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THE BIHAMILTONIAN STRUCTURE OF THE DRINFEL'D–SOKOLOV HIERARCHIES RELATED TO THE AFFINE TWISTED LIE ALGEBRAS

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ABSTRACT. This paper presents a bihamiltonian construction of the Drinfel'd–Sokolov hierarchies corresponding to the twisted affine Lie algebras. We show, indeed, that these hierarchies can be framed in the general theory of bihamiltonian manifolds, by applying an extended bihamiltonian reduction theorem stated and proved in the paper.

1. Introduction

At least since the seminal paper of Drinfel'd and Sokolov [1] it is a well known fact, that each affine Lie algebra is associated with an integrable hierarchy of soliton equations, usually referred in the literature as hierarchy of KdV type or GD hierarchies [2] [3] [4], or finally (which seems the more appropriate name) Drinfel'd–Sokolov hierarchies [3], [4]. These hierarchies can be divided by their properties in two classes: the hierarchies corresponding to the non twisted affine Lie algebras, and those corresponding to the twisted ones. While the hierarchies of the first class share (up to some technicalities) all the properties usually owned by the integrable systems, the hierarchies corresponding to the twisted case seem to lack some theoretical features.

In particular, it is widely accepted, that the soliton hierarchies corresponding to the case of the twisted affine Lie algebras do not posses a bihamiltonian structure. This statement is, strictly speaking, obviously true, but it is certainly not the end of the story.

In this paper, it will be shown that actually, also these "exceptional" hierarchies can be seen as reduced theories of bihamiltonian theories defined on loop-algebras, by performing an appropriate bihamiltonian reduction process, which extends to this case the bihamiltonian reduction already developed in [5] [6] [7].

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This new reduction process, which contains the usual one presented in [6] and [7] as particular case, is based on a different geometrical construction. As a consequence of this fact one of the two Poisson tensors of the reduced bihamiltonian theory is simply the trivial structure, and not all the vector fields of the bihamiltonian hierarchy can be projected to the quotient space, but only those belonging to particular subclasses. Therefore on the level of the soliton phase space these theories appear to be only Hamiltonian and not bihamiltonian, although, they descend from bihamiltonian theories, from which they inherit all their remarkable properties, like the existence of an infinite number of commuting flows, or an infinite number of conserved quantities.

This paper is organized as follows. In the next section we will present our extended bihamiltonian reduction theorem. In the third one we will apply this theorem to the bihamiltonian manifold associated with the twisted affine Lie algebras. The last section, is devoted to study of the behavior of the twisted bihamiltonian hierarchies under the reduction process.

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2. The Extended bihamiltonian reduction theorem

This section is devoted to state and prove the main result of the paper, an extended bihamiltonian reduction theorem, which allows a bihamiltonian interpretation of the twisted Drinfel'd–Sokolov hierarchies. Although we are mainly interested in these latter hierarchies, we will formulate the theorem in a more general fashion, which contains both reductions (i.e. for the untwisted and the twisted as well) as particular cases.

Let $\mathfrak{g}$ be a simple (finite–dimensional) Lie algebra, and let us denote by $\tilde{\mathfrak{g}}$ the loop–algebra $C^\infty(S^1, \mathfrak{g})$. This algebra is an infinite dimensional flat manifold (see for example [10]), which, therefore can be identified with its tangent space at any point: $\tilde{\mathfrak{g}} \simeq T_U\tilde{\mathfrak{g}}$, $\forall U \in \tilde{\mathfrak{g}}$. Moreover by means of the separating bilinear form:

$$\langle V, U\rangle = \int_{S^1} (V(x), U(x))_{\mathfrak{g}} dx \qquad U, V \in \tilde{\mathfrak{g}}\,, \tag{2.1}$$

induced on $\tilde{\mathfrak{g}}$ by the Killing form $(\cdot,\cdot)_{\mathfrak{g}}$ of $\mathfrak{g}$, the cotangent spaces $T^*_U\tilde{\mathfrak{g}}$, $\forall U \in \tilde{\mathfrak{g}}$ can be also identified with the tangent spaces $T_U\tilde{\mathfrak{g}}$. So we have the following chain of identifications:

$$\tilde{\mathfrak{g}} \simeq T_U\tilde{\mathfrak{g}} \simeq T^*_U\tilde{\mathfrak{g}} \simeq \tilde{\mathfrak{g}}^* \qquad \forall U \in \tilde{\mathfrak{g}}\,, \tag{2.2}$$

which allows us to define on the manifold $\tilde{\mathfrak{g}}$ the two Poisson tensors:

$$(P_0)_U\, V = [A, V] \tag{2.3}$$

$$(P_1)_U\, V = V_x + [V, U] \tag{2.4}$$

where $V \in T^*_U\tilde{\mathfrak{g}} \simeq T_U\tilde{\mathfrak{g}} \simeq \tilde{\mathfrak{g}}, U \in \tilde{\mathfrak{g}}$, $A \in \mathfrak{g}$ is a constant loop, and V_x is the derivative of the loop V along S^1. It is well–known that, for any choice of A, P_0 and P_1 are compatible (i.e. the Poisson pencil $P_0 - \lambda P_1$ is still a Poisson tensor $\forall\lambda \in \mathbb{R}$), so that $\tilde{\mathfrak{g}}$ has several bihamiltonian structures (see, for instance, [5] [11]).

This means that the corresponding Poisson brackets $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$ satisfy for any triple of functions (f, g, h) on M the cyclic compatibility condition [11]:

$$\begin{aligned}\{f,\{g,h\}_0\}_1 + \{h,\{f,g\}_0\}_1 + \{g,\{h,f\}_0\}_1 +& \\ +\{f,\{g,h\}_1\}_0 + \{h,\{f,g\}_1\}_0 + \{g,\{h,f\}_1\}_0 &= 0\,.\end{aligned} \tag{2.5}$$

Both theories of bihamiltonian reductions are based on the Marsden–Ratiu reduction theorem for a Poisson manifold [12]. This theorem asserts that if M is a Poisson manifold with Poisson tensor P and Poisson bracket $\{\cdot,\cdot\}$; S is a submanifold of M; $i_S : S \hookrightarrow M$ is the canonical immersion of S into M; and finally $\mathcal{D}$ is a distribution on M such that:

(1) $\mathcal{E} = \mathcal{D} \cap TS$ is an integrable distribution of S;
(2) the foliation induced by $\mathcal{E}$ on S is regular, so that $N = S/\mathcal{E}$ is a manifold and $\pi : S \to N$ is a submersion;
(3) the Poisson bracket $\{F, G\}^M$ of functions F, G which are constant along $\mathcal{D}$, is constant along $\mathcal{D}$;
(4) $P(D^0) \subset TS + \mathcal{D}$, where $\mathcal{D}^0$ is the annihilator of $\mathcal{D}$;

then N is a Poisson manifold with bracket $\{\cdot,\cdot\}^N$ given by

$$\{f, g\}^N \circ \pi = \{F, G\}^M \circ i_S\,, \tag{2.6}$$

where F, G are extensions of $f \circ \pi$, $g \circ \pi$ which are constant along $\mathcal{D}$.

As for the usual bihamiltonian reduction theorem, the central idea of the extended one is to use the pair of compatible Poisson tensor in order to determine the geometry of the Marsden–Ratiu reduction, or more precisely, to choose the submanifold S and the distribution $\mathcal{D}$. Moreover in our case we need an additional algebraic structure, which is the decomposition of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \tag{2.9}$$

where $\mathfrak{g}^0$ is a subalgebra of $\mathfrak{g}$ such that $\mathfrak{g}^0 \cap (\mathfrak{g}^0)^\perp = \{0\}$ (or in other words $(\mathfrak{g}^0)^\perp = \mathfrak{g}^1$, so that $\mathfrak{g}^0$ is a reductive subalgebra of $\mathfrak{g}$). Observe that any simple Lie algebra $\mathfrak{g}$ admits at least the trivial decomposition: $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^1 = \{0\}$, and that from the ad–invariance of the Killing form follows immediately $[\mathfrak{g}^0, \mathfrak{g}^1] \subset \mathfrak{g}^1$. The decomposition (2.9) can obviously extended to the whole loop–algebra $\tilde{\mathfrak{g}}$ by setting

$$\tilde{\mathfrak{g}}_i = C^\infty(S^1, \mathfrak{g}_i) \qquad i = 0, 1\,. \tag{2.10}$$

We are now in the position to define the submanifold S and the distribution $\mathcal{D}$. Let us start, by characterizing the symplectic foliation of $\tilde{\mathfrak{g}}$ induced by the tensor P_0.

In our case, P_0 is a skew–symmetric tensor with respect to the bilinear separating form (2.2) on $\tilde{\mathfrak{g}}$ and the characteristic distribution of P_0 is given by

$$\tilde{\mathfrak{g}}_A^{\perp} = \{v \in \tilde{\mathfrak{g}} | \langle V, U\rangle = 0 \quad \forall U \in \tilde{\mathfrak{g}}_A\} \tag{2.11}$$

where $\tilde{\mathfrak{g}}_A$ is the kernel of the Poisson tensor P_0: $\tilde{\mathfrak{g}}_A = \mathrm{Ker}(\mathrm{ad}\, A)$. This distribution $\tilde{\mathfrak{g}}_A^{\perp}$, (see [13]) is obviously integrable, and because $\tilde{\mathfrak{g}}_A^{\perp}$ is a flat manifold its integral leaves are the affine submanifolds S'_V of $\tilde{\mathfrak{g}}$ modeled on $\tilde{\mathfrak{g}}_A^{\perp}$ passing through any point of $\tilde{\mathfrak{g}}$. As submanifold S we take any submanifold of S'_V:

$$S = S'_V \cap \tilde{\mathfrak{g}}^0 = B + S'_B \cap \tilde{\mathfrak{g}}^0 \tag{2.12}$$

where B is any point of $\tilde{\mathfrak{g}}^0$ so that S is the intersection of the symplectic leaf S'_B with the loop–subalgebra $\tilde{\mathfrak{g}}^0$.

It remains to define the distribution $\mathcal{D}$ such that condition (4) holds for both Poisson tensors. Since we have restricted the submanifold S and therefore its tangent bundle TS, our idea is to enlarge the distribution $\mathcal{D}$. So let us define

$$\mathcal{D} = P_1(\tilde{\mathfrak{g}}_A) \cap \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}^1 . \tag{2.13}$$

Then the following new bihamiltonian reduction holds

Theorem 2.1. *Let $(M = \tilde{\mathfrak{g}}, P_0, P_1)$ be the bihamiltonian manifold defined above, where A belongs either to $\tilde{\mathfrak{g}}^0$ or $\tilde{\mathfrak{g}}^1$, let S be any submanifold of the type* (2.12), *and let $\mathcal{D}$ be the distribution* (2.13). *Then*

(1) *$\mathcal{E} = \mathcal{D} \cap TS = P_1(\tilde{\mathfrak{g}}_A) \cap TS$ is an integrable distribution;*
(2) *Points (3) and (4) of the Marsden–Ratiu theorem hold for both Poisson tensors, so that, if $N/\mathcal{E}$ is a manifold, then N is a bihamiltonian manifold and the corresponding reduced Poisson brackets $\{\cdot,\cdot\}_0^N$ and $\lambda\{\cdot,\cdot\}_1^N$ are obtained from those of M by means of formula* (2.6).

Proof. 1) $\mathcal{E} = \mathcal{D} \cap TS = P_1(\tilde{\mathfrak{g}}_A) \cap TS$ is an integrable distribution since it is the intersection of two integrable ones: $P_1(\tilde{\mathfrak{g}}_A)$ which is integrable because it is the characteristic distribution of the Poisson tensor P_1 of the subalgebra $\tilde{\mathfrak{g}}_A$; and TS which is obviously integrable.

2) a) In order to prove that the functions constant on $\mathcal{D}$ build a Poisson algebra with respect both brackets $\{\cdot,\cdot\}_0$, $\{\cdot,\cdot\}_1$ it is sufficient to prove that

$$L_X\{F,G\}_i = 0 \qquad \forall X \in \mathcal{D} \qquad i = 0,1 \tag{2.14}$$

for all F and G constant on $\mathcal{D}$ (i.e. such that $dF, dG \in \mathcal{D}^0$), where L_X denote the Lie derivative along X. Let us decompose the vector field X as

$$X = X^0 + X^1 \qquad \text{where} \quad X^0 \in \tilde{\mathfrak{g}}^0, \quad \text{and} \quad X^1 \in \tilde{\mathfrak{g}}^1$$

and let us prove (2.14) separately for X^0 and X^1. First of all, $\tilde{\mathfrak{g}}^1 \subset \mathcal{D}$, implies that $X^1 \in \mathcal{D}$, and therefore, since $X \in \mathcal{D}$ also that $X^0 \in \mathcal{D}$. Moreover $dF \in \mathcal{D}^0 \subset \tilde{\mathfrak{g}}^0$ implies

$$\{F,G\}_i = \langle dF, (P_i dG)^0\rangle \qquad i = 0,1 \tag{2.15}$$

(where $(P_i dG)^0$ denotes the component in $\tilde{\mathfrak{g}}^0$ of $(P_i dG)$). Thus to prove that $L_{X^1}\{F,G\}_i = 0$ it is sufficient to show that

$$L_{X^1}\langle dF, Y\rangle = 0 \qquad \forall dF \in \mathcal{D}^0 \qquad \forall Y \in \tilde{\mathfrak{g}}^0 .$$

But this is easily checked writing $L_{X^1}\langle dF, Y\rangle$ as (see [14])

$$L_{X^1}\langle dF, Y\rangle = \langle L_{X^1}(dF), Y\rangle + \langle dF, L_{X^1}(Y)\rangle ,$$

because $dF \in \mathcal{D}^0$ and $X^1 \in \mathcal{D}$ imply that $L_{X^1}(dF) = 0$, and thus that $L_{X^1}\langle dF, Y\rangle = \langle dF, L_{X^1}(Y)\rangle$, and finally

$$\langle dF, [X^1, Y]\rangle = 0$$

because $[X^1, Y] \in \tilde{\mathfrak{g}}_1$, whereas $dF \in \mathcal{D}^0 \subset \tilde{\mathfrak{g}}^0$.

It remains to prove that

$$L_{X^0}\{F,G\}_i = 0$$

but in this case $X^0 \in \mathcal{D} \cap \tilde{\mathfrak{g}}^0 = P_1(\mathrm{Ker}\, P_0) \cap \tilde{\mathfrak{g}}^0$ and we can therefore write X^0 as $X^0 = P_1 dk$ where k is a Casimir of P_0, (recall that $\mathrm{Ker}\, P_0$ is integrable see [5]). Then

$$L_{P_1(dk)}\{F,G\}_1 = \{L_{P_1(dk)}(F), G\}_1 + \{dF, L_{P_1(dk)}(G)\}_1 = 0$$

because $L_{P_1(dk)}(F) = L_{P_1(dk)}(G) = 0$. Similarly since $L_{P_0(dk)} = 0$,

$$\begin{aligned} L_{P_1(dk)}\{F,G\}_0 &= L_{P_1(dk)}\{F,G\}_0 - \{L_{P_1(dk)}(F), G\}_0 - \{F, L_{P_1(dk)}(G)\}_0 = \\ &\overset{(2.5)}{=} -L_{P_0(dk)}\{F,G\}_1 + \{L_{P_0(dk)}(F), G\}_1 + \{F, L_{P_0(dk)}(G)\}_1 = \\ &= 0 . \end{aligned}$$

b) Let us finally check condition (4) of the Marsden Ratiu Theorem. Since A is either in $\tilde{\mathfrak{g}}^0$ or in $\tilde{\mathfrak{g}}^1$ we have that either $P_0(X) \in \tilde{\mathfrak{g}}^1 \subset \mathcal{D}\ \forall X \in \mathcal{D}^0$, or $P_0(X) \in \tilde{\mathfrak{g}}^0 \cap \tilde{\mathfrak{g}}_A^\perp\ \forall X \in \mathcal{D}^0$, and thus in both cases $P_0(\mathcal{D}^0) \subset \mathcal{D} + TS$.
Further under our assumption on A any $Y \in \tilde{\mathfrak{g}}_A$ can be decomposed as $Y = Y^0 + Y^1$ with $Y^0 \in \tilde{\mathfrak{g}}^0 \cap \tilde{\mathfrak{g}}_A$ and $Y^1 \in \tilde{\mathfrak{g}}^1 \cap \tilde{\mathfrak{g}}_A$. Now, since on the points of S, $P_1(\tilde{\mathfrak{g}}_i) \subset \tilde{\mathfrak{g}}_i$, we have that $P_1(X) \in \tilde{\mathfrak{g}}^0$, and thus

$$\langle P_1(X), Y\rangle = \langle P_1(X), Y^0 + Y^1\rangle = \langle P_1(X), Y^0\rangle = -\langle P_1(Y^0), X\rangle = 0$$

because $P_1(X) \in P_1(\mathrm{Ker}\, P_0) \cap \tilde{\mathfrak{g}}_A$. This equation implies finally that $P_1(X) \in \tilde{\mathfrak{g}}_A^\perp \cap \tilde{\mathfrak{g}}^0 = TS$, and therefore that condition (4) of Marsden Ratiu theorem holds in our case. □

Let us close this section by the remark that if the decomposition (2.9) of $\mathfrak{g}$ is the trivial one then Theorem 2.1 coincides with the usual bihamiltonian reduction theorem [5].

3. The "Twisted" Bihamiltonian Manifolds

The aim of this section is to apply our extended bihamiltonian reduction theorem to a particular case of bihamiltonian manifolds, which turns out to be strictly related to the twisted Drinfel'd Sokolov hierarchies.

These bihamiltonian manifolds, defined on loop–algebras, are constructed by making a opportune choice of the gradation $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ (2.9), and of the elements A and B considered in the previous section. These choices will depend heavily on the algebraic data carried by an associated twisted affine Lie algebra. Therefore, we need some results of the theory of the twisted affine Lie algebras, which for the convenience of the reader are briefly summarized in the next subsection. A complete and well made exposition of these algebras can be found in the well known book by Kac ([8] chapter VIII).

The twisted Kac–Moody affine Lie algebras.

Let $\hat{\mathfrak{g}}$ be an affine (twisted) Lie algebra, whose Dynkin diagram is denoted by $X_n^{(k)}$, let $\mathfrak{g}$ be the (finite) simple Lie algebra associated with the Dynkin diagram X_n, and let us finally denote by $\{E_1, \dots E_n, F_1, \dots F_n\}$ a set of Chevalley generators of $\mathfrak{g}$, then the following holds

Theorem 3.1. *Let us label with $(0, 1, \dots, n)$ the dots of the Dynkin diagram $X_n^{(k)}$ and let $s = (s_0, s_1, \dots, s_n)$ be a corresponding sequence of non negative relatively prime integers, define $m = k \sum_{i=0}^{n} a_i s_i$ (where a_i is the Kac label of $X_n^{(k)}$ associated with the dot i see* [8] *and* [9]*), then*

(1) *the relations*

$$\begin{aligned} \sigma_{s,k}(E_j) &= e^{2\pi s_j/m} E_j \\ \sigma_{s,k}(F_j) &= e^{-2\pi s_j/m} F_j \qquad j = 1, 2, \dots n \end{aligned} \tag{3.1}$$

define a m–th automorphism $\sigma_{s,k}$ of $\mathfrak{g}$ (i.e. $\sigma^m = 1$, $\sigma^q \neq 1$ if $q \neq 0$ mod *m);*

(2) *up to conjugation by an automorphism of $\mathfrak{g}$, the automorphisms $\sigma_{s,k}$ exhaust all m–the order automorphism of $\mathfrak{g}$;*

(3) *the automorphisms $\sigma_{s,k}$ and $\sigma_{s',k'}$ are conjugate by an automorphism of $\mathfrak{g}$ if and only if $k = k'$ and the sequence s can be transformed into the sequence s' by an automorphism of the diagram $X_n^{(k)}$.*

The Lie twisted affine algebra $\hat{\mathfrak{g}}$ can be constructed starting from the Lie algebra $\mathfrak{g}$, by means of this Theorem of $\mathfrak{g}$ as follows. An automorphism $\sigma_{s,k}$ of order m of $\mathfrak{g}$ induces on $\mathfrak{g}$ a $\mathbb{Z}/m\mathbb{Z}$–gradation:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_j \tag{3.2}$$

where $\mathfrak{g}_j$ is the eigenspace of $\sigma_{s,k}$ corresponding to the eigenvalue $e^{\frac{2\pi i}{m}}$.

One elementary properties of the gradation (3.2), which will be useful in the following, is that $(\mathfrak{g}_i, \mathfrak{g}_j)_{\mathfrak{g}} = 0$ if $i+j \neq 0 \mod m$, and $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are nondegenerate paired if $i+j = 0 \mod m$.

With the automorphism σ of $\mathfrak{g}$, we associate the subalgebra $L_\lambda(\mathfrak{g}, \sigma_{s,k})$ of the loop algebra $L_\lambda(\mathfrak{g})$:

$$L_\Lambda(\mathfrak{g}, \sigma_{s,k}) = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_{j \mod m} \otimes \lambda^j \,, \tag{3.3}$$

which is the subalgebra of $L_\Lambda(\mathfrak{g})$ formed by the fixed point of the automorphism $\tilde{\sigma}_{s,k}$ of $L_\lambda(\mathfrak{g})$ extending to the whole loop algebra $L_\lambda(\mathfrak{g})$ the automorphism $\sigma_{s,k}$ as follows:

$$\tilde{\sigma}_{s,k}(X \otimes \lambda^j) = \sigma_{s,k}(X) \otimes \epsilon^{-j}\lambda^j \qquad \forall j \in \mathbb{Z} \quad X \in \mathfrak{g}\,. \tag{3.4}$$

Then it can be proved (see for example [8] for the details) that the Lie algebra $\hat{\mathfrak{g}}$ is isomorph to the algebra

$$\hat{\mathfrak{g}} \simeq \hat{L}_\lambda(\mathfrak{g}, \sigma_{s,k}) = L_\lambda(\mathfrak{g}, \sigma_{s,k}) \oplus \mathbb{C}c \oplus \mathbb{C}d\,, \tag{3.5}$$

where c and d are respectively the central extensions and the derivation of $L_\lambda(\mathfrak{g}, \sigma_{s,k})$ defined in [8]. Therefore if $X_i = X \otimes \lambda^i$ (with obviously $X \in \mathfrak{g}_{i \mod m}$) is a typical element of $\hat{L}_\lambda(\mathfrak{g}, \sigma)_{s,k})$ then the algebra–multiplication of $\hat{\mathfrak{g}}$ may be written as

$$\begin{aligned} &[X_i, Y_j] = [X, Y]_{i+j} + (X, Y)_{\mathfrak{g}} c\delta_{i+j,0} \\ &[d, X_i] = iX_i \\ &[d, d] = [c, d] = [c, c] = [c, X_n] = 0\,. \end{aligned} \tag{3.6}$$

The $\mathbb{Z}/m\mathbb{Z}$–gradation associated with $\sigma_{s,k}$ can be obviously extended to a $\mathbb{Z}$–gradation of the whole affine Lie $\hat{L}_\lambda(\mathfrak{g}, \sigma_{s,k})$ setting:

$$\hat{L}_\lambda(\mathfrak{g}, \sigma_{s,k}) = \bigoplus_{j\in\mathbb{Z}} \hat{L}_\lambda(\mathfrak{g}, \sigma_{s,k})_j \tag{3.7}$$

where $\hat{L}_\lambda(\mathfrak{g}, \sigma_{s,k})_j = \mathfrak{g}_{j \mod m} \otimes \lambda^j$ and $\hat{L}_\lambda(\mathfrak{g}, \sigma_{s,k})_0 = \mathfrak{g}_0 \otimes \lambda^j \otimes c \otimes d$.

This gradation can be also introduced from another point of view, perhaps more useful for us, as follows.

Let $\{E_0, \dots E_n, F_0, \dots F_n\}$ a set of Chevalley generators of $\hat{\mathfrak{g}} = \hat{L}_\lambda(\mathfrak{g}, \sigma_{s,k})$ (which can be constructed starting from a set of Chevalley generators of $\mathfrak{g}$ (see [8] chapt. VII and VIII) then the gradation corresponding to the automorphism $\sigma_{s,k}$, with $s = (s_0, s_1, \dots, s_n)$ is given by the formula:

$$\deg(E_i) = s_i \qquad \deg(F_i) = -s_j \qquad i = 0, 1, \dots n\,. \tag{3.8}$$

Among them only the following two will be considered in the paper:

(1) the principal gradation, for which $s_0 = s_1 = \cdots = s_n = 1$;
(2) the standard gradations (also called "minimal gradation" see [3]), for which, for some j, $s_j = 1$, while $s_i = 0$ if $i \neq j$.

The above collected facts on the twisted affine Lie algebras allow us to define a bihamiltonian manifold, which satisfies the hypothesis of Theorem 2.1. This amounts to describe the choice of the decomposition (2.9), and of the element A and B depends on the twisted affine Lie algebra $\hat{\mathfrak{g}}$.

Let $\mathfrak{g}$ be a simple Lie algebra, whose Dynkin diagram X_n can be extended to a twisted affine Dynkin diagram $X_n^{(k)}$ $(k = 2, 3)$, (thus $\mathfrak{g}$ is of the type A_n, D_n, E_6, because the only twisted affine Lie algebra are $A_n^{(2)}$, $D_n^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$).

We have seen in the previous subsection, how to realize explicitly the affine Lie algebra $\hat{\mathfrak{g}}$ associated to the Dynkin diagram $X_n^{(k)}$ starting from the Lie algebra $\mathfrak{g}$ as

$$\hat{\mathfrak{g}} = L_\lambda(\mathfrak{g}, \sigma_{s,k}) \tag{3.9}$$

where $L_\lambda(\mathfrak{g}, \sigma_{s,k})$ is the subloop–algebra (3.5), and $\sigma_{s,k}$ is supposed here and in what follows to be a standard automorphism of $\mathfrak{g}$, which will be simply denoted by σ. The automorphism σ induces a $\mathbb{Z}/m\mathbb{Z}$–gradation (3.2) of $\mathfrak{g}$ and of $\tilde{\mathfrak{g}}$ as well (where m is the finite order of σ, defined in the Theorem 3.1 and $\tilde{\mathfrak{g}}_{i \mod m} = C^\infty(S^1, \mathfrak{g}_{i \mod m})$). Thanks to the elementary properties recalled above, this gradation allows us to define a nontrivial decomposition (2.9) of $\mathfrak{g}$ by setting

$$\mathfrak{g}^0 = \mathfrak{g}_{0 \mod m} \qquad \mathfrak{g}^1 = \bigoplus_{\substack{i \in \mathbb{Z}/m\mathbb{Z} \\ i \neq 0 \mod m}} \mathfrak{g}_i\,. \tag{3.10}$$

Once the decomposition (2.9) is given, it remains only to define the elements A and B. This is made as follows.

With respect to the chosen standard gradation we can find a Chevalley basis of $\hat{\mathfrak{g}}$: $\{E_0, E_1, \dots, E_n\}$, $\{F_0, F_1, \dots, F_n\}$ such that $\deg(E_j) = s_j$ and $\deg(F_j) = -s_j$, where, being σ a standard automorphism, $s_j = \delta_{ij}$ for some i, $(j = 0, 1, \dots, n)$. Beside the standard gradation of $\hat{\mathfrak{g}}$, we can endowed $\hat{\mathfrak{g}}$ by the principal gradation obtained by putting:

$$\deg(E_i) = 1 \qquad \deg(F_i) = -1 \qquad \forall i = 0, 1, \dots, n\,. \tag{3.11}$$

Then the element Λ defined by

$$\Lambda = \sum_{i=0}^{n} E_i \tag{3.12}$$

is an element of degree 1 with respect to the principal gradation, while with respect to the standard gradation Λ can be written as the sum of two homogeneous elements Λ_0, Λ_1 of degree 0 and 1 respectively. The properties of the element Λ, which play a crucial role in our construction, are collected in the following

Proposition 3.3. *The element* Λ *of* $\hat{\mathfrak{g}}$ *(where* $\hat{\mathfrak{g}}$ *is of the type* $A_n^{(2)}, D_n^{(2)}, E_6^{(2)}$, $D_4^{(3)}$*) defined by* (3.12) *is a regular element (i.e., (see* [1], [3]*),* $\mathrm{Ker}(\mathrm{ad}\,\Lambda)$ *is a Heisenberg subalgebra of* $\hat{\mathfrak{g}}$*, or in other words its restriction* $\mathrm{Ker}\,(\mathrm{ad}\,\Lambda|_{\mathfrak{g}})$ *to* $\mathfrak{g}$ *is a Cartan subalgebra). It has degree* 1 *with respect to the principal gradation of* $\hat{\mathfrak{g}}$*. Its decomposition in the standard gradation used to realized explicitly* $\hat{\mathfrak{g}}$ *from* $\mathfrak{g}$*, is*

$$\Lambda = \Lambda_0 + \Lambda_1 \tag{3.13}$$

with Λ_0 *in* $\mathfrak{g}$*, whereas* Λ_1 *can be written in the form*

$$\Lambda_1 = A \otimes \lambda \tag{3.14}$$

where $A \in \mathfrak{g}$*.* A *and* Λ_1 *are elements of* $\mathfrak{g}$ *of degree* 1 *and* 0 *respectively, in the* $\mathbb{Z}/m\mathbb{Z}$*–gradation of* $\mathfrak{g}$ (3.2) *induced from the standard automorphism* σ*.*

Proof. We need only to prove the last assertion, but this follows easily from the properties of the standard gradation of $\hat{\mathfrak{g}}$ and the fact proved by Kac [8] that it is always possible to find a Chevalley basis with the properties described in the paragraph before the proposition. Indeed by definition $\Lambda_0 \in \hat{\mathfrak{g}}_0 \simeq \mathfrak{g}_0$ and $\Lambda_1 \in \hat{\mathfrak{g}}_1 \simeq \mathfrak{g}_1 \otimes \lambda$. □

The above stated proposition allows us to make explicitly our choice of A and B in such a way that all the hypothesis of the Theorem 2.1 are satisfied. We namely choose A as defined by the formula (3.14) and put $B = \Lambda_0$.
We can therefore apply the general bihamiltonian reduction Theorem 2.1, in our particular case obtaining a reduced "bihamiltonian" manifold N, which coincides (see [7]) with the phase space of the soliton hierarchies associated with the twisted affine Lie algebras.

Unfortunately the choice made of the element A implies $P_0(\tilde{\mathfrak{g}}_i) \subset \tilde{\mathfrak{g}}_{i+1}$ $i = 0, 1$ mod 2 and then from equation (2.8) follows immediately that the reduced Poisson tensor P_0^N is identically equal to zero. Therefore the bihamiltonian reduction theorem for the twisted case is not so powerful as in the non twisted case. On the phase space level the bihamiltonian structure virtually disappears. The soliton hierarchies of the twisted case still, however, remain the "shadows" of bihamiltonian hierarchies defined on the loop-algebra level. Moreover each remarkable properties of the twisted soliton hierarchies can be recovered by the above described reduction process from the properties usually owned by the bihamiltonian hierarchies. In this paper we limit ourself to study the reduction process of the bihamiltonian hierarchy.

4. The twisted reduced hierarchy

In this section we want study what happens to a twisted bihamiltonian hierarchy defined at the level of the loop–algebra $\tilde{\mathfrak{g}}$, when one tries to reduced it on the manifold N. Unfortunately, since one of the Poisson tensors, namely P_0, is reduced to the trivial one, we may not expect that the whole bihamiltonian hierarchies can be recovered on the reduced space, as happens in the non–twisted case.

A bihamiltonian hierarchy on a bihamiltonian manifold (M, P_0, P_1) is a sequence of functions $\{H_k\}_{k\geq 0}$ verifying the Lenard's recursion relation

$$P_0(dH_{k+1}) = P_1(H_k) \quad \text{or} \quad \{F, H_{k+1}\}_0 = \{F, H_k\}_1 \qquad \forall F \in C^\infty(M, \mathbb{R}) \tag{4.1}$$

together with $P_0(dH_0) = 0$ or $\{F, H_0\}_0 = 0$ $\forall F \in C^\infty(M, \mathbb{R})$.
The main property of these hierarchies is that the functions H_k are in involution with respect to both Poisson brackets [5]:

$$\{H_j, H_k\}_1 = \{H_j, H_k\}_0 = 0\,. \tag{4.2}$$

A bihamiltonian hierarchy gives rise to a hierarchy of bihamiltonian vector fields, pairwise commuting, defined by the relations $X_k = P_1 dH_k = P_0 dH_{k+1}$. This vector fields generate (if independent) the whole hierarchy of partial differential equations of the integrable system associated with the bihamiltonian manifold (M, P_0, P_1).

These remarks are the basis of the bihamiltonian approach to integrable systems.

Unfortunately the problem to construct a bihamiltonian hierarchy for a given bihamiltonian variety is perhaps the hardest one of the whole theory, and it is not yet solved in the most general case. Therefore we do not tackle it in this paper preferring to refer to the papers [1], [3] and [16], where this problem is discussed and solved in the case that the bihamiltonian variety is a loop–algebra $\tilde{\mathfrak{g}} = C^\infty(S^1, \mathfrak{g})$ with $\mathfrak{g}$ simple, and the elements A and B defined in Sect. 2 are such that $A+B$ is a regular element (as it happens in our case, as stated by Proposition 3.3). In this important case one has to find first a Casimir of the Poisson pencil $P_\lambda = P_1 - \lambda P_0$ i.e., a 1–form $V(\lambda) = \sum_{k=-j}^{\infty} V_k \lambda^{-k}$ which solves the stationary equation:

$$P_\lambda(V(\lambda)) = 0\,, \tag{4.3}$$

and then showing that $V(\lambda)$ is exact 1–form, so that a Laurent series $H(\lambda) = \sum_{k=-j}^{\infty} H_k \lambda^{-k}$ exists such that $dH(\lambda) = V(\lambda)$. Then the coefficients H_k of the Laurent series $H(\lambda)$ satisfy the Lenard recursion relations (4.1). (Here we have changed the enumeration of the bihamiltonian hierarchies in such a way that it matches with that of the powers of λ.) In the particular case studied in this work the results of those papers can be reformulated stating, that there exist on S exact Casimirs $V(\lambda) = \sum_{k=-j}^{\infty} V_k \lambda^{-k}$ of the pencil P_λ which belong to the algebra $\hat{\mathfrak{g}}$ so that $V_k \in \tilde{\mathfrak{g}}_k$ $\forall k = -j, -j+1, -j+2, \ldots$. Since the Casimir $V(\lambda)$ is exact, there exists a corresponding Laurent series of functions $H(\lambda) = \sum_{k=-j}^{\infty} H_k \lambda^{-k}$ with $dH(\lambda) = V(\lambda)$. To each of these Casimirs $V(\lambda)$ corresponds a infinite hierarchy of vector fields defined by $X_k = P_1 dV_k = P_0 dV_{k+1}$ $k = -j, -j+1, \ldots$. The vector fields of these hierarchies which can be reduced on the quotient submanifold N form the integrable hierarchy of the solitonic theory. These last vector fields are selected by the algebraic data encoded in the twisted affine Lie algebra $\hat{\mathfrak{g}}$. It holds, indeed

Theorem 4.1. *Let $V = \sum_{k=-j}^{\infty} V_k \lambda^{-k}$ be a exact Casimir of the stationary equation* (4.3) *belonging to $\hat{\mathfrak{g}}$, and let $X_k = P_1 V_k = P_0 V_{k+1}$ the corresponding hierarchy of vector fields. Then only the vector fields X_k with $k = 0 \mod m$ are reducible on N.*

Proof. Obviously the vector fields X_k with $k \neq 0 \mod m$ cannot be reduced on the quotient space N, because they are not even tangent to the leave S, belonging to $\bigoplus_{k=1}^{m-1} \tilde{\mathfrak{g}}_k \simeq \tilde{\mathfrak{g}}^1$ while $TS \in \tilde{\mathfrak{g}}_0 \simeq \tilde{\mathfrak{g}}^0$.

The vector fields on S: $X_k = P_0(V_{k+1}) = P_1(V_k)$ with $k = 0 \mod m$ are reducible on N. In order to be reducible the vector fields X_k have to be on S tangent to the leave S, and projectable with respect to the projection $\pi : S \to N$. The first condition is immediately satisfied. Indeed, each X_k belongs to the image of P_0 by the relation $X_k = P_0 V_{k+1}$, but being X_k bihamiltonian $k = -j, -j+1, \ldots$, we can write it also as $X_k = P_1(V_k)$, so that $X_k \in \tilde{\mathfrak{g}}^0$ if $k = 0 \mod m$, because $V_k \in \tilde{\mathfrak{g}}_0$, and moreover P_1 has degree 0 on S as observed in the proof of Theorem 2.1, (part (2) b).

In order to prove the second one, we have to show [11] that if $k = 0 \mod m$ then

$$[X_k, Y] \in \mathcal{E} \qquad \forall Y \in \mathcal{E}\,. \tag{4.4}$$

From the definition of $\mathcal{E}$ follows that its elements have the form $Y = P_1(\eta)$ with $\eta \in \mathrm{Ker}(P_0)$ and therefore we get [11] [16] that

$$[X_k, Y] = [P_1(V_k), P_1(\eta)] = P_1\{V_k, \eta\}_1\,.$$

Thus, since $[X_k, Y]$ belongs to TS being S a submanifold, it remains only to prove that $\{V_k, \eta\}_1$ is in $\mathrm{Ker}(P_0)$. But this follows immediately from the compatibly condition [11]

$$\{V_k, \eta\}_1 = \{V_{k+1}, \eta\}_0\,.$$

Indeed

$$P_0(\{V_k, \eta\}_1) = P_0(\{V_{k+1}, \eta\}_0) = [P_0(V_k), P_0(\eta)] = 0\,,$$

because $\eta \in \mathrm{Ker}(P_0)$. □

References

[1] Drinfel'd, V. G., Sokolov, V. V., *Lie Algebras and Equations of Korteweg–de Vries Type*, J. Sov. Math. **30** (1985), 1975–2036.

[2] Dickey, L. A., *Soliton Equations and Hamiltonian Systems*, Adv. Series in Math. Phys 12 World Scientific, Singapore 1991.

[3] de Groot, M. F., Hollowood, T. J., Miramontes, J. L., *Generalized Drinfel'd–Sokolov Hierarchies*, Commun. Math. Phys. **145** (1992), 57–84.

[4] de Groot, M. F., Hollowood, T. J., Miramontes, J. L., *Generalized Drinfel'd–Sokolov Hierarchies. II The Hamiltonian Structures*, Commun. Math. Phys. **153** (1993), 187–215.

[5] Casati, P., Magri, F., Pedroni, M., *Bihamiltonian Manifolds and τ–Function*, Proceedings of the 1991 Joint Summer Research Conference on Mathematical Aspects of Classical Field Theory (M. J. Gotay, J. E. Marsden, and V. E. Moncrief, America Mathematical Society, Providence (RI) **132**, 213–234, eds.).

[6] Casati, P., Pedroni, M., *Drinfel'd–Sokolov Reduction on a Simple Lie Algebra from the Bihamiltonian Point of View*, Lett. Math. Phys. **25** (1992), 89–101.

[7] Pedroni, M., *Equivalence of the Drinfel'd–Sokolov Reduction to a Bihamiltonian Reduction*, Lett. Math. Phys. **35** (1995), 291-302.

[8] Kac, V. G., *Infinite Dimensional Lie Algebras. Third edition*, Cambridge University Press, Cambridge, 1990.

[9] Kac, V. G., *Automorphism of Finite Order of Semi–Simple Lie Algebras*, Funct. Anal. Its Appl. **3** (1969), 252-254.

[10] Pressley, A. N., Segal, A. N., *Loop Groups and their Representations,*, Oxford University Press, Oxford, 1985.

[11] Magri, F., Morosi, C., *A geometrical Characterization of Integrable Hamiltonian Systems Through The Theory of Poisson–Nijenhuis Manifolds*, Quaderno S/19, Dipartimento di Matematica, Università di Milano (1984).

[12] Marsden, J. E., Ratiu, T., *Reduction of Poisson Manifolds*, Lett. Math. Phys. **11** (1986), 161–169.

[13] Liebermann, P., Marle, C. M., *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht, 1987.

[14] Abraham, L. R., Marsden, J. E., *Foundations of Mechanics. Second edition*, Addison Wesley, Singapore, 1978.

[15] Casati, P., Falqui, G., Magri, F., Pedroni, M., *The KP Theory Revisited. III. The bihamiltonian Action and the Gel'fand–Dickey Equations*, preprint SISSA 4/96/FM, SISSA (1996).

[16] Dorfman, I., *Dirac Structures and Integrability of Nonlinear Evolution Equations*, John Wiley and Sons, New York, 1993.

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