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VARIATIONAL COMPATIBILITY OF FORCE LAWS

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Abstract. The paper presents a simplified version of conditions for the variational compatibility of general forces in equations of motions of physical systems with the finite number of degrees of freedom. The forces are classified with respect to their transformation properties, and the classification is discussed from the physical point-of-view.

1. Introduction

The variational principle is the fundamental element of most physical theories, particularly mechanics, classical electrodynamics as well as the theory of gravitation. However, the interpretation of phenomena which can be explained within the framework of these disciplines need not be restricted to the use of the variational principle. Alternative approaches are often admitted including the wider region of phenomena than those based on the variational principle. As the typical example, the classical Newtonian mechanics can be mentioned: it enables us to take into account the more extensive class of force laws compared with the Lagrangean (variational) mechanics. Thus, the aim can be formulated: what are the conditions for the variational compatibility of force laws in classical physics? The answer is, of course, contained in the solution of the *inverse variational problem* formulated and solved by Helmholtz [2] and then treated by many authors from various points-of-view. In the presented contribution, this problem is solved by the use of a very simple integration procedure, and its physical aspects are pointed out. Especially, the variationally compatible force laws in classical physics are characterized, and the role of coordinate transformations is discussed in detail. The force laws are

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classified into basic classes containing the so-called *real*, *inertial*, *geometrical* and *metric* forces. The invariance aspects are not studied here.

2. Basic notations

We briefly summarize the basic concepts and notations which are used in our considerations. Throughout the paper, we adopt the commonly used notation as presented e. g. in [3,5,8].

Consider a mechanical system with m degrees of freedom. We can describe all reasonable cases of the time development of its configuration by means of the fibered manifold (Y,π,X), with the one-dimensional *base* X, *total space* Y, $\dim Y = m+1$, and the *projection* π. For the dynamical description of the system, we introduce the *r–jet prolongation* of the fibered manifold denoted by (J^rY,π^r,X), $r\geq 1$, and the spaces Ω^r_k of differential k–forms, $k\geq 1$, functions for $k=0$, defined on J^rY. In the standard notation, $\pi^{r,s}: J^rY \to J^sY$, $r\geq s\geq 0$, are the canonically defined projections. Let $V\subset Y$ be an open set. We denote $(V,\psi = (t,q^\sigma))_{\sigma=1,\dots,m}$ a *fibered chart* on Y, $(U,\varphi=(t))$, $U=\pi(V)$, the *associated chart* on X and $(V^r,\psi^r=(t,q^\sigma,q^\sigma_1,\dots,q^\sigma_r))$, $V^r=(\pi^{r,0})^{-1}(V)$, the *associated fibered chart* on J^rY. By Ω^r_kV, the space of differential k–forms defined on V^r is denoted. Let $\lambda = L\ \mathrm{d}t \in \Omega^1_1V$ be a *Lagrangian* with the *Lagrange function* $L\in\Omega^1_0V$, (π,λ) the corresponding *Lagrange structure* and $\mathcal{E}_\lambda = E_\sigma\ \mathrm{d}q^\sigma\wedge\mathrm{d}t\in\Omega^s_2V$ ($s\geq r$ is the appropriate integer), the associated *Euler-Lagrange form.* Functions E_σ of the order s are called *Euler-Lagrange expressions*. Equations $E_\sigma=0$ define the system of ordinary differential equations of the order s. The solutions of this system are sections of the underlying fibered manifold, so-called *extremals* of the given Lagrange structure (π,λ). They represent the real motion of the studied physical system. For $r=1$, it holds $s=2$, and in the fibered chart $(V,\psi=(t,q^\sigma,q^\sigma_1,q^\sigma_2))$, we have

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q^\sigma_1}\right) . \tag{1}$$

The quantities q^σ, q^σ_1, q^σ_2 are the *generalized coordinates, velocities* and *accelerations*, respectively.

3. Variational equations

Let $\mathcal{E} = E_\sigma\ \mathrm{d}q^\sigma\wedge\ \mathrm{d}t\in\Omega^2_2V$ be a so-called *dynamical form.* The form $\mathcal{E}$ and, equivalently, the system of equations $E_\sigma=0$ is *variational*, i. e. arising from a Lagrangian, if and only if it satisfies the well-known *Helmholtz conditions* (Helmholtz [2]) formulated for the expressions E_σ themselves. Variational expressions E_σ are of the form $E_\sigma = A_\sigma + B_{\sigma\nu}q^\nu_2$ with A_σ, $B_{\sigma\nu}\in\Omega^1_0V$ satisfying certain additional conditions (Havas [1]). The alternative approach to the variationality conditions leading to the explicite formulae for *tensor of mass* $(-B_{\sigma\nu})$ and *general forces* (A_σ), based on geometrical concepts, was advised by Novotný [9]. The entirely correct and complete procedure for obtaining these formulae was first given by Krupková [6] and then still treated in more detailed form by the same author [7,8]. The problem was also treated by Sarlet [10]. Our following results are equivalent to those of Krupková. However, our calculation procedure as well as the results themselves are simpler.

Let $V \subset \mathcal{R}^m$ be an open convex set containing the origin $0_{\mathcal{R}^m}$. For the purposes of two following lemmas, we define the mapping

$$\eta : [0,1] \times V \ni (s, y^k) \longrightarrow \eta(s, y^k) = (sy^k) \in V \ , \ 1 \le k \le m \ .$$

Lemma 1. *Let* $(B_{ij})_{i,j=1,\dots,m}$, $B_{ij} : V \ni (y^k) \longrightarrow B_{ij}(y^k) \in \mathcal{R}$, $1 \le k \le m$, *be a given system of differentiable functions. Then the system of equations*

$$B_{ij} = \frac{\partial A_i}{\partial y^j} - \frac{\partial A_j}{\partial y^i} \ , \ 1 \le i, j \le m \tag{2}$$

is integrable with respect to the unknown functions $A_i : V \to \mathcal{R}$ *if and only if it holds*

$$\frac{\partial B_{ij}}{\partial y^k} + \frac{\partial B_{jk}}{\partial y^i} + \frac{\partial B_{ki}}{\partial y^j} \ , \ B_{ij} + B_{ji} = 0 \ . \tag{3}$$

The general solution is

$$A_i = y^j \int_0^1 (B_{ij} \circ \eta)\, s \ \mathrm{d}s + \frac{\partial F}{\partial y^i} \ , \tag{4}$$

with the arbitrarily chosen function $F : V \to \mathcal{R}$.

Proof. Lemma 1 is the direct consequence of the well-known Poincaré lemma applied to the form $\omega = B_{ij} \ \mathrm{d}y^i \wedge \ \mathrm{d}y^j$. □

Lemma 2. *Let* p *be an integer and let* $(B_{i_1 \dots i_p})$, $1 \le i_1, \dots, i_p \le m$, $B_{i_1 \dots i_p} : V \ni (y^k) \longrightarrow B_{i_1 \dots i_p}(y^k) \in \mathcal{R}$, $1 \le k \le m$, *be a given system of differentiable functions. Then the system of equations*

$$B_{i_1 \dots i_p} = \frac{\partial^p A}{\partial y^{i_1} \dots \partial y^{i_p}} \tag{5}$$

is integrable with respect to the unknown function $A : V \longrightarrow \mathcal{R}$ *if and only if the set of quantites* $B_{i_1 \dots i_p}$ *is completely symmetrical and the following conditions are satisfied:*

$$\frac{\partial B_{i_1 \dots i_{p-1} i_p}}{\partial y^{i_{p+1}}} = \frac{\partial B_{i_1 \dots i_{p-1} i_{p+1}}}{\partial y^{i_p}} \ . \tag{6}$$

Then the general solution of the system (5) is $A = A^H + A^P$ where

(7)

$$A^H = \sum_{l=0}^{p-1} C_{i_1 \dots i_l} y^{i_1} \dots y^{i_l} \ , \ A^P = y^{i_1} \dots y^{i_p} \int_0^1 \sum_{l=1}^{p} \frac{(-1)^{p-l}}{(p-l)!(l-1)!} s^{p-l} (B_{i_1 \dots i_p} \circ \eta) \ \mathrm{d}s,$$

with arbitrary constants $C_{i_1\dots i_l}$, $0\leq l\leq p-1$.

Proof. A function $A : V \longrightarrow \mathcal{R}$ is the solution of the system (5) if and only if it satisfies the following system of first order equations:

$$\frac{\partial A}{\partial y^{i_1}} = B_{i_1}\ ,\ \frac{\partial B_{i_1}}{\partial y^{i_2}} = B_{i_1 i_2}\ ,\ \dots\ ,\ \frac{\partial B_{i_1\dots i_{p-1}}}{\partial y^{i_p}} = B_{i_1\dots i_p}\ . \tag{8}$$

The symmetry conditions for $B_{i_1\dots i_p}$ follow directly from (5) and the conditions (6) are immediately connected with the Poincaré lemma applied to the last set of equations (8). A^P is the particular integral of the system (5). This can be proved by the use of Poincaré lemma for $p = 1$ and then by induction with respect to p. A^H is the general solution of the homogenized system (5). □

The following theorem gives the possibility to recognize the variationally compatible general force laws.

Theorem. *Let $\mathcal{E} = E_\sigma\ dq^\sigma \wedge\ dt \in \Omega_2^2 V$ be a dynamical form. Then $\mathcal{E}$ is variational if and only if its components are of the form*

$$E_\sigma(t, q^\lambda, q_1^\lambda, q_2^\lambda) = A_\sigma(t, q^\lambda, q_1^\lambda) + B_{\sigma\nu}(t, q^\lambda, q_1^\lambda)q_2^\nu \tag{9}$$

in which functions A_σ, $B_{\sigma\nu} \in \Omega_0^1 V$ obey the following conditions:

$$B_{\sigma\nu} = B_{\nu\sigma}\ ,\ \frac{\partial B_{\sigma\nu}}{\partial q_1^\lambda} = \frac{\partial B_{\sigma\lambda}}{\partial q_1^\nu}$$

(10)

$$A_\sigma = q_1^\nu q_1^\lambda \int_0^1 \left(\frac{\partial B_{\sigma\nu}}{\partial q^\lambda}\circ\eta - \frac{\partial B_{\lambda\nu}}{\partial q^\sigma}\circ\eta(1-s)\right) ds + q_1^\nu \int_0^1 \frac{\partial B_{\sigma\nu}}{\partial t}\circ\eta\ \ ds + N_{\sigma\nu}q_1^\nu + M_\sigma$$

$$N_{\sigma\nu}, M_\sigma \in \Omega_0^0 V, N_{\sigma\nu} = -N_{\nu\sigma}, \frac{\partial N_{\sigma\nu}}{\partial q^\lambda} + \frac{\partial N_{\nu\lambda}}{\partial q^\sigma} + \frac{\partial N_{\lambda\sigma}}{\partial q^\nu} = 0, \frac{\partial M_\sigma}{\partial q^\nu} - \frac{\partial M_\nu}{\partial q^\sigma} - \frac{\partial N_{\sigma\nu}}{\partial t} = 0\,.$$

The corresponding Lagrange function is

(11)

$$L = -q_1^\sigma q_1^\nu \int_0^1 (B_{\sigma\nu}\circ\eta)(1-s)\mathrm{d}s + q^\sigma q_1^\nu \int_0^1 (N_{\sigma\nu}\circ\overline{\eta})\, s\ \mathrm{d}s + q^\sigma \int_0^1 (M_\sigma\circ\overline{\eta})\ \ \mathrm{d}s + \frac{\mathrm{d}F}{\mathrm{d}t}$$

with the arbitrary function $F \in \Omega_0^0 V$. The mappings η and $\overline{\eta}$ in equations (10,11) are defined by

$$\overline{\eta} : [0,1]\times\psi(V) \ni (s, t, q^\lambda) \longrightarrow \overline{\eta}(s, t, q^\lambda) = (t, sq^\lambda) \in \psi(V)\,,$$

$$\eta : [0,1]\times\psi^1(V^1) \ni (s, t, q^\lambda, q_1^\lambda) \longrightarrow \eta(s, t, q^\lambda, q_1^\lambda) = (t, q^\lambda, sq_1^\lambda) \in \psi^1(V^1)\,.$$

Proof. The variationality of the dynamical form $\mathcal{E} = E_\sigma \, \mathrm{d}q^\sigma \wedge \mathrm{d}t$ is equivalent to the integrability of equations

$$(12) \qquad E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q_1^\sigma}\right) = \left(\frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial q_1^\sigma} - q_1^\nu \frac{\partial^2 L}{\partial q^\nu \partial q_1^\sigma}\right) - q_2^\nu \frac{\partial^2 L}{\partial q_1^\nu \partial q_1^\sigma}$$

with respect to the unknown function $L \in \Omega_0^1 V$. The conditions for functions $B_{\sigma\nu} = -\frac{\partial^2 L}{\partial q_1^\nu \partial q_1^\sigma}$ and the function L are immediately given by the use of lemma 2 for $p = 2$:

$$L = -q_1^\sigma q_1^\nu \int_0^1 (B_{\sigma\nu} \circ \eta)(1-s) \, \mathrm{d}s + C_\sigma q_1^\sigma + D \; ; \; C_\sigma, \, D \in \Omega_0^0 V \; .$$

Functions C_σ and D can be easily obtained from (12). Denoting $M_\sigma = \frac{\partial D}{\partial q^\sigma} - \frac{\partial C_\sigma}{\partial t}$ and $N_{\sigma\nu} = \frac{\partial C_\nu}{\partial q^\sigma} - \frac{\partial C_\sigma}{\partial q^\nu}$ and using lemma 1, we can finish the proof. □

Note that the Lagrange function L and the variationally compatible general force laws (A_σ) are expressed through the variationally compatible tensor of mass $(B_{\sigma\nu})$ by the single integral only, i.e. by the simpler way than in [6,7,8]. By the relation (10), the forces (A_σ) are decomposed into two terms: the term containing $B_{\sigma\nu}$ corresponds to the so-called *canonical connection* on $\mathcal{R} \times TM$, and the contribution $N_{\sigma\nu} q_1^\nu + M_\sigma$ is connected with the *soldering form*, both introduced by Krupková [7]. The function L given by the relation (11) defines the so-called *minimal Lagrangian* (see e.g. [8]) which is of the first order for the second order equations of motion.

4. Variational forces

Let us suppose that $\frac{\partial B_{\sigma\nu}}{\partial q_1^\lambda} = 0$, $1 \leq \sigma, \nu, \lambda \leq m$ and denote

$$(13) \qquad g_{\sigma\nu} = -B_{\sigma\nu} \; , \; \Gamma_{\sigma\nu\mu} = \frac{1}{2}\left(\frac{\partial g_{\sigma\nu}}{\partial q^\lambda} + \frac{\partial g_{\sigma\lambda}}{\partial q^\nu} - \frac{\partial g_{\lambda\nu}}{\partial q^\sigma}\right) \; , \; \Lambda_{\sigma\nu} = \frac{\partial g_{\sigma\nu}}{\partial t} \; .$$

Then the variational expressions can be written as $E_\sigma = -g_{\sigma\nu} q_2^\nu + A_\sigma$ where

$$(14) \qquad A_\sigma = -q_1^\nu q_1^\lambda \Gamma_{\sigma\nu\lambda} - q_1^\nu \Lambda_{\sigma\nu} + q_1^\nu N_{\sigma\nu} + M_\sigma .$$

Let (V, t, q^σ) and $(\overline{V}, \overline{t}, \overline{q}^\sigma)$ be two fibered charts on Y and $q^\sigma = q^\sigma(\overline{t}, \overline{q}^\nu)$, $t = t(\overline{t})$, the corresponding coordinate transformation. (These relations represent the most general coordinate transitions preserving fibres of the underlying fibered manifold. They do not contain e.g. Lorentzian transformations.) We denote

$$T = \frac{\mathrm{d}t}{\mathrm{d}\overline{t}}, \; \overline{T} = \frac{\mathrm{d}^2 t}{\mathrm{d}\overline{t}^2} \; ,$$

$$C_\nu^\sigma = \frac{\partial q^\sigma}{\partial \overline{q}^\nu}, \; C_{\nu\lambda}^\sigma = \frac{\partial^2 q^\sigma}{\partial \overline{q}^\nu \partial \overline{q}^\lambda}, \; C_T^\sigma = \frac{\partial q^\sigma}{\partial \overline{t}}, \; C_{TT}^\sigma = \frac{\partial^2 q^\sigma}{\partial \overline{t}^2}, \; C_{T\nu}^\sigma = \frac{\partial^2 q^\sigma}{\partial \overline{t} \partial \overline{q}^\nu} \; .$$

For the coordinate transformation it holds, of course, $\det(C^{\sigma}_{\nu}) \neq 0$ and $T \neq 0$ on $W = V \cup \overline{V}$. The quantities $\Gamma_{\sigma\nu\lambda}$, $\Lambda_{\sigma\nu}$, $N_{\sigma\nu}$ and M_{σ} in the relation (14) for variational general forces (A_{σ}) obey the following transformation rules:

$$\overline{g}_{\sigma\nu} = g_{\lambda\mu} C^{\lambda}_{\sigma} C^{\mu}_{\nu} T^{-1} ,$$

$$\overline{\Gamma}_{\sigma\nu\lambda} = \left(\Gamma_{\alpha\beta\gamma} C^{\alpha}_{\sigma} C^{\beta}_{\nu} C^{\gamma}_{\lambda} + g_{\alpha\beta} C^{\alpha}_{\sigma} C^{\beta}_{\nu\lambda} \right) T^{-1} ,$$

$$\overline{\Lambda}_{\sigma\nu} = \Lambda_{\alpha\beta} C^{\alpha}_{\sigma} C^{\beta}_{\nu} + g_{\alpha\beta} (C^{\alpha}_{\sigma} C^{\beta}_{T\nu} + C^{\alpha}_{\nu} C^{\beta}_{T\sigma}) T^{-1} + g_{\alpha\beta} C^{\alpha}_{\sigma} C^{\beta}_{\nu} \overline{T} , \tag{15}$$

$$\overline{N}_{\sigma\nu} = N_{\alpha\beta} C^{\alpha}_{\sigma} C^{\beta}_{\nu} - \Gamma_{\alpha\beta\gamma} C^{\alpha}_{\sigma} (C^{\gamma}_{T} C^{\beta}_{\nu} + C^{\beta}_{T} C^{\gamma}_{\nu}) + g_{\alpha\beta} (C^{\beta}_{T\sigma} C^{\alpha}_{\nu} - C^{\alpha}_{\sigma} C^{\beta}_{T\nu}) T^{-1} ,$$

$$\overline{M}_{\sigma} = M_{\alpha} C^{\alpha}_{\sigma} T + N_{\alpha\beta} C^{\alpha}_{\sigma} C^{\beta}_{T} - \Gamma_{\alpha\beta\gamma} C^{\alpha}_{\sigma} C^{\beta}_{T} C^{\gamma}_{T} T^{-1} -$$

$$- \Lambda_{\alpha\beta} C^{\alpha}_{\sigma} C^{\beta}_{T} - g_{\alpha\beta} C^{\alpha}_{\sigma} C^{\beta}_{T} \overline{T} - g_{\alpha\beta} C^{\alpha}_{\sigma} C^{\beta}_{TT} T^{-1} .$$

Note that the equation (13) for $\Gamma_{\sigma\nu\lambda}$ is similar to the definition of the *Christoffel symbols* of the well-known *Levi-Civita connection.* Thus, we can identify $(g_{\sigma\nu})$ with the metric tensor of the fibre $\pi^{-1}\{t\}$. As the equations of motion have the form $g_{\sigma\nu} q^{\nu}_{2} = A_{\sigma}$, we can see, by analogy with the second Newton law, that the expressions (A_{σ}) are related to the forces. However, the family of functions (A_{σ}) is not in general the geometrical object, in contradiction to $(g_{\sigma\nu})$.

If we denote $A^{M}_{\sigma} = -q^{\nu}_{1} q^{\lambda}_{1} \Gamma_{\sigma\nu\lambda} - q^{\nu}_{1} \Lambda_{\sigma\nu}$ and $A^{D}_{\sigma} = q^{\nu}_{1} N_{\sigma\nu} + M_{\sigma}$, we can again decompose the general forces (A_{σ}) into two terms. The first of them is related to the canonical connection, and from the physical point-of-view, it corresponds to the free particle; the second one is given by the soldering form, after [7],

$$A_{\sigma} = A^{M}_{\sigma} + A^{D}_{\sigma} .$$

Let $\mathcal{M}$ be the set of all expressions (A_{σ}). We say that two families of expressions from this set are equivalent if it is possible to transform one to another by a change of coordinates. We factorize the set $\mathcal{M}$ by this equivalence. We can classify the equivalence classes into four types, according to the transformation behaviour of sets of functions $(\Gamma_{\sigma\nu\lambda})$, $(\Lambda_{\sigma\nu})$, $(N_{\sigma\nu})$ and (M_{σ}).

- Type I. is given by the only class containing zero expressions (A_{σ}) on V.
- Type II. is generated by classes containing the expressions (A_{σ}) with all $\Gamma_{\sigma\nu\lambda} = 0$, $\Lambda_{\sigma\nu} = 0$ on V (but some $N_{\sigma\nu} q^{\nu}_{1} + M_{\sigma} \neq 0$).

Next two types are given by classes which contain no expressions (A_{σ}) with all $\Gamma_{\sigma\nu\lambda} = 0$, $\Lambda_{\sigma\nu} = 0$.

- Type III. classes contain expressions (A_{σ}) with all $N_{\sigma\nu} = 0$, $M_{\sigma} = 0$ (but some $\Gamma_{\sigma\nu\lambda} \neq 0$ or $\Lambda_{\sigma\nu} \neq 0$)
- Type IV. contains all other classes.

The forces (A^{M}_{σ}) and (A^{D}_{σ}) are called, in the agreement with the presented classification and in the physical context, as follows:

Type	A_σ^M	A_σ^D
I.	geometrical	inertial
II.	geometrical	real
III.	metric	inertial
IV.	metric	real

The set of functions (A_σ^D) is, in physical terminology, considered as representing the so-called *inertial* and *real* forces, whereas the family (A_σ^M) corresponds to *geometrical* and *metric* forces. The expressions (M_σ) represent the *electric-type* forces, and the quantities $(N_{\sigma\nu})$ define by the expressions $(N_{\sigma\nu}q_1^\nu)$ the *magnetic-type* forces. *Geometrical* forces have no actual physical basis, because they can be removed by the appropriate change of coordinates on fibres, $t = \overline{t}$, $q^\sigma = q^\sigma(\overline{q}^\nu)$. Metric forces can be made zero along extremals. On the other hand, the difference between inertial and real forces is not so outstanding. For eliminating the inertial forces, a more general transformation $t = \overline{t}$, $q^\sigma = q^\sigma(\overline{q}^\nu, \overline{t})$ must be performed, containing $\overline{t}$ explicitely. The sum of the real and inertial forces is again real. Any real forces can be always decomposed into the sum of inertial and other real forces. Let us discuss this problem somewhat in detail.

Now let us take into account only such coordinate systems in which $(A_\sigma^M) = 0$ on V for the I and II type classes of expressions (A_σ), or the coordinate systems in which this condition is satisfied along some extremals, for the types III and IV. These coordinate systems are preferred in the certain sense. We choose one of them and declare the corresponding (in general nonzero) expressions (A_σ^D) as *true forces*. The choice if such a coordinate system is, of course, arbitrary from the mathematical point-of-view. However, in real physical situations it can be supported by physical arguments, e.g. a usual inertial system of classical mechanics can be chosen. So, in the physical analogy, we call such an arbitrarily chosen system the *inertial-like coordinate system*. In all coordinate systems related to the inertial-like one by Galilean transformations, $t = \overline{t}$, $q^\sigma = \overline{q}^\sigma + V^\sigma \overline{t}$, $V^\sigma =$ const., the expressions (A_σ^D) remain unchanged and the expressions $(A_\sigma) = (A_\sigma^D)$ are again the true forces. Such systems can be called also inertial-like and remaining ones are considered as *non-inertial* ones. In non-inertial systems the expressions $(A_\sigma) = (A_\sigma^D)$ include non only the true forces, but the inertial forces as well. In the following example we illustrate the concept of true, real and inertial forces. The example shows that there are the physical situations in which the true forces can be compensated by the inertial ones in an appropriately chosen non-inertial system.

5. Example

Let us take into account two particles with masses and charges m_1, e_1 and m_2, e_2, respectively, in the electric field with the intensity $\vec{E}$ and the gravitational field with the intensity $\vec{G}$. Let time be the parameter t, $X = \mathcal{R}$. The configuration of the system is given by cartesian coordinates of radius-vectors of particles $\vec{r}_1$, $\vec{r}_2$, i.e. $m = 6$, $Y = \mathcal{R} \times \mathcal{R}^6$, $(q^\sigma)_{\sigma=1,\dots,6}$, $q^\sigma = x_1^\sigma$ for $1 \le \sigma \le 3$ and $q^\sigma = x_2^{\sigma-3}$ for $4 \le \sigma \le 6$, $g_{\sigma\nu} = \delta_{\sigma\nu}$. Denote as $\vec{v}_1$, $\vec{v}_2$ and $\vec{a}_1$, $\vec{a}_2$ velocities and accelerations of particles, respectively. The equations of motion in the reference frame which is inertial in the usual sense accepted in Newtonian mechanics are

$$m_1\vec{G} + e_1\vec{E} - m_1\vec{a}_1 = \vec{0}\ ,\ m_2\vec{G} + e_2\vec{E} - m_2\vec{a}_2 = \vec{0}\ .$$

These equations are variational. It is clear, that $A_\sigma^M = 0$ and then $A_\sigma = A_\sigma^D$. We choose this system as the inertial-like one and thus we declare the expressions

$$A_\sigma = m_1 G^\sigma + e_1 E^\sigma \quad \text{for} \quad 1 \le \sigma \le 3\ ,\ A_\sigma = m_2 G^{\sigma-3} + e_2 E^{\sigma-3} \quad \text{for} \quad 4 \le \sigma \le 6$$

as the true forces. We discuss two cases:

• $\frac{e_1}{m_1} = \frac{e_2}{m_2}$. The true forces $(A_\sigma)_{\sigma=1,\dots,6}$ are of the type I and they vanish in the new (non-inertial) coordinate system defined by the transformation $t = \bar{t}$ and
$q^\sigma = \bar{q}^\sigma + \frac{1}{2}\left(G^\sigma + \frac{e_1}{m_1}E^\sigma\right)t^2$, $1 \le \sigma \le 3$; $q^\sigma = \bar{q}^\sigma + \frac{1}{2}\left(G^{\sigma-3} + \frac{e_2}{m_2}E^{\sigma-3}\right)t^2$, $4 \le \sigma \le 6$.
(The true forces occurring in the inertial system were compensated by the inertial forces.)

• $\frac{e_1}{m_1} \neq \frac{e_2}{m_2}$. The true forces $e_2\vec{E}$, $e_2\vec{E}$ cannot be entirely removed by any coordinate transition (type II) and thus they are the real forces. On the other hand, the tranformation

$$q^\sigma = \bar{q}^\sigma + \tfrac{1}{2}G^\sigma t^2 \ \text{ for } 1 \le \sigma \le 3\ ,\ q^\sigma = \bar{q}^\sigma + \tfrac{1}{2}G^{\sigma-3}t^2\ ,\ \text{ for } 4 \le \sigma \le 6\ ,\ t = \bar{t}\ ,$$

leads to the absence of forces $m_1\vec{G}$, $m_2\vec{G}$ which are compensated by the inertial forces $-m_1\vec{G}$, $-m_2\vec{G}$ occurring in the new non-inertial system.

Summarizing the results of the example, we can conclude: The forces (A_σ^D) are declared as true by the choice of some of coordinate systems with $(A_\sigma^M) = 0$ which is then taken as inertial-like. The forces of various physically specified types (electrical, magnetical, gravitational, centrifugal, etc.) can contribute to (A_σ^D). These contributions or their appropriately chosen sums which can be removed by a coordinate transformations and thus compensated by inertial forces, are purely inertial, the remaining ones are real. Only purely inertial forces can be recognized in the complete expressions (A_σ^D) by finding the coordinate transitions eliminating them. This is e.g. the case of forces $m_1\vec{G}$, $m_2\vec{G}$ in both alternatives of the presented example and the case of forces $e_1\vec{E} + m_1\vec{G}$, $e_2\vec{E} + m_2\vec{G}$ for $\frac{e_1}{m_1} = \frac{e_2}{m_2}$. For different specific charges of particles the forces $e_1\vec{E}$, $e_2\vec{E}$ are real because there is no coordinate transition removing them entirely.

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