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ON THE GEOMETRY OF NON–HOLONOMIC MECHANICAL SYSTEMS *)

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ABSTRACT. A geometric approach to the theory of non-holonomic systems is presented. General first order mechanical systems (not necessarily regular or Lagrangian) are studied. A constraint structure on J^1Y is given by a submanifold fibered over Y, naturally endowed with a distribution (of a constant rank, possibly non-integrable). Constraint force is studied and geometric arguments for obtaining the Chetaev formula are presented. Constrained mechanical systems are geometrically described, their motion equations in intrinsic form are found, and dynamics are characterized by means of distributions on the constraint submanifold.

1. Non-holonomic constraints in classical mechanics

In classical mechanics one naturally encounters different kinds of constraints on the motion, mostly *holonomic* and *linear non-holonomic constraints* (integrable or non-integrable). While mechanics of holonomic systems is well-established both from the physical and geometric point of view, in non-holonomic mechanics still many questions remain open. The main problems studied concern namely geometric structures arising due to presence of constraints, a direct geometric characterization of constrained systems (without help of Lagrange multipliers), variationality and regularity of constrained systems, the concept of constraint force, a generalization of the concept of virtual displacements, Principle of virtual work, etc., all for the case of *general non-holonomic constraints* (which are not supposed linear in the first derivatives, as it is done within classical mechanics). Following classical works (e.g. [4]), recently there have appeared many papers dealing with geometric aspects of the problem (see [3], [5], [8]–[21] and references therein).

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Let us turn to a brief characterization of systems subject to non-holonomic constraints. To this purpose consider mechanical systems modeled on a fibered manifold $\pi : Y \to X$ where $\dim X = 1$ and $\dim Y = m + 1$. Denote by $(t, q^1, \ldots, q^m)$ fiber coordinates on Y, and by dotted resp. doubledotted q's the associated coordinates on the first and second jet prolongation of π. Within this scheme, dynamics is usually represented by a semispray distribution on J^1Y. It should be stressed that such a geometric description is suitable only for *regular* mechanical systems; generalization to non-regular cases requires to consider distributions on J^1Y of rank greater than one [7].

A system of k *non-holonomic constraints* is a submanifold Q of codimension k in J^1Y, where $1 \leq k < m$, fibered over Y; locally it is given by a system of k equations

$$f^i(t, q^1, \ldots, q^m, \dot{q}^1, \ldots, \dot{q}^m) = 0, \quad 1 \leq i \leq k\,, \tag{1.1}$$

where

$$\operatorname{rank}\Big(\frac{\partial f^i}{\partial \dot{q}^\sigma}\Big) = k\,.$$

Consider a mechanical system on π locally represented by motion equations

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} = F_\sigma\,,$$

where L is the Lagrange function of the system (in classical mechanics $L = T - V$, the kinetic minus potential energy), and F_σ have the meaning of components of the sum of non-conservative forces. The motion of the corresponding constrained system is then described by the equations

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} = F_\sigma + \mu_i\,\Phi^i_\sigma\,, \tag{1.2}$$

where μ_i, $1 \leq i \leq k$, are *Lagrange multipliers* and $\Phi_\sigma = \mu_i\,\Phi^i_\sigma$ are the components of the so called *non-holonomic constraint force*, usually considered to be given by the formula

$$\Phi^i_\sigma = \frac{\partial f^i}{\partial \dot{q}^\sigma}\,, \tag{1.3}$$

proposed by *Chetaev* in 1932 [4]. Equations (1.2) together with the equations of constraints (1.1) represent a system of $m + k$ mixed first and second order ordinary differential equations for $m + k$ unknown functions $q^\sigma(t)$, $\mu_i(t)$. Solving them one finds both the explicit form of the Lagrange multipliers (as functions of the parameter t) and the trajectories of the constrained system. The classical procedure of excluding Lagrange multipliers from the motion equations with help of the equations of constraints means in fact a procedure to express the constrained system as a *mechanical system on the fibered manifold* $Q \to R$.

Let us recall briefly at least a few points which have been of interest in non-holonomic mechanics.

– First of all, the equations (1.2) are not the only possibility for motion equations of non-holonomic systems proposed so far. Different equations (leading to different description of the motion) have been studied within *vakonomic dynamics* ([1], [2]). However, it seems that the vakonomic equations do not reflect the non-holonomic motions correctly.

– Also Chetaev's approach, though widely accepted, carries some open problems, especially when non-linear non-holonomic constraints are considered. This concerns namely the *constraint force*: the formula (1.3) proposed by Chetaev is heuristic, and, at least at the heuristic level, not the only possibility which could be taken into account. A closely related question is a generalization of the *Principle of virtual work*. One needs more arguments, both physical and mathematical, to support the theory.

– Recently, studies of a *geometric framework* for non-holonomic mechanics have been a "hot topic" of research. This concerns geometric models for constraints and constrained systems, intrinsic forms of the constrained equations of motion, geometric characterization of properties of constrained systems (as e.g. regularity), etc. In particular, a geometric background for the procedure of excluding Lagrange multipliers has been found, and constrained systems have been modeled as mechanical systems on constraint manifolds. Apparently, geometric studies are helpful in clarifying problems of not only mathematical but also of physical nature.

– Many questions which remain open concern *Lagrangian systems* subject to non-holonomic constraints. Contrary to the case of holonomic constraints where a constrained system arising from a Lagrangian system remains Lagrangian with the Cartan form pulled-back to the constraint submanifold, non-holonomic constraints do not possess this lovely property. (Consult e.g. [3], [8], [19], [21]).

– Another remarkable property which can be violated under constraints is the property of *regularity*. More precisely, a regular mechanical system (i.e., defined by a regular Lagrangian, or, more generally, described by a semispray vector field) can become singular (i.e., not representable by means of a semispray on the constraint submanifold), and vice versa, a singular mechanical system can possibly become regular under constraints (the procedure of *regularization via constraints*).

– Last but not least, one should notice that *equivalent* motion equations (i.e., whose solutions are the paths of the same semispray) subject to the same constraints may result in essentially *different* constrained systems (with different dynamics on the constraint submanifold).

The aim of this paper is to explain and further develop the approach to the theory of non-holonomic systems presented in [8]. The theory is proposed for *general* (not only regular or Lagrangian) *mechanical systems* on fibered manifolds, and for *general non-holonomic constraints*. The key-points are the following:

– the concept of a *mechanical system* as an *equivalence class of* 2*-forms* on J^1Y, called *Lepagean* 2*-forms*, which, roughly speaking, represents a geometric model generalizing e.g. the symplectic model of regular Lagrangian mechanics on tangent spaces,

– the concept of *constraint structure* as a submanifold Q of J^1Y, fibered over Y, endowed with a distribution $\mathcal{C}$ (*canonical distribution* or *Chetaev bundle*), naturally generated by the constraint (similarly as in [8] and [15]). $\mathcal{C}$ is a distribution on Q of minimal possible rank such that the motion of *every* mechanical system constrained to Q is described by a subdistribution of $\mathcal{C}$. In this sense, the canonical distribution has the meaning of an "universal covering distribution" for all possible dynamical distributions on Q. (In the unconstrained case the role of such a distribution is played simply by the tangent distribution to J^1Y. In the holonomic case, where $Q = J^1Q_0$ of a fibered submanifold Q_0 of Y, $\mathcal{C}$ is the tangent distribution TJ^1Q_0).

A constrained system then can be modeled either by means of a local deformation of the original mechanical system in a neighborhood of the constraint submanifold ("physical description") or as a mechanical system on the constraint structure ("geometrical description"). In both the cases we present equations of motion in intrinsic form. Within this scheme we also study constraint force and give geometric arguments for the Chetaev formula (1.3). We also deal with regularity and variationality of constrained systems.

2. Mechanical systems on fibered manifolds

Throughout the paper, manifolds and mappings are smooth, and the summation convention is used. We denote by d the exterior derivative of differential forms, by T and J^r, $r = 1, 2$, the tangent and the prolongation functor, respectively.

We consider a fibered manifold $\pi : Y \to X$ with $\dim X = 1$, $\dim Y = m + 1$, and the jet prolongations $\pi_1 : J^1Y \to X$ and $\pi_2 : J^2Y \to X$ of π. (Local) fiber coordinates on Y are denoted by (t, q^σ) where $1 \leq \sigma \leq m$. The associated coordinates on J^1Y and J^2Y are denoted by $(t, q^\sigma, \dot{q}^\sigma)$ and $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$, respectively. We also use the projections $\pi_{1,0} : J^1Y \to Y$, or $\pi_{2,1} : J^2Y \to J^1Y$, etc.

A section δ of π_1 is called *holonomic* if there exists a section γ of π such that $\delta = J^1\gamma$.

By a *distribution* on J^1Y we shall mean a mapping $\mathcal{D}$ assigning to every point $z \in J^1Y$ a vector subspace $\mathcal{D}(z)$ of the vector space T_zJ^1Y. A distribution is said to be of a *constant rank* if $\dim \mathcal{D}(z)$ does not depend on z. If $\mathcal{D}$ is a distribution, we denote by $\mathcal{D}^0$ its annihilator, i.e., the set of all 1-forms η_κ on J^1Y such that $i_{\xi_\iota}\eta_\kappa = 0$ for every vector field ξ_ι belonging to $\mathcal{D}$. Recall that a section δ of π_1 is called an *integral section* of $\mathcal{D}$ if $\delta^*\eta = 0$ for every 1-form η belonging to $\mathcal{D}^0$.

A differential p-form η on J^1Y is called π_1-*horizontal* (resp. $\pi_{1,0}$-horizontal) if $i_\xi\eta = 0$ for every π_1-vertical (resp. $\pi_{1,0}$-vertical) vector field ξ on J^1Y.

We denote by h the *horizontalization* mapping, assigning to every 1-form η on Y (resp. on J^1Y) a π-horizontal (resp. π_1-horizontal) 1-form $h\rho$ on J^1Y (resp. on J^2Y).

A form η on J^1Y is called *contact* if $J^1\gamma^*\eta = 0$ for every section γ of π. In particular, a 2-form η is called 1-*contact* if for every π_1-vertical vector field ξ the 1-form $i_\xi\eta$ is horizontal, and it is called 2-*contact* if $i_\xi\eta$ is contact.

In what follows, we use the following contact 1-forms

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma\, dt, \quad \dot{\omega}^\sigma = d\dot{q}^\sigma - \ddot{q}^\sigma\, dt, \quad 1 \leq \sigma \leq m,$$

and the bases $(dt,\, \omega^\sigma,\, d\dot q^\sigma)$, or $(dt,\, \omega^\sigma,\, \dot\omega^\sigma,\, d\ddot q^\sigma)$ of 1-forms on J^1Y (resp. on J^2Y).

Every 1-form η on J^1Y admits a unique decomposition into a sum of a horizontal and contact form. In fiber coordinates, if $\eta = f\,dt + f_\sigma\, dq^\sigma + \tilde f_\sigma\, d\dot q^\sigma$, this decomposition reads

$$\pi_{2,1}^*\eta = (f + f_\sigma \dot q^\sigma + \tilde f_\sigma \ddot q^\sigma)dt + f_\sigma\, \omega^\sigma + \tilde f_\sigma\, \dot\omega^\sigma \,.$$

Similarly, every 2-form η on J^1Y admits a unique decomposition into a sum of a 1-contact and 2-contact form.We denote by $p\eta$ (resp. $p_1\eta$) the contact (resp. 1-contact) part of η.

A 2-form E on J^2Y is called a *dynamical form* if it is 1-contact and $\pi_{2,0}$-horizontal. In this paper we restrict to dynamical forms on J^2Y with the components *affine in the second derivatives*, i.e., we suppose that in every fiber chart on J^2Y,

$$E = E_\sigma(t, q^\nu, \dot q^\nu, \ddot q^\nu)\, dq^\sigma \wedge dt\,,$$

where

$$E_\sigma = A_\sigma(t, q^\nu, \dot q^\nu) + B_{\sigma\rho}(t, q^\nu, \dot q^\nu)\ddot q^\rho\,. \tag{2.1}$$

A section γ of π is called a *path* of E if

$$E \circ J^2\gamma = 0\,.$$

In fiber coordinates this equation represents a system of m second-order ordinary differential equations

$$A_\sigma\Big(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}\Big) + B_{\sigma\rho}\Big(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}\Big)\frac{d^2\gamma^\rho}{dt^2} = 0$$

for the components $\gamma^\nu(t)$, $1 \le \nu \le m$, of γ. Notice that these equations need not be equations for paths of a semispray connection. However, they can be represented in terms of a distribution on J^1Y as follows.

Let α be a 2-form on J^1Y. Following [8] we say that α is a *(generalized) Lepagean 2-form* associated to a dynamical form E if $p_1\alpha = E$. Immediately one gets for E the following *equivalence class* of associated Lepagean 2-forms:

$$[\alpha] = A_\sigma\omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot q^\nu \bmod F \tag{2.2}$$

where F runs over the 2-contact 2-forms of order one, $F = F_{\sigma\nu}(t, q^\rho, \dot q^\rho)\omega^\sigma \wedge \omega^\nu$. This equivalence class is then called a *first-order mechanical system* on π. Notice that 2-forms equivalent in the above sense are equivalent also in the sense of the Krupka variational sequence [6].

For every Lepagean 2-form α there is a distribution Δ_α on J^1Y defined by

(2.3)
$$\begin{aligned}\Delta^0_\alpha &= \mathrm{span}\{i_\xi\alpha \,|\, \xi \text{ runs over the set of all } \pi_1\text{-vertical vector fields on } J^1Y\} \\ &= \mathrm{span}\{A_\sigma dt + B_{\sigma\nu} d\dot q^\nu + 2F_{\sigma\nu}\omega^\nu,\ B_{\sigma\nu}\omega^\nu\}\,,\end{aligned}$$

and called the *dynamical distribution* of α. At each point of J^1Y, $\mathrm{rank}\,\Delta_\alpha \ge 1$, however, the rank need not be constant. Hence, we get the following:

Proposition 1.

(1) *For any two equivalent Lepagean 2-forms the holonomic integral sections of the corresponding dynamical distributions coincide.*

(2) *The set of holonomic integral sections of any dynamical distribution of E coincides with the set of paths of E.*

(3) *A section γ of π is a path of a dynamical form E if and only if for every π_1-vertical vector field ξ on J^1Y*

$$J^1\gamma^* i_\xi \alpha = 0\,,$$

where α is a Lepagean 2-form associated to E.

The class of dynamical distributions related to a mechanical system $[\alpha]$ will be denoted by $[\Delta_\alpha]$.

Now, we get in a natural way a geometric concept of *regularity of a mechanical system* (cf. [7] and [8]): A mechanical system $[\alpha]$ is called *regular* if there exists a dynamical distribution $\Delta_\alpha \in [\Delta_\alpha]$ such that $\operatorname{rank} \Delta_\alpha = 1$.

We obtain the following useful characterization of regular mechanical systems:

Theorem 1 [8]. *Let $[\alpha]$ be the mechanical system related to E, and let $[\Delta_\alpha]$ be the corresponding class of dynamical distributions. Denote $E = (A_\sigma + B_{\sigma\nu}\ddot{q}^\nu)dq^\sigma \wedge dt$. The following conditions are equivalent:*

(1) *The mechanical system $[\alpha]$ is regular.*

(2) *The matrix $(B_{\sigma\nu})$ is everywhere regular.*

(3) *Each of the dynamical distributions belonging to $[\Delta_\alpha]$ has rank one.*

(4) *All the dynamical distributions belonging to $[\Delta_\alpha]$ coincide, and*

$$\Delta_\alpha = \operatorname{span}\Big\{ \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} - B^{\sigma\rho} A_\rho \frac{\partial}{\partial \dot{q}^\sigma} \Big\}\,,$$
$$\Delta^0_\alpha = \operatorname{span}\{ A_\sigma dt + B_{\sigma\nu} d\dot{q}^\nu\,, \omega^\sigma, \quad 1 \le \sigma \le m \}\,,$$

where $(B^{\sigma\nu})$ is the inverse matrix to $(B_{\sigma\nu})$ (i.e., Δ_α is a semispray distribution).

(5) *The equations for paths of E have an equivalent form*

$$\ddot{q}^\sigma = -B^{\sigma\rho} A_\rho \text{ along } J^2\gamma, \quad 1 \le \sigma \le m\,.$$

For every regular mechanical system $[\alpha]$ the dynamical distribution Δ_α is the characteristic distribution of any of 2-forms $\alpha \in [\alpha]$. This means that for $\Gamma \in \Delta_\alpha$ and all $\alpha \in [\alpha]$,

$$i_\Gamma \alpha = 0\,.$$

Another particular but highly important case concerns mechanical systems arising from Lagrangians.

Theorem 2 [7]. *Suppose that the dynamical form E is locally variational, i.e., that in a neighborhood of every point of J^1Y there exists a Lagrange function $L(t, q^\sigma, \dot{q}^\sigma)$ such that*

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma}.$$

Then the equivalence class $[\alpha]$ of Lepagean 2-forms of E contains a unique closed 2-form.

Conversely, if the equivalence class $[\alpha]$ of Lepagean 2-forms of E contains a closed 2-form then this is unique, and E is locally variational.

The unique closed representative of a locally variational form E is called the *Lepagean equivalent* of E, and the corresponding mechanical system is called a *Lagrangian system* (cf. [7] for more details).

3. The constraint structure

In what follows, we denote by Q a fibered submanifold $\pi_{1,0}|_Q : Q \to Y$ of the fibered manifold $\pi_{1,0} : J^1Y \to Y$ of codimension k, where $1 \le k \le m-1$. Locally, Q is defined by the equations

$$f^i = 0, \quad 1 \le i \le k, \tag{3.1}$$

where

$$\operatorname{rank}\Big(\frac{\partial f^i}{\partial \dot{q}^\sigma}\Big) = k. \tag{3.2}$$

Q is called a *constraint submanifold*, and the equations (3.1) are called a *a system of k (local) non-holonomic constraints.*

Let U be a fiber chart at a point $x \in Q$ with fiber coordinates $(t, q^\sigma, \dot{q}^\sigma)$. The equations (3.1) can be "locally solved" with respect to k of the functions $\dot{q}^\sigma$. Without loss of generality one can suppose that it is possible to express the $\dot{q}^{m-k+1}, \dots, \dot{q}^m$ as functions of the $(t, q^\sigma, \dot{q}^1, \dots, \dot{q}^{m-k})$. This means that we have an adapted chart (U, χ) to the submanifold Q at x of the form $(t, q^\sigma, \dot{q}^1, \dots, \dot{q}^{m-k}, f^1, \dots, f^k)$. Such coordinates will be called *adapted fiber coordinates.* In particular, adapted fiber coordinates such that Q is described by the equations

$$\dot{q}^{m-k+i} - g^i(t, q^\sigma, \dot{q}^1, \dots, \dot{q}^{m-k}) = 0 \tag{3.3}$$

are called *normal coordinates.*

A section γ of π defined on an open set $I \subset X$ is called a *holonomic path in Q* if for every $x \in I$, $J^1\gamma(x) \in Q$.

Consider a cover of Q by an atlas $\mathcal{A}$ of adapted fiber charts. For every $(U, \chi) \in \mathcal{A}$ put

$$\mathcal{C}^0_{U,\varphi} = \operatorname{span}\{\varphi^i, \, df^i, \, 1 \le i \le k\},$$

where φ^i are linearly independent 1-forms such that

$$h\varphi^i = f^i\, dt, \quad 1 \le i \le k\,. \tag{3.4}$$

Then $\mathcal{C}_{U,\varphi}$ is a distribution of rank $2m+1-2k$ on U, in general not completely integrable. At each point $x \in Q \cap U$ it is a subdistribution of the tangent distribution to $Q \cap U$ (generated by means of the 1-forms df^i, $1 \le i \le k$). Therefore, $\mathcal{C}_{U,\varphi}$ along the submanifold $Q \cap U$ is a distribution of corank k on $Q \cap U$.

Holonomic integral sections of the distribution $\mathcal{C}_{U,\varphi}$ obviously coincide with the holonomic paths in $Q \cap U$. Indeed, they are solutions of the equations $J^1\gamma^*\varphi^i = J^1\gamma^* h\varphi^i = 0$, $J^1\gamma^* df^i = 0$, $1 \le i \le k$, i.e., of the "constraint equations" $f^i \circ J^1\gamma = 0$ and their prolongations $(df^i/dt) \circ J^2\gamma = 0$.

The above construction provides us, for a fixed adapted fiber chart (U, χ), with a *family* of distributions $\mathcal{C}_{U,\varphi}$ with the same holonomic integral sections. Indeed, the condition (3.4) means that the *contact* parts of the φ^i's remain undetermined (unless all the forms φ^i are $\pi_{1,0}$-projectable). Namely, we have

$$\varphi^i = f^i\, dt + \varphi^i_\sigma\, \omega^\sigma\,, \tag{3.5}$$

where φ^i_σ are arbitrary functions on U such that the matrix (f^i, φ^i_σ) with k rows and $m+1$ columns has rank k on U. However, there is a natural choice for the φ^i's, picking up a unique distribution from the family $\mathcal{C}_{U,\varphi}$, giving a geometric sense to the physical concept of "possible generalized displacements". To this end denote by $\mathcal{V}_U(\pi)$ (resp. $\mathcal{V}_U(\pi_{1,0})$) the module of π-vertical vector fields on $\pi_{1,0}(U)$ (resp. of $\pi_{1,0}$-vertical vector fields over the ring of $\pi_{1,0}$-projectable functions on U).

Definition. 1-forms φ^i, $1 \le i \le k$, on U are called *constraint 1-forms* if

$$h\varphi^i = f^i\, dt, \quad 1 \le i \le k\,,$$

and for $1 \le i \le k$, and every π-vertical vector field ξ on $\pi_{1,0}(U)$,

$$i_{J^1\xi}\varphi^i = i_{F(\xi)}df^i\,, \tag{3.6}$$

where $F(\xi)$ is the image of ξ under the canonical morphism $\mathcal{V}_U(\pi) \to \mathcal{V}_U(\pi_{1,0})$, defined by $F(\partial/\partial q^\sigma) = \partial/\partial \dot{q}^\sigma$, $F(g\xi) = (g \circ \pi_{1,0})F(\xi)$.

The condition (3.6) gives a unique solution

$$\varphi^i_\sigma = \frac{\partial f^i}{\partial \dot{q}^\sigma}, \quad 1 \le i \le k \tag{3.7}$$

– the Chetaev expressions (1.3). Consequently, by (3.2), the obtained 1-forms are linearly independent. The corresponding distribution $\mathcal{C}_{U,\varphi}$ on U is called the *constraint distribution*, and denoted by $\mathcal{C}_U$. It has the meaning of "possible generalized displacements", its π_1-vertical subdistribution represents "virtual generalized displacements", and its $\pi_{1,0}$-vertical subdistribution could be called a distribution of "virtual velocities".

Summarizing the results, *in a neighborhood of the constraint submanifold Q there exists a canonical, unique system of local constraint distributions, subordinate to a cover of Q by adapted fiber charts. The holonomic paths in Q (piecewise) coincide with holonomic integral sections of these distributions.*

It is important to notice that if in a neighborhood U of $x \in Q$, $\bar{f}^i = 0$ are other equations of Q, we get a different constraint distribution $\bar{\mathcal{C}}_U$ on U. However, the holonomic integral sections of the distributions $\mathcal{C}_U$ and $\bar{\mathcal{C}}_U$ coincide. In this sense, the description of holonomic paths in Q by means of a system of local constraint distributions does not depend on a choice of an atlas of adapted fiber charts to Q.

In what follows we denote by $\mathcal{I}(\varphi^i)$ the ideal generated by the constraint 1-forms φ^i, $1 \le i \le k$, on U.

Physical constraint structure. A dynamical form Φ_U on U will be called a *local constraint force* or *Chetaev force* if $\Phi_U \in \mathcal{I}(\varphi^i)$. We have

$$\Phi_U = \varphi^i \wedge \mu_i = \mu_{0i} \frac{\partial f^i}{\partial \dot{q}^\sigma} \, dq^\sigma \wedge dt \,. \tag{3.8}$$

The functions μ_{0i} on U are called *Lagrange multipliers*. The Chetaev force satisfies the *Principle of virtual work*

$$i_\xi \Phi_U = 0$$

for every π_1-vertical vector field belonging to the constraint distribution $\mathcal{C}_U$.

It can be shown using partition of unity techniques that local constraint forces define a constraint force Φ in a neighborhood of the constraint submanifold Q. This constraint force depends on the choice of an atlas $\mathcal{A}$ of adapted fiber charts. However, this dependence is unessential, since it is realized via Lagrange multipliers (see [8] for details).

A pair (Q, Φ) where $Q \subset J^1Y$ is a constraint submanifold and Φ is a constraint force related to Q will be called a *physical constraint structure* on J^1Y.

Geometrical constraint structure. Consider on Q an atlas $\mathcal{A}$ of adapted fiber charts, and the subordinate system of local constraint distributions; recall that for a chart $(U, \chi) \in \mathcal{A}$ we have

$$\mathcal{C}^0_U = \mathrm{span}\{\varphi^i, \, df^i, \, 1 \le i \le k\}, \quad \varphi^i = f^i dt + \frac{\partial f^i}{\partial \dot{q}^\sigma} \omega^\sigma \,. \tag{3.9}$$

Denote $\iota : Q \to J^1Y$ the canonical embedding.

Theorem 3 [8]. *For each $(U, \chi) \in \mathcal{A}$ consider the distribution $\mathcal{C}_U$ restricted to $Q \cap U$, i.e., the distribution on $Q \cap U$ whose annihilator is spanned by the 1-forms $\iota^* \varphi^i$, $1 \le i \le k$. If $\bar{\varphi}^i$, $1 \le i \le k$, are the constraint 1-forms related to a chart $(\bar{U}, \bar{\chi})$, $\bar{U} \cap U \ne \emptyset$, then there is a regular matrix (a^i_j) on $\bar{U} \cap U$ such that $\iota^* \bar{\varphi}^i = a^i_j \, \iota^* \varphi^j$.*

By the above theorem, putting

$$\mathcal{C}^0 = \mathrm{span}\{\iota^* \varphi^i, 1 \le i \le k\} \,, \tag{3.10}$$

where φ^i runs over all constraint 1-forms subordinate to the cover $\mathcal{A}$ of Q by adapted fiber charts, we get a distribution on Q of corank k (with respect to Q). This distribution does not depend on the choice of the atlas $\mathcal{A}$. It is denoted by $\mathcal{C}$ and called the *canonical distribution* [8] or *Chetaev bundle* [15] of the constraint Q. Obviously, the holonomic integral sections of $\mathcal{C}$ coincide with the holonomic paths in Q.

Notice that the holonomic paths in Q are integral sections of (local) vector fields belonging to $\mathcal{C}$ which in *normal* coordinates, i.e., where Q is locally described by the equations $\dot{q}^{m-k+i} - g^i(t, q^\sigma, \dot{q}^1, \ldots, \dot{q}^{m-k}) = 0$, are given by

$$\Gamma_Q = \frac{\partial}{\partial t} + \sum_{l=1}^{m-k} \dot{q}^l \frac{\partial}{\partial q^l} + \sum_{i=1}^{k} g^i \frac{\partial}{\partial q^{m-k+i}} + \sum_{l=1}^{m-k} \Gamma^l \frac{\partial}{\partial \dot{q}^l}\,, \tag{3.11}$$

where Γ^l are some functions. Such vector fields are called *constraint semisprays.*

A pair $(Q, \mathcal{C})$ where $Q \subset J^1Y$ is a constraint submanifold and $\mathcal{C}$ is the canonical distribution on Q will be called a *geometrical constraint structure* on J^1Y. The ideal on Q generated by the 1-forms annihilating the canonical distribution will be called the *constraint ideal* and will be denoted by $\mathcal{I}(\mathcal{C}^0)$.

4. Geometrical models for classical constraints

Let us show relations of the above model of constraints with some frequently used geometric representations of classical (i.e., holonomic and linear non-holonomic) constraints.

Submanifold of Y. Constraints which are realized as a fibered submanifold Q_0 of Y of codimension $k < m$ are called *holonomic.* Locally they are given by equations $u^i(t, q^\sigma) = 0$, $1 \leq i \leq k$, such that $\operatorname{rank}(\partial u^i/\partial q^\sigma) = k$. Within our scheme this means that $f^i = du^i/dt$, and

$$\varphi^i = \frac{du^i}{dt}\, dt + \frac{\partial u^i}{\partial q^\sigma} \omega^\sigma = \pi_{1,0}^* du^i\,,$$

i.e., all the constraint 1-forms are $\pi_{1,0}$-projectable and closed. Consequently, $\mathcal{I}(\mathcal{C}^0)$ is a *differential ideal*, the canonical distribution $\mathcal{C}$ is *completely integrable*, and along J^1Q_0 it is nothing but the tangent distribution TJ^1Q_0. Its projection is the tangent distribution to Q_0, giving sense to the concepts of *degrees of freedom*, *possible displacements*, and *virtual displacements* defined to be the dimension of the manifold Q_0, the tangent distribution to Q_0, and its π-vertical subdistribution, respectively. The geometrical and the physical constraint structure for holonomic systems becomes (J^1Q_0, TJ^1Q_0), and (J^1Q_0, Φ), respectively, where

$$\Phi = \varphi^i \wedge \mu_i = \mu_{0i} \frac{\partial u^i}{\partial q^\sigma} \omega^\sigma \wedge dt\,. \tag{4.1}$$

Completely integrable distribution on Y. Consider a completely integrable distribution $\mathcal{D}$ of corank $k < m$ on Y. This means that in a neighborhood of every point in Y there are k linearly independent 1-forms du^i spanning the annihilator $\mathcal{D}^0$. Such a distribution represents constraints called *semiholonomic* or *linear integrable* (referring to the equations of the constraints which are affine in velocities).

In our notations, $f^i = du^i/dt$, $\varphi^i = \pi^*_{1,0} du^i$. Thus, the constraint 1-forms are $\pi_{1,0}$-projectable and closed, i.e., $\mathcal{I}(\mathcal{C}^0)$ is a *differential ideal*, and the corresponding constraint distribution is *completely integrable* and projects onto $\mathcal{D}$. The geometrical constraint structure on J^1Y is $(Q, \mathcal{C})$, where Q is locally defined by

$$\frac{du^i}{dt} = \frac{\partial u^i}{\partial t} + \frac{\partial u^i}{\partial q^\sigma}\dot{q}^\sigma = 0, \quad 1 \le i \le k\,,$$

and $\mathcal{C}^0 = \mathrm{span}\{\iota^*\pi^*_{1,0}du^i,\, 1 \le i \le k\}$. The physical constraint structure is (Q, Φ), where Φ is given by (4.1).

Distribution on Y. In this case the constraints are given by a distribution $\mathcal{D}$ of corank $k < m$ on Y which is not supposed to be completely integrable. They are called *simple non-holonomic* [8] or *linear nonintegrable.*

Denote $\mathcal{D}^0 = \mathrm{span}\{\eta^i = a^i\,dt + b^i_\sigma dq^\sigma, 1 \le i \le k\}$. Then in our notations,

$$\varphi^i = \pi^*_{1,0}\eta^i = (a^i + b^i_\sigma \dot{q}^\sigma)dt + b^i_\sigma \omega^\sigma\,, \quad f^i = a^i + b^i_\sigma \dot{q}^\sigma\,,$$

i.e., the constraint 1-forms are $\pi_{1,0}$-projectable (the corresponding constraint distribution projects onto $\mathcal{D}$). The geometrical constraint structure on J^1Y is $(Q, \mathcal{C})$, where Q is locally defined by

$$a^i + b^i_\sigma \dot{q}^\sigma = 0, \quad 1 \le i \le k\,,$$

and $\mathcal{C}^0 = \mathrm{span}\{\iota^*\pi^*_{1,0}\eta^i,\, 1 \le i \le k\}$. The physical constraint structure is (Q, Φ), where

$$\Phi = \varphi^i \wedge \mu_i = \mu_{0i}\, b^i_\sigma\, \omega^\sigma \wedge dt\,.$$

5. Constrained systems

The point of view of physics: systems deformed by constraints. Consider on J^1Y a constraint structure (Q, Φ), where Q is a (fibered over Y) submanifold of codimension $k < m$ and Φ is a corresponding constraint force defined in a neighborhood of Q. Let $[\alpha]$ be a mechanical system on J^1Y, E its dynamical form. Putting

$$\alpha_\Phi = \alpha - \Phi\,,$$

we get a new mechanical system $[\alpha_\Phi]$ defined in a neighborhood of Q, called a *deformation* of $[\alpha]$ by Φ [8]. If $E = (A_\sigma + B_{\sigma\nu}\ddot{q}^\nu)dq^\sigma \wedge dt$, the $f^i = 0$ are local equations of the submanifold Q, and Φ is a Chetaev force (3.8) then

$$\alpha_\Phi = \Big(A_\sigma - \mu_{i0}\,\frac{\partial f^i}{\partial \dot{q}^\sigma}\Big)\omega^\sigma \wedge dt + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu\,.$$

The deformed mechanical system is regular if and only if the original mechanical system is regular.

Deformed dynamical distributions are defined by means of the 1-forms

$$\Big(A_\sigma - \mu_{i0}\,\frac{\partial f^i}{\partial \dot q^\sigma}\Big)dt + 2F_{\sigma\nu}\omega^\nu + B_{\sigma\nu}d\dot q^\nu\,, \quad B_{\sigma\nu}\omega^\nu\,.$$

The *deformed equations of motion* are $J^1\gamma^* i_\xi \alpha_\Phi = 0$ for every π_1-vertical vector field ξ on J^1Y. In fiber coordinates,

$$f^i \circ J^1\gamma = 0, \quad A_\sigma + B_{\sigma\nu}\ddot q^\nu = \mu_{i0}\,\frac{\partial f^i}{\partial \dot q^\sigma} \quad \text{along } J^2\gamma\,.$$

The point of view of geometry: constrained systems as mechanical systems on constraint submanifolds. With help of the canonical distribution $\mathcal{C}$ one can obtain an intrinsic description of mechanical systems constrained to a submanifold Q of J^1Y.

Let $[\alpha]$ be a mechanical system, and let $(Q,\mathcal{C})$ be a constraint structure on J^1Y. Denote by ι the embedding of Q into J^1Y, and by $\mathcal{I}(\mathcal{C}^0)$ the constraint ideal on Q. In keeping with [8], for $\alpha \in [\alpha]$ put

$$\alpha_Q = \iota^*\alpha \mod \mathcal{I}(\mathcal{C}^0)\,.$$

Denote by $[\alpha_Q]$ the mechanical system generated by the class α_Q. It is easy to see that if $\alpha_1,\, \alpha_2 \in [\alpha]$ then $[(\alpha_1)_Q] = [(\alpha_2)_Q]$. The class $[\alpha_Q]$ is called the *constrained system* related to the mechanical system $[\alpha]$ and the constraint structure $(Q,\mathcal{C})$.

Computing the coordinate expression in normal coordinates we get a representative

$$\alpha_Q = \sum_{l=1}^{m-k} A'_l \omega^l \wedge dt + \sum_{l,s=1}^{m-k} B'_{ls}\omega^l \wedge d\dot q^s\,,$$

where

$$\begin{aligned} A'_l = \Big(A_l &+ \sum_{p=1}^{k} A_{m-k+p}\,\frac{\partial g^p}{\partial \dot q^l} \\ &+ \sum_{s=1}^{k}\Big(B_{l,m-k+s} + \sum_{j=1}^{k} B_{m-k+j,m-k+s}\,\frac{\partial g^j}{\partial \dot q^l}\Big)\Big(\frac{\partial g^s}{\partial t} + \frac{\partial g^s}{\partial q^\sigma}\dot q^\sigma\Big)\Big) \circ \iota\,, \\ B'_{ls} = \Big(B_{ls} &+ \sum_{r=1}^{k}\Big(B_{l,m-k+r}\,\frac{\partial g^r}{\partial \dot q^s} + B_{m-k+r,s}\,\frac{\partial g^r}{\partial \dot q^l}\Big) \\ &+ \sum_{r,j=1}^{k} B_{m-k+j,m-k+r}\,\frac{\partial g^j}{\partial \dot q^l}\,\frac{\partial g^r}{\partial \dot q^s}\Big) \circ \iota \end{aligned}$$

(see also [8]). Now, the equations of motion have the following form:

Proposition 2. *A section γ of π is a path of the constrained system $[\alpha_Q]$ if and only if $J^1\gamma$ is an integral section of the canonical distribution $\mathcal{C}$, and for every π_1-vertical vector field $\xi \in \mathcal{C}$ it satisfies the equation*

$$J^1\gamma^* i_\xi \alpha_Q = 0 \, ,$$

where α_Q is (any) 2-form belonging to $[\alpha_Q]$.

In fiber coordinates this gives the following system of $m-k$ second order ODE, and k first order ODE for the components $\gamma^1, \dots \gamma^m$ of γ:

$$f^i \circ J^1\gamma = 0 \, , \quad \Big(A'_l + \sum_{s=1}^{m-k} B'_{ls} \ddot{q}^s \Big) \circ J^2\gamma = 0 \, .$$

Now, a *constrained dynamical distribution* Δ_{α_Q} is a subdistribution of the canonical distribution $\mathcal{C}$, generated by means of the 1-forms $i_\xi \alpha_Q$, where ξ runs over all π_1-vertical vector fields on Q belonging to $\mathcal{C}$.

In analogy with the unconstrained case, the constrained system is called *regular* if the equivalence class $[\Delta_{\alpha_Q}]$ contains a representative of rank 1. In this case, by Theorem 1, the class $[\Delta_{\alpha_Q}]$ contains only one element—a constraint semispray distribution, and regularity is equivalent with the condition that the $(m-k) \times (m-k)$-matrix (B'_{sl}) is regular. Denote by (B'^{sl}) the inverse matrix. Apparently, $\Delta_{\alpha_Q} = \operatorname{span} \Gamma_Q$, where Γ_Q is given by (3.11) with $\Gamma^l = B'^{ls} A'_s$, and we can see that

$$i_{\Gamma_Q} \alpha_Q \in \mathcal{C}^0 \, .$$

Notice that a constrained system of a regular mechanical system need not be regular.

Let $[\alpha]$ be a *Lagrangian system*. By Theorem 2 this means that there exists a (unique) 2-form $\alpha_E \in [\alpha]$ such that $d\alpha_E = 0$. If λ is a Lagrangian and θ_λ its Cartan form, we have (at least locally) $\alpha_E = d\theta_\lambda$. Unfortunately, a corresponding constraint 2-form $(\alpha_E)_Q = (d\theta_\lambda)_Q$ need not be closed. However, by Theorem 2, we have

Theorem 4. *A constrained system $[\alpha]$ is Lagrangian if and only if the class $[\alpha_Q]$ contains a closed representative.*

Corollary. *Let $(Q, \mathcal{C})$ be a non-holonomic constraint structure on J^1Y. A constrained system arising from a Lagrangian system on J^1Y is Lagrangian if and only if there exists a 2-form $\beta \in \mathcal{I}(C^0)$ and a 2-contact 2-form F on Q such that $d(\beta + F) = 0$.*

In particular, if the ideal $\mathcal{I}(C^0)$ is closed then every constrained system arising from a Lagrangian system on J^1Y is Lagrangian.

Thus, trivially, Lagrangian systems subject to *holonomic* or *semiholonomic* (=affine integrable non-holonomic) constraints remain Lagrangian.

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