DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 511–523

A NOTE TO THE REPRESENTATION OF THE VARIATIONAL SEQUENCE IN MECHANICS

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ABSTRACT. The problem of the representation of the physically important terms in the variational sequence in the higher order mechanics (Euler-Lagrange and Helmholtz-Sonin mappings) is completely solved in the local case. The physically adequate representation of classes of 1–forms, 2–forms and 3–forms is found. Moreover, the conditions for recognizing the forms which play the role of representatives of classes in the variational sequence are given. Finally, the complete reconstruction of classes of forms based on the knowledge of the representative is presented.

1. Introduction

The most effective tools for studying the local and global aspects of the calculus of variations are on the one hand the *variational bicomplex* constructed on infinite jet prolongation of a fibered manifold (Vinogradov [13], Takens [11], Tulczyjew [12], Anderson and Duchamp [1]) and its finite order counterpart – the *variational sequence* on the other hand (Krupka [3–7]). The last mentioned concept is rather adequate in situations corresponding to practical problems of calculus of variations or in physics.

The variational sequence is constructed as the quotient sequence of the well-known De Rham sequence defined on the finite jet prolongation (J^rY, π^r, X) of an underlying fibered manifold (Y, π, X), with respect to its subsequence defined by contact forms which is chosen by such a way that both the Euler-Lagrange and Helmholtz-Sonin mappings are contained in the quotient sequence as its quotient

1991 *Mathematics Subject Classification.* 49F05, 58A15, 58E99.

Key words and phrases. Fibered manifold, r–jet prolongation, variational sequence, Lagrange structure, Euler–Lagrange mapping, Helmholtz–Sonin mapping.

Research supported by the Grant No. 201/98/0853 of the Czech Grant Agency, and Project No. VS 96003 ("Global Analysis") of the Ministry of Education, Youth and Sports of the Czech Republic.

This paper is in final form and no version of it will be submitted for publication elsewhere.

mappings. The theory developed for the $1-$dimensional or $n-$dimensional ($n > 1$) base X is called the *higher order mechanics* or the *higher order field theory*, respectively.

The general concept of the variational sequence was presented and investigated from the point-of-view of its global aspects by Krupka in his earlier works (see e.g. [3]). There occured a question of the representation of the variational sequence, i.e. the question of replacing the sequence of modules of classes of forms, by a suitable sequence of representing spaces of forms themselves. Their choice is, of course, not unique. Such a representation was obtained again by Krupka [6,7] for the first order mechanics. The problem of the higher order mechanics was solved independently by Musilová [8] and Štefánek [9,10]. The results presented in [8] are obtained for classes of $1-$forms, $2-$forms and $3-$forms and they show that these classes can be represented by such a way that the corresponding mappings of spaces of representatives are directly the Euler-Lagrange mapping and the Helmholtz-Sonin mapping. (The completely analogous conclusions are made by Krupka [3] in higher order field theory.) The problem of $k-$forms with $k > 3$ is not discussed in [8]. In [9] and [10] a case of the complete representation of the variational sequence in higher order mechanics is presented, the Euler–Lagrange mapping being directly contained in the sequence of representatives as well. However, the Helmholtz-Sonin mapping is there replaced by the equivalent mapping only (i.e. the exterior derivatives of $2-$forms are not represented by Helmholtz–Sonin forms in the commonly accepted sense). The problem of the reconstruction of classes of forms from given representatives is not solved and questions concerned the globality are not answerred in [8,9,10]. (For the first order mechanics the problem of globality is solved in [7].)

In our paper the chart expressions of the physically appropriate (global for $k = 1, 2$) representatives of the physically meaningfull terms ($1, 2, 3-$forms) in the variational sequence in higher order mechanics is presented. The Euler-Lagrange and Helmholtz-Sonin mappings considered in the standard sense are directly contained in the representing sequence. Moreover, the necessary and sufficient conditions for recognizing the representatives are studied and the complete reconstruction of classes of forms from the chart expressions of the known representatives is performed. All necessary definitions and chart expressions of geometrical objects used in the paper are adapted for the practical purposes.

2. Underlying structures and basic notations. Throughout the paper we use the standard notation commonly used in higher order mechanics (see e.g. Krupka [6,7]): (Y, π, X) is a *fibered manifold* with the $1-$*dimensional base* X and *projection* π. We denote $m = \dim Y - 1$. For an arbitrary integer $r > 0$, (J^rY, π^r, X) is the $r-$*jet prolongation* of (Y, π, X), π^r and $\pi^{r,s}$ for $r > s \geq 0$ are the *canonical projections* of J^rY on X and J^sY, respectively, $N = N_r = \dim J^rY = 1 + m(r + 1)$. Let $1 \leq \sigma \leq m$ and (V, ψ), $\psi = (t, y^\sigma)$, be a *fibered chart* on Y. Then we denote (U, φ) and (V^r, ψ^r) the *associated chart* on X and the *associated fibered chart* on J^rY, respectively. Here $U = \pi(V)$, $\varphi = (t)$, $V^r = (\pi^{r,0})^{-1}(V)$, $\psi^r = (t, y^\sigma, y^\sigma_1, \dots, y^\sigma_r)$. Let $\Omega^r_0 V$ be the ring of smooth functions on V^r. We denote by $\Omega^r_k V$ the Ω^r_0-*module of smooth differential $k-$forms* on V^r, $\Omega^r_{k,c}V \subset \Omega^r_k V$ the submodule of $k-$*contact* $k-$*forms* and $\mathrm{d}\Omega^r_{k-1,c}V \subset \Omega^r_k V$ the subset of exterior derivatives of $(k-1)-$contact $(k-1)-$forms. $\Omega^r_k V$, $\Omega^r_{k,c}V$ and $\mathrm{d}\Omega^r_{k-1,c}V$ are, of course, Abelian groups of forms. Let $\Theta^r_k V = \mathrm{d}\Omega^r_{k-1,c}V + \Omega^r_{k,c}V$ for $k > 1$ and $\Theta^r_1 V = \Omega^r_{1,c}V$. In addition, we denote by $\omega^\sigma_j = \mathrm{d}y^\sigma_j - y^\sigma_{j+1}\,\mathrm{d}t$, $0 \le j \le r-1$, and $\omega^\sigma_0 = \omega^\sigma$ the contact 1−forms. It holds $\omega^\sigma_j \wedge \mathrm{d}t = -\mathrm{d}\omega_{j-1}$ for $1 \le j \le r-1$. Let $k \ge 1$. We remind that the $k-$form $\varrho \in \Omega^r_k V$ is called *contact* if $J^r\gamma^*\varrho = 0$ for any section γ of the fibered manifold Y. For $k \ge 2$ any $k-$form is contact. A form ϱ is called *horizontal* or $0-$*contact* if its contraction $i_\xi\,\varrho$ by a π^r-vertical vector field ξ on J^rY is the identically zero form. A contact 1−form is called $1-$*contact* $1-$*form.* Let $k \ge 2$. A $k-$form ϱ is called $q-$*contact*, $k-1 \le q \le k$, if the $(k-1)-$form $i_\xi\,\varrho$ is $(q-1)-$contact, ξ being the π^r-vertical vector field on J^rY.

3. Decompositions of differential forms. As we discuss here the local properties of the variational sequence only, we use the shortened notation $\Omega^r_k V = \Omega^r_k$, $\Theta^r_k V = \Theta^r_k$, etc. Any $k-$form $\varrho \in \Omega^r_k$ can be generated by forms $(\mathrm{d}t, \mathrm{d}y^\sigma_j)$, $0 \le j \le r$, or, alternatively, by forms $(\mathrm{d}t, \omega^\sigma_j, \mathrm{d}y^\sigma_r)$, $0 \le j \le r-1$, generating the so-called *contact base.* For our purposes the contact base is more convenient:

$$\varrho = \frac{1}{(k-1)!} A^{j_1\dots j_{k-1}}_{\sigma_1\dots\sigma_{k-1}} \omega^{\sigma_1}_{j_1} \wedge \dots \wedge \omega^{\sigma_{k-1}}_{j_{k-1}} \wedge \mathrm{d}t +$$

$$+ \sum_{1\le l\le k-1} \frac{1}{l!(k-1-l)!} A^{r\dots\dots r j_{l+1}\dots j_{k-1}}_{\sigma_1\dots\sigma_l\sigma_{l+1}\dots\sigma_{k-1}} \mathrm{d}y^{\sigma_1}_r \wedge\dots\wedge \mathrm{d}y^{\sigma_l}_r \wedge \omega^{\sigma_{l+1}}_{j_{l+1}} \wedge\dots\wedge \omega^{\sigma_{k-1}}_{j_{k-1}} \wedge \mathrm{d}t +$$

$$+\frac{1}{k!} B^{j_1\dots j_k}_{\sigma_1\dots\sigma_k} \mathrm{d}y^{\sigma_1}_{j_1} \wedge \dots \wedge \mathrm{d}y^{\sigma_k}_{j_k} + \tag{1}$$

$$+ \sum_{1\le l\le k} \frac{1}{l!(k-l)!} B^{r\dots\dots r j_{l+1}\dots j_k}_{\sigma_1\dots\sigma_l\sigma_{l+1}\dots\sigma_k} \mathrm{d}y^{\sigma_1}_r \wedge \dots \wedge \mathrm{d}y^{\sigma_l}_r \wedge \omega^{\sigma_{l+1}}_{j_{l+1}} \wedge \dots \wedge \omega^{\sigma_k}_{j_k} \;,$$

where $0 \le \sigma_q \le m$, $0 \le j_q \le r-1$, $1 \le q \le k$. The coefficients $A^{r\dots\dots r j_{l+1}\dots j_{k-1}}_{\sigma_1\dots\sigma_l\sigma_{l+1}\sigma_{k-1}}$, $B^{r\dots\dots r j_{l+1}\dots j_k}_{\sigma_1\dots\sigma_l\sigma_{l+1}\dots\sigma_k} \in \Omega^r_0$, $0 \le l \le k$, are completely antisymmetrical in multiindices $\binom{j_q}{\sigma_q}$, $l+1 \le q \le k$, and $\binom{r}{\sigma_p}$, $1 \le p \le l$.

There exists the unique decomposition

$$(\pi^{r+1,r})^*\varrho = p_{k-1}\varrho + p_k\varrho \tag{2}$$

of a form $\varrho \in \Omega^r_k$ into its $(k-1)-$*contact component* $p_{k-1}\varrho \in \Omega^{r+1}_k$ and $k-$*contact component* $p_k\varrho \in \Omega^{r+1}_{k,c}$. It holds

$$p_{k-1}\varrho = P^{j_1\dots j_{k-1}}_{\sigma_1\dots\sigma_{k-1}} \omega^{\sigma_1}_{j_1} \wedge \dots \wedge \omega^{\sigma_{k-1}}_{j_{k-1}} \wedge \mathrm{d}t \;,$$

$$P^{j_1\dots j_{k-1}}_{\sigma_1\dots\sigma_{k-1}} = \frac{1}{(k-1)!} \left(A^{j_1\dots j_l j_{l+1}\dots j_{k-1}}_{\sigma_1\dots\sigma_l\sigma_{l+1}\dots\sigma_{k-1}} + B^{j_1\dots j_l j_{l+1}\dots j_{k-1} r}_{\sigma_1\dots\sigma_l\sigma_{l+1}\dots\sigma_{k-1}\sigma_k} y^{\sigma_k}_{r+1} \right)_A \;, \tag{3}$$

(4) $$p_k\varrho = \frac{1}{k!}(B^{j_1\dots j_k}_{\sigma_1\dots\sigma_k})_A\,\omega^{\sigma_1}_{j_1}\wedge\dots\wedge\omega^{\sigma_k}_{j_k}\,,\ 0\le j_q\le r\,,\ 1\le q\le k\,.$$

where $(\,)_A$ denotes the antisymmetrization in all multiindices. The coefficients $P^{j_1\dots j_{k-1}}_{\sigma_1\dots\sigma_{k-1}}\in\Omega^{r+1}_0$ are then antisymmetrical in multiindices $\binom{j_q}{\sigma_q}$, $0\le q\le r$.

For $k=1$ we denote by $p_0\varrho = h\varrho$ the *horizontal component* of the form ϱ. The mapping $h:\Omega^r_1\ni\varrho\to h\varrho\in\Omega^{r+1}_{1,X}$ is called *horizontalization*, $\Omega^{r+1}_{1,X}$ being the submodule of all horizontal 1−forms of the $(r+1)$−th order. It is evident that a k−form ϱ is k−contact if and only if it holds $(\pi^{r+1,r})^*\varrho = p_k\varrho$, i.e. ϱ is of the form $\varrho=\frac{1}{k!}B^{j_1\dots j_k}_{\sigma_1\dots\sigma_k}\omega^{\sigma_1}_{j_1}\wedge\dots\wedge\omega^{\sigma_k}_{j_k}$, $0\le j_q\le r-1$, $1\le q\le k$, $1\le\sigma_q\le m$. A k−form is $(k-1)$−contact if and only if $(\pi^{r+1,r})^*\varrho = p_{k-1}\varrho$ and thus $\varrho = \frac{1}{(k-1)!}A^{j_1\dots j_{k-1}}_{\sigma_1\dots\sigma_{k-1}}\,\omega^{\sigma_1}_{j_1}\wedge\dots\wedge\omega^{\sigma_{k-1}}_{j_{k-1}}\wedge\mathrm{d}t$, $0\le j_q\le r-1$, $1\le\sigma_q\le m$, $1\le q\le k-1$.

4. Variational sequence. Now let us construct the variational sequence. Recall that $N=m(r+1)+1=\dim J^rY$ and denote $P=mr+1$ the index of the last non-trivial subgroup Θ^r_P. Consider the well–known *De Rham sequence*

(5) $$0\to\mathcal{R}\to\Omega^r_0\to\Omega^r_1\to\dots\to\Omega^r_N\to 0\,.$$

First two arrows represent the canonical injections and the remaining ones are the exterior derivatives d. The sequence (5) is, of course, exact. Let $\varrho\in\Theta^r_kV$. Then there exists the unique decomposition $\varrho=\varrho_0+\mathrm{d}\varrho'$, $\varrho_0\in\Omega^r_{k,c}$, $\varrho'\in\Omega^r_{k-1,c}$, see [6,7]. Thus, it can be immediately seen that the subsequence

(6) $$0\to\Theta^r_1\to\Theta^r_2\to\dots\to\Theta^r_P\to 0$$

of the De Rham sequence (5) is exact as well, the first arrow being the canonical injection and all remaining ones represent the exterior derivatives. It is evident that the corresponding quotient sequence

(7) $$0\to\mathcal{R}\to\Omega^r_0\to\Omega^r_1/\Theta^r_1\to\dots\to\Omega^r_P/\Theta^r_P\to\Omega^r_{P+1}\to\dots\to\Omega^r_N\to 0$$

is also exact. It is called the *variational sequence of the r−th order*. The *quotient mappings*

$$E^r_k:\Omega^r_k/\Theta^r_k\ni[\varrho]\to E^r_k([\varrho])\in\Omega^r_{k+1}/\Theta^r_{k+1}$$

for $1\le k\le N$ and $E^r_0:\Omega^r_0\ni f\to E^r_0(f)\in\Omega^r_1/\Theta^r_1$ are defined by

(8) $$E^r_k([\varrho]) = [\mathrm{d}\varrho]$$

where $[\varrho]$ denotes the class of the form ϱ. The following diagram shows the complete scheme of the variational sequence.

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & & & & \\
 & & 0 \arrow[r, "d"] \arrow[d] & \Theta_1^r \arrow[r, "d_1"] \arrow[d] & \Theta_2^r \arrow[r, "d_2"] \arrow[d] & \cdots \arrow[r, "d_{P-1}"] & \Theta_P^r \arrow[r, "d"] \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathcal{R} \arrow[r] & \Omega_0^r \arrow[r, "d_0"] \arrow[dr, "E_0^r"] & \Omega_1^r \arrow[r, "d_1"] \arrow[d, "F_1^r"] & \Omega_2^r \arrow[r, "d_2"] \arrow[d, "F_2^r"] & \cdots \arrow[r, "d_{P-1}"] & \Omega_P^r \arrow[r, "d_P"] \arrow[d, "F_P^r"'] & \Omega_{P+1}^r \arrow[r, "d_{P+1}"] & \cdots \arrow[r, "d_N"] & \Omega_N^r \arrow[r] & 0 \\
 & & & \Omega_1^r/\Theta_1^r \arrow[r, "E_1^r"] \arrow[d] & \Omega_2^r/\Theta_2^r \arrow[r, "E_2^r"] \arrow[d] & \cdots \arrow[r, "E_{P-1}^r"] & \Omega_P^r/\Theta_P^r \arrow[ur, "E_P^r"] \arrow[d] & & & & \\
 & & & 0 & 0 & & 0 & & & &
\end{tikzcd}
$$

The general properties of the variational sequence as well as its global properties resulting from its cohomology are discussed elsewhere (see e.g. Krupka [3,6,7]). Our main purpose is to obtain the explicite coordinate formulas for appropriately chosen representatives of classes $[\varrho]$ or $E_k^r([\varrho])$ and thus to construct the representing sequence of the variational sequence.

5. Representation of the variational sequence. Two forms $\varrho, \omega \in \Omega_k^r$ are called *equivalent* (we write $\varrho \sim \omega$) if $\varrho - \omega \in \Theta_k^r$. The relation $\sim$ is the equivalence relation on Ω_k^r. The corresponding decomposition defines the quotient group Ω_k^r/Θ_k^r of the group Ω_k^r with respect to the (normal) subgroup Θ_k^r. Let us introduce the following notions: The *class generated by the form* ϱ is $[\varrho] = \{\omega \in \Omega_k^r \,|\, \omega \sim \varrho\}$, the *representative of the class* $[\varrho]$ is an arbitrary form $\omega \in [\varrho]$. The mapping $F_k^r : \Omega_k^r \ni \varrho \longrightarrow F_k^r(\varrho) = [\varrho] \in \Omega_k^r/\Theta_k^r$ is called the *factorization.* Any mapping $\Phi_k^r : \Omega_k^r/\Theta_k^r \ni [\varrho] \longrightarrow \varrho_0 \in [\varrho] \subset \Omega_k^r$ which assigns to every class $[\varrho]$ some its representative is called the *representation* of the quotient group Ω_k^r/Θ_k^r. The representation is, of course, not unique. Any representation is injective.

The forms $\varrho \in \Omega_k^r$ and $\omega \in \Omega_k^t$ are called *equivalent in the generalized sense* if there exists an integer $s \geq r, t$ for which $(\pi^{s,r})^*\varrho - (\pi^{s,t})^*\omega \in \Theta_k^s$. We then write $\varrho \approx \omega$. It is evident that the forms ϱ and ω for which $\varrho \sim \omega$ are equivalent in the generalized sense too. Let $\varrho \approx \omega$, i.e. $\eta = (\pi^{s,r})^*\varrho - (\pi^{s,t})^*\omega \in \Theta_k^s$. Then ϱ and ω can be understood as equivalent in the usual sense on Ω_k^s. We can define the corresponding quotient group Ω_k^s/Θ_k^s in which we shall work with classes $[\varrho]$ generated by forms $\varrho \in (\pi^{s,r})^*(\Omega_k^r)$, i.e. by $\pi^{s,r}-$projectable forms. The concept of the *generalized reprezentation* is closely related to the generalized equivalence. Let r be an integer. Any mapping $\overline{\Phi}_k^{s,r} : \Omega_k^r/\Theta_k^r \ni [\varrho] \longrightarrow \overline{\Phi}_k^{s,r}([\varrho]) = \varrho_0 \in \Omega_k^s$ for which the relation $\varrho_0 \approx \omega$ hold for any $\omega \in [\varrho]$ is called the *generalized representation* of the quotient group Ω_k^r/Θ_k^r. Any generalized representation is also injective.

Let us now present the following theorem which ensures the possibility to construct the generalized representations from the usual ones by the uniquely defined way. The theorem shows that the $\pi^{r+1,r}-$projectability of a sum $\varrho = \vartheta_c + \mathrm{d}\theta_c \in \Theta_k^{r+1}$, where $\vartheta_c \in \Omega_{k,c}^{r+1}$ and $\theta_c \in \Omega_{k-1,c}^{r+1}$ implies the $\pi^{r+1,r}-$projectability of both forms ϑ_c and θ_c:

Theorem 1. *Let $\varrho \in \Theta_k^{r+1}$ be $\pi^{r+1,r}-$projectable, i.e. it is of the form $\varrho = (\pi^{r+1,r})^*\eta$. Then $\eta \in \Theta_k^r$.*

For the proof we refer the reader to the papers [6,7]. Theorem 1. has some important consequences: Let us take into account the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \Theta_k^{r+1} \arrow[r, "\iota_k^{r+1}"] & \Omega_k^{r+1} \arrow[r, "F_k^{r+1}"] & \Omega_k^{r+1}/\Theta_k^{r+1} \arrow[r] & 0 \\
0 \arrow[r] & \Theta_k^{r} \arrow[r, "\iota_k^{r}"'] \arrow[u, "(\pi^{r+1,r})^*"] & \Omega_k^{r} \arrow[r, "F_k^{r}"'] \arrow[u, "(\pi^{r+1,r})^*"] & \Omega_k^{r}/\Theta_k^{r} \arrow[r] \arrow[u, "Q_k^{r,r+1}"] & 0
\end{tikzcd}
$$

in which ι_k^{r+1}, ι_k^r are the canonical injections, F_k^{r+1}, F_k^r are the factorizations and $Q_k^{r,r+1} : \Omega_k^r/\Theta_k^r \ni [\varrho] \longrightarrow Q_k^{r,r+1}([\varrho]) \in \Omega_k^{r+1}/\Theta_k^{r+1}$ is the quotient mapping

$$Q_k^{r,r+1}([\varrho]) = F_k^{r+1}((\pi^{r+1,r})^*\varrho) = [(\pi^{r+1,r})^*\varrho] \;\Leftrightarrow\; [\varrho] = F_k^r(\varrho)\ , \ \textit{i.e.}$$

$$Q_k^{r,r+1} \circ F_k^r \;=\; F_k^{r+1} \circ (\pi^{r+1,r})^* \quad .$$

Corollary 1. The quotient mapping $Q_k^{r,r+1}$ is injective.

Proof: As the mapping $Q_k^{r,r+1}$ is linear it is sufficient to prove that $\operatorname{Ker} Q_k^{r,r+1} = 0_{\Omega_k^r/\Theta_k^r}$. Let $Q_k^{r,r+1}([\varrho]) = 0_{\Omega_k^{r+1}/\Theta_k^{r+1}}$. Then $[(\pi^{r+1,r})^*\varrho] = F_k^{r+1} \circ (\pi^{r+1,r})^*\varrho = 0_{\Omega_k^{r+1}/\Theta_k^{r+1}}$ and, according to the theorem 1, $\varrho \in \Theta_k^r$. $\square$

By the completely analogous way we can define the mappings $Q^{r,s} : \Omega_k^r/\Theta_k^r \ni [\varrho] \longrightarrow Q_k^{r,s}([\varrho]) = [(\pi^{s,r})^*\varrho] \in \Omega_k^s/\Theta_k^s$, i.e. $Q_k^{r,s} \circ F_k^r = F_k^s \circ (\pi^{s,r})^*$, which is, of course, also injective.

Corollary 2. Let (Φ_k^s) be a family of representations of quotient groups Ω_k^s/Θ_k^s. Denote

$$\overline{\Phi}_k^{s,r} : \Omega_k^r/\Theta_k^r \ni [\varrho] \to \overline{\Phi}_k^{s,r}([\varrho]) = \Phi_k^s \circ Q_k^{r,s}([\varrho]) \in \Omega_k^s \ . \tag{9}$$

Then every mapping $\overline{\Phi}_k^{s,r}$ is the generalized representation of Ω_k^r/Θ_k^r.

$$\overline{F}_k^{s,r} = \overline{\Phi}_k^{s,r} \circ F_k^r \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \Theta_k^r & \longrightarrow & \Omega_k^r & \xrightarrow{F_k^r} & \Omega_k^r/\Theta_k^r & \longrightarrow & 0 \\ & & & & {\scriptstyle \overline{F}_k^{s,r}}\downarrow & \swarrow {\scriptstyle \overline{\Phi}_k^{s,r}} & & & \\ & & & & \Omega_k^s & & & & \end{array}$$

Let $(E_k^r)_{k=0,1,\dots,P}$ be the representation of the $r-$th order variational sequence and let $(\overline{\Phi}_k^{s,r})$ be the family of generalized representations of quotient groups Ω_k^r/Θ_k^r. The following diagram defines the *associated generalized representation* $(\overline{E}_k^{s,r})_{k=0,\dots,P}$ of the given variational sequence.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{E_{k-1}^r} & \Omega_k^r/\Theta_k^r & \xrightarrow{E_k^r} & \Omega_{k+1}^r/\Theta_{k+1}^r & \xrightarrow{E_{k+1}^r} & \cdots \\ & \overline{\Phi}_k^{s,r} & \downarrow & & \downarrow & \overline{\Phi}_{k+1}^{s,r} & \\ \cdots & \xrightarrow[\overline{E}_{k-1}^{s,r}]{} & \Omega_k^s & \xrightarrow[\overline{E}_k^{s,r}]{} & \Omega_{k+1}^s & \xrightarrow[\overline{E}_{k+1}^{s,r}]{} & \cdots \end{array}$$

For the *associated* mappings $\overline{E}_k^{s,r} : \overline{\Phi}_k^{s,r}(\Omega_k^r/\Theta_k^r) \longrightarrow \Omega_{k+1}^s$, $1 \le k \le P$, it holds

$$\overline{E}_k^{s,r} = \overline{\Phi}_{k+1}^{s,r} \circ E_k^r \circ (\overline{\Phi}_k^{s,r})^{-1} \quad \text{and} \quad \overline{E}_0^{s,r} = \overline{\Phi}_1^{s,r} \circ E_0^r \, .$$

Now we can formulate the *representation problem*: For a given integer r we look for the appropriate integer s and such a generalized representation $\overline{\Phi}_k^{s,r}$ for which the representatives $\overline{\Phi}_k^{s,r}([\varrho])$ of classes $[\varrho] \in \Omega_k^r/\Theta_k^r$, can be identified with the corresponding terms in the variational sequence having the usual physical meaning. Especially, it would be convenient to represent the class of forms containing a $r-$order Lagrangian by the Lagrangian itself (up to a projection at most), the class containing the exterior derivative of the Lagrangian by the corresponding Euler-Lagrange form, and the class containing a dynamical form $\mathcal{E} = E_\sigma \omega^\sigma \wedge \mathrm{d}t \in \Omega_2^t, t \le 2r$ by its Helmholtz-Sonin form the coefficient of which are the left-hand-sides of Helmholtz-Sonin variationality conditions:

$$(10) \qquad \overline{\Phi}_k^{s,r}([\lambda]) = (\pi^{s,r})^* \lambda \, , \; \overline{\Phi}_k^{s,r}([\mathrm{d}\lambda]) = \mathcal{E}_\lambda \, , \; \overline{\Phi}_k^{s,t}([\mathrm{d}\mathcal{E}]) = \mathcal{H}_\lambda \, .$$

Moreover, we wish find the globally defined representatives. Thus the procedure leading to the construction of the representatives must give the resulting chart expressions connected by coordinate transformation formulas in various fibered charts.

6. Physical representation. We have shown in our previous work [8] that the following representatives of quotient groups Ω_k^r/Θ_k^r for $k = 1, 2, 3$ have the required physical meaning specified by the relations (10). The results agree with those of Krupka [6,7] obtained for the first order mechanics.

(a) $k=1:$ $\quad \omega = A\,\mathrm{d}t + B^j_\sigma\,\omega^\sigma_j + B^r_\sigma \mathrm{d}y^\sigma_r\,,\ 0\le j\le r-1,\ A, B^j_\sigma, B^r_\sigma \in \Omega^r_0$

$$h\omega = p_0\omega = (A + B^r_\sigma\, y^\sigma_{r+1})\mathrm{d}t = P\ \mathrm{d}t\ ,\ P\in\Omega^{r+1}_0$$

$$\overline{\Phi}^{r+1,r}_1([\omega]) = P\ \mathrm{d}t \in \Omega^{r+1}_1 \tag{11}$$

(b) $k=2:$

$$\omega = A^j_\sigma \omega^\sigma_j \wedge \mathrm{d}t + A^r_\sigma \mathrm{d}y^\sigma_r \wedge \mathrm{d}t +$$

$$\frac{1}{2}B^{jl}_{\sigma\nu}\omega^\sigma_j \wedge \omega^\nu_l + B^{rj}_{\sigma\nu}\mathrm{d}y^\sigma_r \wedge \omega^\nu_j + \frac{1}{2}B^{rr}_{\sigma\nu}\ \mathrm{d}y^\sigma_r \wedge \mathrm{d}y^\nu_r$$

$$p_1\omega = (A^j_\sigma + B^{jr}_{\sigma\nu}\ y^\nu_{r+1})\,\omega^\sigma_j \wedge \mathrm{d}t = P^j_\sigma\,\omega^\sigma_j \wedge \mathrm{d}t\ ,\ P^j_\sigma \in \Omega^{r+1}_0$$

$$\overline{\Phi}^{2r+1,r}_2([\omega]) = \left(\sum_{j=0}^{r}(-1)^j \frac{\mathrm{d}^j P^j_\sigma}{\mathrm{d}t^j}\right)\ \omega^\sigma_0 \wedge \mathrm{d}t \tag{12}$$

(c) $k=3:$ $\omega = \frac{1}{2}A^{jl}_{\sigma\nu}\omega^\sigma_j \wedge \omega^\nu_l \wedge \mathrm{d}t + A^{jr}_{\sigma\nu}\omega^\sigma_j \wedge \mathrm{d}y^\nu_r \wedge \mathrm{d}t + \frac{1}{2}A^{rr}_{\sigma\nu}\mathrm{d}y^\sigma_r \wedge \mathrm{d}y^\nu_r \wedge \mathrm{d}t +$

$$\frac{1}{3!}B^{jkl}_{\sigma\nu\lambda}\omega^\sigma_j \wedge \omega^\nu_k \wedge \omega^\lambda_l + \frac{1}{2}B^{jkr}_{\sigma\nu\lambda}\omega^\sigma_j \wedge \omega^\nu_k \wedge \mathrm{d}y^\lambda_r +$$

$$\frac{1}{2}B^{jrr}_{\sigma\nu\lambda}\omega^\sigma_j \wedge \mathrm{d}y^\nu_r \wedge \mathrm{d}y^\lambda_r + \frac{1}{3!}B^{rrr}_{\sigma\nu\lambda}\mathrm{d}y^\sigma_r \wedge \mathrm{d}y^\nu_r \wedge \mathrm{d}y^\lambda_r$$

$$p_2\omega = P^{jl}_{\sigma\nu}\omega^\sigma_j \wedge \omega^\nu_l \wedge \mathrm{d}t\ ,\ P^{jl}_{\sigma\nu} = \frac{1}{2}\left(A^{jl}_{\sigma\nu} + B^{jkr}_{\sigma\nu\lambda}q^\lambda_{r+1}\right)_A\ ,\ P^{jl}_{\sigma\nu} = -P^{lj}_{\nu\sigma} \in \Omega^{r+1}_0$$

$$\overline{\Phi}^{2r+1,r}_3([\omega]) = \underbrace{\left(\sum_{j=0}^{r}\sum_{l=0}^{r}(-1)^l \binom{l}{j}\frac{\mathrm{d}^{l-j}}{\mathrm{d}t^{l-j}}P^{k-j,l}_{\sigma\nu}\right)}_{H^k_{\sigma\nu}}\ \omega^\sigma_k \wedge \omega^\nu_0 \wedge \mathrm{d}t \tag{13}$$

For $k=1,2$ the obtained representations (11) and (12) are global. The question of the globality of the representation (13) for $k=3$ is not yet completely answerred, with the exception of the case $r=1$ for which a global representation is presented in [7].

7. Class reconstruction. The problem of the class reconstruction can be formulated by the following way: Let the system of $(\overline{\Phi}^{s,r}_k)$ of mappings (9) is the generalized representation of quotient groups (Ω^r_k/Θ^r_k) and let the form $\varrho_0 \in \Omega^s_k$ be the representative of the class $[\varrho] \in \Omega^r_k/\Theta^r_k$. The injectivity of mappings $\overline{\Phi}^{s,r}_k$ ensures the possibility for obtaining the class $[\varrho]$ in the following form

$$[\varrho] = \overline{\varrho} + \Theta^r_k$$

where the form $\overline{\varrho} \in \Omega^r_k$ is given exclusively by the coefficients of the representative ϱ_0. We now formulate the conditions for the identification of representatives of the "physical" representation (10) defined in the previous section as well as the relations for reconstructed classes.

Theorem 2. *Let $(\overline{\Phi}_k^{s,r})_{k=1,2,3}$ be the physically appropriate generalized representations of the quotient groups $(\Omega_k^r/\Theta_k^r)_{k=1,2,3}$ defined by (10).*

(a) *Let $k=1$. The form $\varrho_0 \in \Omega_1^s$ is the representative of a class of the quotient group Ω_1^r/Θ_1^r if and only if it is of the form $\varrho_0 = P\,\mathrm{d}t$ where $P \in \Omega_0^{r+1}$ (i.e. $s=r+1$) is the polynomial $P = A + B_\sigma^r y_{r+1}^\sigma$, $A, B_\sigma^r \in \Omega_0^r$. The reconstructed class satisfying the relation $\varrho_0 = \overline{\Phi}_1^{s,r}([\varrho])$ is*

$$[\varrho] = A\,\mathrm{d}t + B_\sigma^r\,\mathrm{d}y_r^\sigma + \Theta_1^r\,. \tag{14}$$

(b) *Let $k=2$. The form $\varrho_0 \in \Omega_2^s$ is the representative of a class of the quotient group Ω_2^r/Θ_2^r if and only if it is of the form $\varrho_0 = F_\sigma\,\omega_0^\sigma \wedge \mathrm{d}t$ where $F_\sigma \in \Omega_0^{2r+1}$ (i.e. $s=2r+1$) is the polynomial*

$$F_\sigma = \alpha_\sigma + \beta_{\sigma\nu}^{(0)} y_{r+1}^\nu + \cdots + \beta_{\sigma\nu}^{(r)} y_{2r+1}^\nu +$$
$$+\sum_{p=2}^{r+1} \sum_{(j_1+\ldots+j_p \leq r, 1\leq j_q \leq r)} \gamma_{\sigma\nu_1\ldots\nu_p}^{j_1\ldots j_p} y_{r+j_1}^{\nu_1} \ldots y_{r+j_p}^{\nu_p} \quad , \quad \alpha_\sigma,\, \beta_{\sigma\nu}^{(k)},\, \gamma_{\sigma\nu_1\ldots\nu_p}^{j_1\ldots j_p} \in \Omega_0^r$$

where the coefficients $\gamma_{\sigma\nu_1\ldots\nu_p}^{j_1\ldots j_p}$ are unambiguously given by α_σ, $\beta_{\sigma\nu}^{(k)}$. The reconstructed class $[\varrho] \in \Omega_2^r/\Theta_2^r$ satisfying the relation $\varrho_0 = \overline{\Phi}_2^{s,r}([\varrho])$ is as follows

$$[\varrho] = \alpha_\sigma\omega_0^\sigma \wedge \mathrm{d}t + \sum_{j=0}^{r-1} \left(\sum_{k=j}^{r} \binom{k}{j} (-1)^k \frac{\mathrm{d}'^{k-j}}{\mathrm{d}t^{k-j}} \beta_{\sigma\nu}^{(k)} \right) \omega_j^\sigma \wedge \mathrm{d}y_r^\nu + \tag{15}$$
$$+\frac{1}{2}(-1)^r \beta_{\sigma\nu}^{(r)}\;\mathrm{d}y_r^\sigma \wedge \mathrm{d}y_r^\nu + \Theta_2^r$$

where $\frac{\mathrm{d}'f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_{j=0}^{r-1} \frac{\partial f}{\partial y_j^\sigma} y_{j+1}^\sigma$ for an integer t and a function $f \in \Omega_0^r$.

(c) *Let $k=3$. The form $\varrho_0 \in \Omega_3^s$ is the representative of a class of the quotient group Ω_3^r/Θ_3^r if an only if it is of the form $\varrho_0 = H_{\sigma\nu}^k \omega_k^\sigma \wedge \omega_0^\nu \wedge \mathrm{d}t \in \Omega_3^s$ where $H_{\sigma\nu}^k \in \Omega_0^{2r+1}$ (i.e. $s=2r+1$) are the polynomials*

$$H_{\sigma\nu}^k = \alpha_{\sigma\nu}^k + \sum_{l=0}^{r} \beta_{\sigma\nu\lambda}^{k(l)} y_{r+1+l}^\lambda + \sum_{p=2}^{r+1} \sum_{(j_1+\ldots+j_p \leq r, 1\leq j_q \leq r)} \gamma_{\sigma\nu\lambda_1\ldots\lambda_p}^{j_1\ldots j_p} y_{r+j_1}^{\lambda_1} \ldots y_{r+j_p}^{\lambda_p}$$
$$\alpha_{\sigma\nu}^k,\, \beta_{\sigma\nu\lambda}^{k(l)},\, \gamma_{\sigma\nu\lambda_1\ldots\lambda_p}^{j_1\ldots j_p} \in \Omega_0^r \quad , \quad 0 \leq k \leq 2r$$

Moreover, the functions $\gamma_{\sigma\nu\lambda_1\ldots\lambda_p}^{j_1\ldots j_p}$ are unambiguously given by $\alpha_{\sigma\nu}^k$ and $\beta_{\sigma\nu\lambda}^{k(l)}$. The reconstructed class $[\varrho] \in \Omega_3^r/\Theta_3^r$ satisfying the relation $\varrho_0 = \overline{\Phi}_3^{s,r}([\varrho])$ is

$$[\varrho] = D_{\sigma\nu}^{j0} \omega_j^\sigma \wedge \omega_0^\nu \wedge \mathrm{d}t + D_{\sigma\nu}^{jr} \omega_j^\sigma \wedge \mathrm{d}y_r^\nu \; wedge \mathrm{d}t + D_{\sigma\nu}^{rr} \mathrm{d}y_r^\sigma \wedge \mathrm{d}y_r^\nu \wedge \mathrm{d}t +$$

$$
(16)\qquad +\sum_{\sigma=1}^{m}\left(\sum_{j=0}^{r-2}F_\sigma^j\omega_j^\sigma\wedge\omega_{j+1}^\sigma\wedge\mathrm{d}t+F_\sigma^{r-1}\omega_{r-1}^\sigma\wedge\mathrm{d}y_r^\sigma\wedge\mathrm{d}t\right)+
$$

$$
+E_{\sigma\nu\lambda}^{jkr}\omega_j^\sigma\wedge\omega_k^\nu\wedge\mathrm{d}y_r^\lambda+E_{\sigma\nu\lambda}^{jrr}\omega_j^\sigma\wedge\mathrm{d}y_r^\nu\wedge\mathrm{d}y_r^\lambda+\frac{1}{3}E_{\sigma\nu\lambda}^{rrr}\mathrm{d}y_r^\sigma\wedge\mathrm{d}y_r^\nu\wedge\mathrm{d}y_r^\lambda+\Theta_3^r
$$

where the coefficients $D_{\sigma\nu}^{jk}$, F_σ^j, $E_{\sigma\nu\lambda}^{jkr}\in\Omega_0^r$ are expressed through $\alpha_{\sigma\nu}^k$, $\beta_{\sigma\nu\lambda}^{k(l)}$ by the following relations

$$
D_{\sigma\nu}^{j0}=\alpha_{\sigma\nu}^j\quad\text{for }\sigma\neq\nu\ ,\ 0\leq j\leq r-1\ ,\ D_{\sigma\sigma}^{j0}=0
$$

$$
D_{\sigma\nu}^{jr}=\sum_{k=j}^{r}(-1)^k\binom{k}{j}\frac{\mathrm{d}'^{(k-j)}}{\mathrm{d}t^{k-j}}\alpha_{\sigma\nu}^{r+k}\quad,\ \sigma\neq\nu\ ,\ 0\leq j\leq r\ ,\ D_{\sigma\sigma}^{jr}=0
$$

$$
F_\sigma^j=(-1)^{j+1}\alpha_{\sigma\sigma}^{2r-1}\quad,\ 0\leq j\leq r-1
$$

$$
(17)\ E_{\sigma\nu\lambda}^{jkr}=\left[\sum_{p=0}^{r-k}\sum_{i=0}^{p}(-1)^{p+k}\binom{k+p}{p}\binom{p}{i}\frac{\mathrm{d}'^{(p-i)}}{\mathrm{d}t^{p-i}}\Gamma_{\sigma\nu\lambda}^{j-i,(k+p)}\right]_A\quad,\ 0\leq j,k\leq r
$$

$$
\Gamma_{\sigma\nu\lambda}^{k(l)}=\beta_{\sigma\nu\lambda}^{k(l)}-\delta_{\sigma\nu\lambda}^{k(l)}\ ,\ 0\leq k\leq 2r\ ;\ \delta_{\sigma\nu\lambda}^{k(l)}=0\ ,\ \sigma\neq\nu\ ,\ 0\leq l\leq r\ ,\ 0\leq k\leq r-1
$$

$$
\delta_{\sigma\nu\lambda}^{k(l)}=\sum_{i=1}^{2r-k-l}\binom{k+l-r-i}{k-r}\frac{\partial\alpha_{\sigma\nu}^{k+l+i}}{\partial y_{r+1-i}^\lambda}\quad,\ \sigma\neq\nu\ ,\ r\leq k\leq 2r
$$

$$
\delta_{\sigma\sigma\lambda}^{k(l)}=\frac{1}{2}\sum_{i=1}^{k}\sum_{p=0}^{k-2i+1}\binom{k-i+1}{i}\binom{k-2i+1}{p}\frac{\mathrm{d}'^{p}}{\mathrm{d}t^{p}}\left(\frac{\partial\alpha_{\sigma\sigma}^{2r-1}}{\partial y_{r+l-k+2i+p}^\lambda}\right)\ ,\ 0\leq k\leq 2r
$$

All terms contained in these summations which are not well defined by their usual definitios are taken as zeros. Note that because of the complexity of the relations expressing the coefficients $\gamma_{\sigma\nu_1\ldots\nu_p}^{j_1\ldots j_p}$ though α_σ and $\beta_{\sigma\nu}^{(k)}$ for $k=2$ and the coefficients $\gamma_{\sigma\nu\lambda_1\ldots\lambda_p}^{j_1\ldots j_p}$ through $\alpha_{\sigma\nu}^k$ and $\beta_{\sigma\nu\lambda}^{k(l)}$ for $k=3$ only the illustration will be given by the example.

Proof: The proof of theorem 2 is very technical and we emphasize here only the key steps leading to the expressions for reconstructed classes.

(a) Let $k=1$ and $\varrho_0=P\,\mathrm{d}t$, $P\in\Omega_0^s$. Suppose that $\varrho_0=h\varrho$, $\varrho=A\mathrm{d}t+B_\sigma^j\omega_j^\sigma+B_\sigma^r\mathrm{d}y_r^\sigma$, A, B_σ^j, $B_\sigma^r\in\Omega_0^r$, $1\leq j\leq r-1$, i.e. $[\varrho]=A\mathrm{d}t+B_\sigma^r y_{r+1}^\sigma+\Theta_1^r$. Then $\varrho_0=(A+B_\sigma^r y_{r+1}^\sigma)\mathrm{d}t$, i.e. $P=(A+B_\sigma^r y_{r+1}^\sigma)\in\Omega_0^{r+1}$. Then $B_\sigma^r=\frac{\partial P}{\partial y_{r+1}^\sigma}$, $A=P-\frac{\partial P}{\partial y_{r+1}^\sigma}y_{r+1}^\sigma\in\Omega_0^r$.

(b) Let $k=2$ and $\varrho_0=F_\sigma\omega_0^\sigma\wedge\mathrm{d}t\in\Omega_2^s$. Suppose that ϱ is of the form

$$
\varrho=A_\sigma^j\omega_j^\sigma\wedge\mathrm{d}t+A_\sigma^r\mathrm{d}y_r^\sigma\wedge\mathrm{d}t+\frac{1}{2}B_{\sigma\nu}^{jk}\omega_j^\sigma\wedge\omega_\nu^k+B_{\sigma\nu}^{jr}\omega_j^\sigma\wedge\mathrm{d}y_r^\nu+\frac{1}{2}B_{\sigma\nu}^{rr}\mathrm{d}y_r^\sigma\wedge\mathrm{d}y_r^\nu
$$

with $0 \le j \le r-1$ and with all usual antisymmetries of coefficients. Then

$$p_1 \varrho = (A^j_\sigma + B^{jr}_{\sigma\nu} y^\nu_{r+1}) \omega^\sigma_j \wedge \mathrm{d}t, \; 0 \le j \le r.$$

The form ϱ_0 is the representative of the class $[\varrho]$ if and only if F_σ is given by (12):

$$F_\sigma = \sum_{j=0}^{r} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} (A^j_\sigma + B^{jr}_{\sigma\nu} y^\nu_{r+1}) \, .$$

By the direct calculation in which we use the relation

$$\frac{\mathrm{d}^j(XY)}{\mathrm{d}t^j} = \sum_{q=0}^{j} \frac{\mathrm{d}^q X}{\mathrm{d}t^q} \frac{\mathrm{d}^{j-q} Y}{\mathrm{d}t^{j-q}} \quad , \quad X, Y \quad \text{functions}$$

which can be easily proved by induction with respect to j, we obtain the polynomial form of the coefficients F_σ as well as the reconstructed class.

(c) Let $k=3$ and $\varrho_0 = H^k_{\sigma\nu} \omega^\sigma_k \wedge \omega^\nu_0 \wedge \mathrm{d}t \in \Omega^s_3$. The fact that the coefficients $H^k_{\sigma\nu}$ are polynomial is the consequence of the relations (13) for the representation $\overline{\Phi}{}^{s,r}_3$. Suppose them to be of the form prescribed in theorem 2. Denote

$$\eta = \frac{1}{2} A^{jk}_{\sigma\nu} \omega^\sigma_k \wedge \omega^\nu_k \wedge \mathrm{d}t + A^{jr}_{\sigma\nu} \omega^\sigma_j \wedge \mathrm{d}y^\nu_r \wedge \mathrm{d}t + \frac{1}{2} A^{rr}_{\sigma\nu} \mathrm{d}y^\sigma_r \wedge \mathrm{d}y^\nu_r \wedge \mathrm{d}t +$$
$$+ \frac{1}{3!} B^{jkl}_{\sigma\nu\lambda} \omega^\sigma_j \wedge \omega^\nu_k \wedge \omega^\lambda_l + \frac{1}{2} B^{jkr}_{\sigma\nu\lambda} \omega^\sigma_j \wedge \omega^\nu_k \wedge \mathrm{d}y^\lambda_r +$$
$$+ \frac{1}{2} B^{jrr}_{\sigma\nu\lambda} \omega^\sigma_j \wedge \mathrm{d}y^\nu_r \wedge \mathrm{d}y^\lambda_r + \frac{1}{3!} B^{rrr}_{\sigma\nu\lambda} \mathrm{d}y^\sigma_r \wedge \mathrm{d}y^\nu_r \wedge \mathrm{d}y^\lambda_r \quad , \quad 0 \le j,k,l \le r-1 \, .$$

Let

$$\theta = \frac{1}{2} \mathrm{d}(C^{jk}_{\sigma\nu} \omega^\sigma_j \wedge \omega^\nu_k) \in \mathrm{d}\Theta^r_2 \quad , \quad 0 \le j,k \le r-1 \, .$$

It is evident that the form η is equivalent to $\varrho = \eta - \theta$ for arbitrarily chosen coefficients $C^{jk}_{\sigma\nu} \in \Omega^r_0$. By the direct calculation we obtain the appropriate choice of the coefficients $C^{jk}_{\sigma\nu}$

$$C^{p,r-1-q}_{\sigma\nu} = \sum_{i=0}^{r-1} \sum_{j=0}^{q-i} \binom{q}{j} (-1)^{i+j} \frac{\mathrm{d}'^j}{\mathrm{d}t^j} A^{p-i,r-q+i+j}_{\sigma\nu}$$

for which $\varrho = \eta - \theta$ is given by the relations (17) presented in theorem 2. □

8. Example $r=1$**.** In the following example we present all conditions for coefficients of representatives of classes $(\Omega^1_k)_{k=1,2,3}$ and the reconstructed classes.

(a) $k=1$: $\quad \varrho_0 = (\alpha + \beta_\sigma y^\sigma_2) \mathrm{d}t \; , \; \alpha \, , \beta_\sigma \in \Omega^1_0 \quad , \quad [\varrho] = \alpha \mathrm{d}t + \beta_\sigma \mathrm{d}y^\sigma_1 + \Theta^1_1$

(b) $k=2$: $\quad \varrho_0 = (\alpha_\sigma + \beta^{(0)}_{\sigma\nu} y^\nu_2 + \beta^{(1)}_{\sigma\nu} y^\nu_3 + \gamma_{\sigma\nu\lambda} y^\nu_2 y^\lambda_2) \, \omega^\sigma_0 \wedge \mathrm{d}t$

$$\alpha_\sigma \, , \, \beta^{(0)}_{\sigma\nu} \, , \, \beta^{(1)}_{\sigma\nu} \, , \, \gamma_{\sigma\nu\lambda} \in \Omega^1_0 \quad , \quad \beta^{(1)}_{\sigma\nu} = -\beta^{(1)}_{\nu\sigma} \; ; , \; \gamma_{\sigma\nu\lambda} = \gamma_{\nu\sigma\lambda}$$

$$\gamma_{\sigma\nu\lambda} = -\frac{1}{2}\left[\left(\frac{\partial\beta^{(1)}_{\sigma\nu}}{\partial y^\lambda_1} + \frac{\partial\beta^{(1)}_{\sigma\lambda}}{\partial y^\nu_1}\right)\right]_{S(\sigma\nu)}$$

$$[\varrho] = \alpha_\sigma\omega^\sigma_0 \wedge \mathrm{d}t + \left(\beta^{(0)}_{\sigma\nu} - \frac{\mathrm{d}'\beta^{(1)}_{\sigma\nu}}{\mathrm{d}t}\right)\omega^\sigma_0 \wedge \,\mathrm{d}y^\nu_1 + \frac{1}{2}\beta^{(1)}_{\sigma\nu}\,\mathrm{d}y^\sigma_1 \wedge \mathrm{d}y^\nu_1 + \Theta^1_2$$

(c) $k = 3:\quad \varrho_0 = H^0_{\sigma\nu}\omega^\sigma_0 \wedge \omega^\nu_0 \wedge \mathrm{d}t + H^1_{\sigma\nu}\omega^\sigma_1 \wedge \omega^\nu_0 \wedge \mathrm{d}t + H^2_{\sigma\nu}\omega^\sigma_2 \wedge \omega^\nu_0 \wedge \mathrm{d}t$

$$H^0_{\sigma\nu} = \alpha^0_{\sigma\nu} + \beta^{0(0)}_{\sigma\nu\lambda}y^\lambda_2 + \beta^{0(1)}_{\sigma\nu\lambda}y^\lambda_3 + \gamma^0_{\sigma\nu\lambda\mu}y^\lambda_2 y^\mu_2$$

$$H^1_{\sigma\nu} = \alpha^1_{\sigma\nu} + \beta^{1(0)}_{\sigma\nu\lambda}y^\lambda_2 + \beta^{1(1)}_{\sigma\nu\lambda}y^\lambda_3 + \gamma^1_{\sigma\nu\lambda\mu}y^\lambda_2 y^\mu_2$$

$$H^2_{\sigma\nu} = \alpha^2_{\sigma\nu} + \beta^{2(0)}_{\sigma\nu\lambda}y^\lambda_2 + \beta^{2(1)}_{\sigma\nu\lambda}y^\lambda_3 + \gamma^2_{\sigma\nu\lambda\nu}y^\lambda_2 y^\mu_2$$

$$\alpha^k_{\sigma\nu},\ \beta^{k(l)}_{\sigma\nu\lambda},\ \gamma^k_{\sigma\nu\lambda\mu} \in \Omega^1_0$$

$$H^0_{\sigma\nu} = -H^0_{\nu\sigma} \Rightarrow \alpha^0_{\sigma\nu},\ \beta^{0(0)}_{\sigma\nu\lambda},\ \beta^{0(1)}_{\sigma\nu\lambda},\ \gamma^0_{\sigma\nu\lambda\mu}\quad \text{antisymmetrical in } \sigma,\ \nu$$

$$\gamma^0_{\sigma\nu\lambda\mu} = \frac{1}{2}\left(\frac{\partial\beta^{0(1)}_{\sigma\nu\lambda}}{\partial y^\mu_1} + \frac{\partial\beta^{0(1)}_{\sigma\nu\mu}}{\partial y^\lambda_1}\right) \quad , \quad \gamma^1_{\sigma\nu\lambda\mu} = -\gamma^1_{\nu\sigma\lambda\mu}$$

$$\gamma^1_{\sigma\nu\lambda\mu} = \frac{1}{2}\left(\frac{\partial\beta^{1(1)}_{\sigma\nu\lambda}}{\partial y^\mu_1} + \frac{\partial\beta^{1(1)}_{\sigma\nu\mu}}{\partial y^\lambda_1}\right) \quad , \quad \gamma^2_{\sigma\nu\lambda\mu} = 0$$

$$\beta^{2(0)}_{\sigma\nu\lambda} = \beta^{1(1)}_{\sigma\nu\lambda}\ \text{ and antisymmetrical in } (\sigma\nu\lambda) \quad , \quad \beta^{2(1)}_{\sigma\nu\lambda} = 0$$

$$\frac{1}{2}(\alpha^1_{\sigma\nu} - \alpha^1_{\nu\sigma}) = \frac{\mathrm{d}'\alpha^2_{\sigma\nu}}{\mathrm{d}t} \quad , \quad \alpha^2_{\sigma\nu} = -\alpha^2_{\nu\sigma}$$

$$\frac{1}{2}(\beta^{1(0)}_{\sigma\nu\lambda} + \beta^{1(0)}_{\nu\sigma\lambda}) = 2(\beta^{0(1)}_{\lambda\sigma\nu} + \beta^{0(1)}_{\lambda\nu\sigma}) \quad , \quad \frac{1}{2}(\beta^{1(0)}_{\sigma\nu\lambda} - \beta^{1(0)}_{\nu\sigma\lambda}) = \frac{\partial\alpha^2_{\sigma\nu}}{\partial y^\lambda_1} + \frac{\mathrm{d}'\beta^{2(0)}_{\sigma\nu\lambda}}{\mathrm{d}t}$$

$$[\varrho] = \alpha^0_{\sigma\nu}\omega^\sigma_0 \wedge \omega^\nu_0 \wedge \mathrm{d}t + \frac{1}{2}(\alpha^1_{\sigma\nu} + \alpha^1_{\nu\sigma})\omega^\sigma_0 \wedge \mathrm{d}y^\nu_1 \wedge \mathrm{d}t - \alpha^2_{\sigma\nu}\mathrm{d}y^\sigma_1 \wedge \mathrm{d}y^\nu_1 \wedge \mathrm{d}t +$$
$$\left(\beta^{0(0)}_{\sigma\nu\lambda} - \frac{\mathrm{d}'\beta^{0(1)}_{\sigma\nu\lambda}}{\mathrm{d}t}\right)\omega^\sigma_0 \wedge \omega^\nu_0 \wedge \mathrm{d}y^\lambda_1 +$$
$$+\beta^{0(0)}_{\sigma\nu\lambda}\,\omega^\sigma_0 \wedge \mathrm{d}y^\nu_1 \wedge \mathrm{d}y^\lambda_1 - \frac{1}{3}\beta^{1(1)}_{\sigma\nu\lambda}\,\mathrm{d}y^\sigma_1 \wedge \mathrm{d}y^\nu_1 \wedge \mathrm{d}y^\lambda_1 + \Theta^1_3\,.$$

Acknowledgement. The authors wish to express their thanks to Prof. Demeter Krupka for his kind interest in their work and for many helpful discussions and comments.

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