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SECTIONAL CURVATURE AND EINSTEIN'S EQUATIONS

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Abstract. Some results regarding the sectional curvature function and its relation to the metric tensor for a space-time are given. They are then used to suggest an alternative viewpoint for Einstein's general theory of relativity.

1. Introduction

Let M be a smooth manifold of dimension $n \geq 2$ admitting a smooth metric g of arbitrary signature and, if $p \in M$, let T_pM denote the tangent space to M at p and let G_p denote the $2(n-2)$-dimensional Grassman manifold of all 2-dimensional subspaces (2-spaces) of T_pM. Consider the real valued function σ_p defined on a certain subset $\bar{G}_p$ of G_p and given, for $F \in \bar{G}_p$, by

$$\sigma_p(F) = \frac{g(R(X,Y)X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} \tag{1}$$

where R is the curvature structure arising from the Levi-Civita connection of g and where F is spanned by $X, Y \in T_pM$. It is easily checked that $\sigma_p(F)$ does not depend on the spanning set X and Y chosen and $\bar{G}_p$ is then taken as the subset of members of G_p for which the denominator in (1) does not vanish (this non-vanishing also being independent of the spanning set chosen for F). The function σ_p is called the *sectional curvature function at* p and $\sigma_p(F)$ the *sectional curvature of* F. If $G = \bigcup_{p \in M} \bar{G}_p$ the function $\sigma : G \to \mathbf{R}$ defined for $F \in G$ by $\sigma(F) = \sigma_p(F)$ if $F \in \bar{G}_p$ is the *sectional curvature function* (of M).

If g is positive definite, $\bar{G}_p = G_p$ and $\sigma_p : G_p \to \mathbf{R}$ is a bounded real valued function on the compact connected Hausdorff manifold G_p.

Now let M be a space-time so that M is a smooth, connected, Hausdorff, 4-dimensional manifold and g is smooth with Lorentz signature (taken as $(+++-)$). For $p \in M$, $\bar{G}_p$ is now the family of non-null 2-spaces at p and is neither compact nor connected. This complication leads to a richer and more interesting structure for σ. Let S_p, T_p and N_p be, respectively, the subsets of spacelike, timelike and null members of G_p defined, respectively, by the conditions that the denominator in (1) is positive, negative or zero. Thus $G_p = S_p \cup T_p \cup N_p$ and $\bar{G}_p = S_p \cup T_p$. Now G_p is 4-dimensional and S_p and T_p are diffeomorphic open (4-dimensional) submanifolds

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of G_p whilst N_p is a 3-dimensional submanifold of G_p which is the boundary of S_p and T_p [1].

2. Space-Time Sectional Curvature

Let M be a space-time manifold with Lorentz metric g and sectional curvature function σ. If $p \in M$, σ_p is defined on $\bar{G}_p$ and is a constant function if and only if R takes the familiar "constant curvature" form at p. Under these conditions σ_p can, of course, be continuously extended to the *whole* of G_p. In [2] it was shown that if σ_p can be extended to the *whole* of G_p then R took the constant curvature form at p and σ_p is a constant function on G_p. However, a much stronger result is available [3]-[5] which says that if σ_p is continuously extendible to a *single* member of N_p then it is a constant function on $\bar{G}_p$ and hence extendible to G_p. Now suppose that the sectional curvature function σ is given on M and one asks how many Lorentz metrics on M could give rise to this sectional curvature. If one imposes the restriction that the given σ is such that for no $p \in M$ is σ_p a constant function, one sees from the above discussion that if g and g' each have σ as their sectional curvature function then g and g' have the same subset of G_p (= domain σ_p) as their set of null 2-spaces for each $p \in M$. As a consequence of this g and g' have the same null cones at each $p \in M$ and hence are conformally related on M so that $g' = \phi g$ for some smooth function $\phi : M \to \mathbf{R}$. But more can be said. If one applies further algebraic analysis and imposes the differential constraints of the Bianchi identities one finds that, apart from some very special cases, $\phi \equiv 1$ on M and so σ uniquely determines the metric that gave rise to it.

Theorem 1. [5],[6] *Let M be a space-time admitting Lorentz metrics g and g' which give rise to the same sectional curvature function σ on M which for no $p \in M$ is a constant function. Then, generically, $g' = g$.*

Here, the word generic should be clarified. In fact if $\Gamma^\infty(L)$ denotes the topological space of all C^∞ Lorentz metrics on M with Whitney C^∞ topology then *there exists an open dense subset of $\Gamma^\infty(L)$ every member of which is uniquely determined by its sectional curvature function* [7],[4]. The special types of sectional curvature functions to which the conclusions of theorem 1 do not apply are essentially the conformally flat null Einstein-Maxwell (or null fluid) plane waves [5] and need not be considered further here. A very important class of space-times in general relativity are the *non-flat vacuum* space-times (for which the Ricci tensor vanishes everywhere on M and the curvature tensor does not vanish over a non-empty open subset of M). For these space-times a much tighter result follows from theorem 1 without clauses.

Theorem 2. [5] *Let M be a space-time and g a non-flat vacuum metric on M with associated sectional curvature function σ. If g' is any other Lorentz metric for M with the same sectional curvature function σ, then $g' \equiv g$.*

The analogue of theorem 1 for positive definite metrics was given in [8].

3. The Description of Gravitational Fields

In general relativity the gravitational field is described by the metric tensor of space-time. However, theorems 1 and 2 suggest that the sectional curvature function may be an alternative field variable for general relativity, especially theorem 2 in view of the extra strength of this result and the importance of vacuum space-times [9].

Two directions for further research present themselves. Firstly (and perhaps most importantly) it would be interesting to study the geometry of M from the point of view of the sectional curvature function σ (σ-based geometry). Secondly one might try to give an interpretation, in terms of σ, of the standard results and techniques that one already has for the geometry of M as described by the Lorentz metric g (g-based geometry). The first of these will probably be difficult since it is an unusual and unexplored avenue of research. Progress in the second direction may be partly misdirected given the general idea of this paper but at least it is easier and may suggest approaches to the first. In the remainder of this section some results in the second direction will be briefly described.

The Petrov classification of the Weyl tensor [10] and the consequent algebraic (Petrov) types for this tensor at a point of space-time have proved of immense value in both the practical and theoretical study of Einstein's theory. If this classification, or something like it, is to be retained one needs a σ-based description of the Petrov types. Following work for the vacuum case in [2] such a scheme was given for general space-times in terms of the critical point structure of the function σ_p [4].

The energy-momentum tensor in general relativity is taken as representative of the sources of the gravitational field. If one wishes to continue to represent such sources in this way a method which is a mixture of σ-based and g-based theory is to consider a function $\phi_p : \bar{G}_p \to \mathbf{R}$ defined by $\phi_p(F) = \sigma_p(F) - \sigma_p(\overset{*}{F})$ where $F \in \bar{G}_p$ and $\overset{*}{F}$ is its orthogonal complement. Then the algebraic (Segre) type of the energy-momentum tensor can be described in terms of the critical point structure of ϕ_p at each $p \in M$ [3]. A special case of this result is known [11] and says that $\sigma_p(F) = \sigma_p(\overset{*}{F})$ ($\Leftrightarrow \phi_p(F) = 0$) for each $F \in \bar{G}_p$ and each $p \in M$ if and only if the space-time is an Einstein space.

There has been much recent interest in various aspects of symmetries in general relativity (for a recent review see [12]). It is hence interesting to define σ-based symmetry concepts and to relate them to the usual g-based ones. For example let X be a smooth vector field on M with associated local flow ϕ_t for suitable $t \in \mathbf{R}$. Then X will be called *sectional curvature preserving* if for each ϕ_t (and with $p \in M$ and $q = \phi_t(p)$) and each $F \in G_p$ with F spanned by $X, Y \in T_pM$

$$\sigma_p(F) = \sigma_q(\phi_{t*}F) \tag{2}$$

where $\phi_{t*}F$ is the member of G_q spanned by $\phi_{t*}X$ and $\phi_{t*}Y$ and in the sense that either neither side of (2) is defined or both are and equality then holds. It turns out that such vector fields are Killing vector fields (the trivial case) or they are essentially a subalgebra of the Lie algebra of conformal vector fields on conformally flat plane wave space-times [13]. As another example let $p \in M$ and let $\mathcal{L}$ be the Lorentz group regarded as a 6-dimensional Lie group of isomorphisms of T_pM. If H is any Lie subgroup of $\mathcal{L}$ one has an action $H \times T_pM \to T_pM$ and hence an action $\psi : H \times \bar{G}_p \to \bar{G}_p$ in an obvious way. If $F \in \bar{G}_p$ and $h \in H$ let $\psi_h(F) = \psi(h, F)$. Then σ_p will be called *H-invariant* if $\sigma_p(F) = \sigma_p(\psi_h(F))$ for each $h \in H$ and $F \in \bar{G}_p$. Using this idea one can show that $SO(3)$-invariance for σ_p for each $p \in M$ essentially leads to the well known space-time cosmological models of Friedmann, Robertson and Walker [4].

It is remarked, in conclusion, that a convenient description of the material described here can be achieved using quadric surface theory [1],[4].

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