dots\lambda_r}(\Gamma^i_{0j}x^j_0 - \Gamma_0{}^i_0) + (\Gamma^{i\,j}_{0\,0} - \Gamma^{j\,i}_{0\,0})\alpha^0_{j|\lambda_1\dots\lambda_r} \\ &\qquad + G_0^{ij}(\partial_j\alpha_{\lambda_1\dots\lambda_r} - r\partial_{\lambda_1}\alpha_{j\lambda_2\dots\lambda_r})\big) \\ X_{\lambda_1\dots\lambda_{r-1}}{}^{0\,i}_{h\,0} &= r(-1)^r G_0^{ij}(\alpha^0{}_{j\lambda_1\dots\lambda_{r-1}}G^0_{hk}x^k_0 + \alpha^0_{h|j\lambda_1\dots\lambda_{r-1}})\,, \end{aligned}$$

with $\Gamma^i_{0j} \equiv G_0^{im}\Gamma_m{}^k_0 G^0_{kj}$ and $\Gamma^{i\,j}_{0\,0} \equiv G_0^{im}\Gamma_m{}^j_0$. □

In particular, we have the following result.

Corollary 3.1. *If f is a quantisable function, then we have the following coordinate expression*

$$\begin{aligned} X^{\uparrow}[f] = H_{f''}[f] &= f^0\,\partial_0 - f^i\,\partial_i \\ &\quad + \big(f^0(\Gamma^i_{0j}x^j_0 - \Gamma_0{}^i_0) + (\Gamma^{i\,j}_{0\,0} - \Gamma^{j\,i}_{0\,0})f^0_j + G_0^{ij}\partial_j f\big)\,\partial_i^0\,. \end{aligned}$$ □

Corollary 3.2. *If α is a quantisable r–form, then we obtain*

$$\begin{aligned} X^{\uparrow}[\alpha] = {} & d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes X^{\uparrow}[\alpha_{\lambda_1\dots\lambda_r}] \\ & - rG_0^{ij}\partial_{\lambda_1}\alpha_{j\lambda_2\dots\lambda_r}d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes\partial_i^0 \\ & + r(-1)^r G_0^{ij}(\alpha^0{}_{j\lambda_1\dots\lambda_{r-1}}G^0_{hk}x^k_0 + \alpha^0_{h|j\lambda_1\dots\lambda_{r-1}})d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r-1}}\wedge d_0^h\otimes\partial_i^0\,. \end{aligned}$$

Definition 3.2. If α is a quantisable r–form, then the projection

$$X[\alpha] := \underline{X^{\uparrow}[\alpha]} : \boldsymbol{E} \to \Lambda^r T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T\boldsymbol{E}$$

of $X^{\uparrow}[\alpha]$ on $\boldsymbol{E}$ is said to be the *tangent lift* of α. □

Theorem 3.2. *If α is a quantisable r–form, then we have the following coordinate expression*

$$X[\alpha] = d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes(\alpha^0{}_{\lambda_1\dots\lambda_r}\,\partial_0 - \alpha^i{}_{\lambda_1\dots\lambda_r}\,\partial_i)\,.$$ □

Corollary 3.3. *If α is a quantisable r–form, then we obtain locally*

$$X[\alpha] = d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes X[\alpha_{\lambda_1 \ldots \lambda_r}] . \qquad \square$$

Example 3.1. Each Poincaré–Cartan form Θ is quantisable. Moreover, we have $\Theta'' = -dt : \boldsymbol{E} \to \mathbb{T} \otimes T^* \boldsymbol{E}$.

Furthermore, we obtain

$$X[\Theta] = -1_{\boldsymbol{E}} = -(d^0 \otimes \partial_0 + d^i \otimes \partial_i) : \boldsymbol{E} \to T^* \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T\boldsymbol{E} .$$

Thus, $X[\Theta]$ does not depend on the gauge of the potential Θ of Ω, as d kills this gauge (of course) and Θ'' does not depend on the gauge. $\square$

3.4. The graded Lie algebra of quantisable forms. Next, we exhibit a natural graded Lie bracket for quantisable forms.

We denote the subsheaf of quantisable forms by

$$\mathcal{Q}(J\boldsymbol{E}) \equiv \underset{0 \leq r \leq 7}{\oplus} \mathcal{Q}^r(J\boldsymbol{E}) \subset \underset{0 \leq r \leq 7}{\oplus} \Lambda^r \mathcal{T}^*(\boldsymbol{E}) .$$

Proposition 3.3. *Let α be a quantisable form and $\phi \in \Lambda \mathcal{T}^*(\boldsymbol{E})$. Then $\phi \wedge \alpha$ is a quantisable form and*

$$(\phi \wedge \alpha)'' = \phi \wedge \alpha'' .$$

So, $\mathcal{Q}(J\boldsymbol{E})$ is a graded module over the graded algebra $\Lambda \mathcal{T}^*(\boldsymbol{E})$.

Moreover

$$X[\phi \wedge \alpha] = \phi \wedge X[\alpha] . \qquad \square$$

Proposition 3.4. *If $\alpha \in \Lambda^r \mathcal{T}^*(\boldsymbol{E})$ and $f \in \mathcal{Q}(J\boldsymbol{E})$, then*

$$X^{\uparrow}[f\alpha] = \alpha \otimes X^{\uparrow}[f] + fH[\alpha] - df \wedge \Lambda^{\sharp}(\alpha) .$$

Proof. It follows from Proposition 2.8. $\square$

Lemma 3.4. *Let $\alpha \in \mathcal{Q}^r(J\boldsymbol{E})$ and $\beta \in \mathcal{Q}^s(J\boldsymbol{E})$, with $r, s \geq 0$. Then, there is a unique (up to a multiplicative factor) (local) $\boldsymbol{E}$–horizontal form $[\alpha, \beta] : J\boldsymbol{E} \to \Lambda^{r+s} T^* \boldsymbol{E}$, which is a linear combination of the forms*

$$i(H[\alpha])d\beta, di(H[\alpha])\beta, \; i(H[\beta])d\alpha, di(H[\beta])\alpha,$$
$$i(\gamma(\alpha''))d\beta, i(\gamma(\beta''))d\alpha, \; di(\gamma(\alpha''))\beta, di(\gamma(\beta''))\alpha .$$

Namely, we have

$$[\alpha, \beta] = L(H_{\alpha''}[\alpha])\beta - (-1)^{sr} L(H_{\beta''}[\beta])\alpha - i(H[\alpha])d\beta . \tag{3.8}$$

Proof. It follows from a very long computation in coordinates. $\square$

Lemma 3.5. *Let $\alpha \in \mathcal{Q}^r(J\boldsymbol{E})$ and $\beta \in \mathcal{Q}^s(J\boldsymbol{E})$. Then we have*

$$[\alpha, \beta] = -(-1)^{rs} [\beta, \alpha] .$$

Proof. It follows from Lemma 3.4 and the identity

$$i(H[\alpha])d\beta = -(-1)^{rs}\, i(H[\beta])d\alpha\,. \qquad \square$$

We can apply the bracket (3.8) also to the case $r = s = 0$.

Lemma 3.6. *Let $f, g \in \mathcal{Q}^0(J\boldsymbol{E})$. Then, the bracket (3.8) becomes*

$$[f, g] = H[f].g + \gamma(f'').g - \gamma(g'').f\,. \qquad \square$$

Theorem 3.3. *Let us consider two quantisable functions f and g. Then, the function $[f, g] = H[f].g + \gamma(f'').g - \gamma(g'').f$ turns out to be quantisable.*

Moreover, we obtain the coordinate expression

$$\begin{aligned}
&[f,g]^0 = f^0\partial_0 g^0 - g^0\partial_0 f^0 - (f^h\partial_h g^0 - g^h\partial_h f^0)\\
&[f,g]^0_i = G^0_{ij}\big(f^0\partial_0 g^j - g^0\partial_0 f^j - (f^h\partial_h g^j - g^h\partial_h f^j)\big)\\
&[f,g]_o = f^0\partial_0 g_o - g^0\partial_0 f_o - (f^h\partial_h g_o - g^h\partial_h f_o) - (f^0 g^j - g^0 f^j)\Phi_{0j} + f^i g^j\Phi_{ij}\,.
\end{aligned}$$

Proof. The computation follows in coordinates. $\square$

Theorem 3.4. *The sheaf $\mathcal{Q}^0(J\boldsymbol{E})$ of quantisable functions turns out to be a Lie algebra with respect to the above bracket.*

Proof. It is sufficient to prove the Jacobi identity, which can be proved in coordinates by using the coordinate expression of Theorem 3.3. $\square$

Lemma 3.7. *The map $f \mapsto X[f]$ is a homomorphism of Lie algebras between the Lie algebra of quantisable functions (with respect to the above Lie bracket) and the Lie algebra of vector fields of $\boldsymbol{E}$ (with respect to the standard Lie bracket).*

Proof. The proof follows by a computation in coordinates by using Theorem 3.2 and 3.3. $\square$

Now we can prove the main theorems. First we have the following result.

Lemma 3.8. *Let us consider two forms $\alpha \in \Lambda^r\mathcal{T}^*(\boldsymbol{E})$ and $\beta \in \Lambda^s\mathcal{T}^*(\boldsymbol{E})$ and two quantisable functions f and g. Then, we have*

$$\begin{aligned}
[f\,\alpha,\, g\,\beta] = {} & [f,\, g]\,\alpha\wedge\beta\\
& + g\,\alpha\wedge L(X[f])\beta - (-1)^{sr} f\,\beta\wedge L(X[g])\alpha\\
& + (-1)^r g\, d\alpha\wedge i(X[f])\beta + (-1)^r f\, i(X[g])\alpha\wedge d\beta\,.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
L(X^{\uparrow}[f\,\alpha])(g\,\beta) = {} & (L(X^{\uparrow}[f])g)\alpha\wedge\beta + g\,\alpha\wedge L(X[f])\beta\\
& - df\wedge L(\Lambda^{\sharp}(\alpha))g\wedge\beta + (-1)^r g\, d\alpha\wedge i(X[f])\beta\\
& + f(L(H[\alpha])g)\wedge\beta\,;
\end{aligned}$$

$$\begin{aligned} L(X^{\uparrow}[g\,\beta])(f\,\alpha) &= (L(X^{\uparrow}[g])f)\beta \wedge \alpha + f\,\beta \wedge L(X[g])\alpha \\ &\quad - dg \wedge L(\Lambda^{\sharp}(\beta))f \wedge \alpha + (-1)^s f\, d\beta \wedge i(X[g])\alpha \\ &\quad + g(L(H[\beta])f) \wedge \alpha\,; \\ i(H[f\,\alpha])d(g\,\beta) &= (i(H[f])dg)\,\alpha \wedge \beta - \alpha \wedge dg \wedge i(H[f])\beta \\ &\quad - df \wedge i(\Lambda^{\sharp}(\alpha))dg \wedge \beta + f i(H[\alpha])dg \wedge \beta \\ &\quad + g\alpha \wedge i(H[f])d\beta\,. \end{aligned}$$

Lemma 3.4 now implies Lemma 3.7. □

Theorem 3.5. *Let $\alpha \in \mathcal{Q}^r(J\boldsymbol{E})$ and $\beta \in \mathcal{Q}^s(J\boldsymbol{E})$, with $r, s \geq 0$. Then, the $(r+s)$-form $[\alpha, \beta]$ turns out to be quantisable. Moreover, we obtain*

$$[\alpha, \beta]'' = Tt \circ [X[\alpha], X[\beta]]\,.$$ □

Theorem 3.6. *The sheaf $\mathcal{Q}(J\boldsymbol{E})$ turns out to be a sheaf of graded Lie algebras with respect to the above bracket.*

Proof. We have to prove the graded Jacobi identity. It suffices to proof it for forms of the type $f\,\alpha$, $g\,\beta$, $h\,\gamma$ where f, g and h are quantisable functions and $\alpha \in \Lambda^r\mathcal{T}^*(\boldsymbol{E})$, $\beta \in \Lambda^s\mathcal{T}^*(\boldsymbol{E})$, $\gamma \in \Lambda^t\mathcal{T}^*(\boldsymbol{E})$. Then, from Theorem 3.6, after a long computation we get

$$\begin{aligned} &[f\,\alpha, [g\,\beta, h\,\gamma]] + (-1)^{t(r+s)}[h\,\gamma, [f\,\alpha, g\,\beta]] - (-1)^{rs}[g\,\beta, [f\,\alpha, h\,\gamma]] \\ &\qquad = \big([f, [g, h]] + [h, [f, g]] - [g, [f, h]]\big)\,\alpha \wedge \beta \wedge \gamma \end{aligned}$$

and the Jacobi identity for quantisable functions implies the Jacobi identity for quantisable forms. □

Eventually, the tangent lift yields an $I\!R$–linear sheaf morphism between the graded Lie algebra of quantisable forms and the Frölicher–Nijenhuis algebra of tangent valued forms on spacetime.

Theorem 3.7. *The map $\alpha \mapsto X[\alpha]$ is a homomorphism of graded Lie algebras between the graded Lie algebra of quantisable forms (with respect to the above graded Lie bracket) and the graded Lie algebra of tangent valued forms of $\boldsymbol{E}$ (with respect to the Frölicher–Nijenhuis bracket).*

Proof. It suffices to proof the theorem for forms of the type $f\,\alpha$, $g\,\beta$, where f and g are quantisable functions and $\alpha \in \Lambda^r\mathcal{T}^*(\boldsymbol{E})$, $\beta \in \Lambda^s\mathcal{T}^*(\boldsymbol{E})$.

The properties of the Frölicher–Nijenhuis bracket yield

$$\begin{aligned} \big[X[f\,\alpha], X[g\,\beta]\big] &= \big[\alpha \otimes X[f],\ \beta \otimes X[g]\big] = \alpha \wedge \beta \otimes \big[X[f], X[g]\big] + \\ &\quad + \alpha \wedge L(X[f])\beta \otimes X[g] - L(X[g])\alpha \wedge \beta \otimes X[f] \\ &\quad + (-1)^r d\alpha \wedge i(X[f])\beta \otimes X[g] + (-1)^r i((X[g])\alpha \wedge d\beta \otimes X[f]\,. \end{aligned}$$

Then, Proposition 3.3, Lemma 3.7 and Lemma 3.8 imply the theorem. □

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