DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 503–509

LOCAL EXPRESSIONS FOR POISSON MANIFOLDS OF GEODESIC ARCS IN LAGRANGIAN MECHANICS

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ABSTRACT. The local expressions for general Poisson manifolds of geodesic arcs are rather complicated. This paper shows that they can be simplified if the Poisson manifold is generated by some Lagrange function. Further, a second simplification is found by changing contravariant velocities to covariant velocities. Finally, local expressions for the corresponding Hamilton function and the corresponding linear connection are given.

1. Fundamental notions

In this paper notions of geodesics, Lagrangian mechanics, linear connections and Poisson manifolds are used in the usual sense. For more information, the reader is referred to [1], [4] and [5]. In all local expressions we use the standard summation convention.

Let us consider the closed interval $[0,1] \subset \mathbb{R}$, a smooth finite-dimensional manifold X, the tangent bundle TX, the canonical projection $\pi : TX \to X$, and a smooth symmetric linear connection Γ on TX. A geodesic $[0,1] \to X$ of the connection Γ is called a *geodesic arc* of the connection Γ. Let $W(X)$ be a set of all geodesic arcs of the connection Γ. It is well known that there exists a bijective mapping $\beta : W(X) \ni \gamma \to \dot{\gamma}(0) \in \operatorname{codom} \beta$, where $\dot{\gamma}$ is the prolongation of geodesic arc γ on tangent bundle TX and $\operatorname{codom} \beta \subset TX$ is the open subset containing the zero section. The set $W(X)$ equipped with a structure of smooth fibered manifold such that β is an isomorphism of smooth fibered manifolds is called a *manifold of geodesic arcs*.

Let M be a set of all polynomial mappings $[0,1] \to [0,1]$ whose degree is less than two. Then it is known that for any $\mu \in M$ there exists the smooth mapping $R_\mu : W(X) \ni \gamma \to \gamma \circ \mu \in W(X)$. Let us consider a Poisson algebra $C^\infty(W(X))$ over $W(X)$ such that for each $\mu \in M$ the mapping $C^\infty(W(X)) \ni f \to f \circ$

1991 *Mathematics Subject Classification.* 17B65, 53B05, 53C15, 53C22, 58D15, 58F05, 70H99.

Key words and phrases. Geodesic arcs, geodesics, Lagrangian mechanics, linear connections, manifolds of geodesic arcs, Poisson algebras, Poisson manifolds, symplectic geometry.

This paper is in final form and no version of it will be submitted for publication elsewhere.

$R_\mu \in C^\infty(W(X))$ is an endomorphism of $C^\infty(W(X))$. A *Poisson manifold of geodesic arcs* is a manifold of geodesic arcs $W(X)$ equipped with a Poisson algebra $C^\infty(W(X))$. General Poisson manifolds of geodesic arcs are the objects of the paper [3].

The objects of this paper are Poisson manifolds in Lagrangian mechanics. A mapping, whose codomain is $\mathbb{R}$, will be called a *function*. Let us consider a smooth regular Lagrange function L, where $\operatorname{dom} L \subset TX$ is an open submanifold equipped with the canonical symplectic structure. We say that a Poisson manifold of geodesic arcs *is generated* by a Lagrange function L if and only if

(1) $\operatorname{dom} L \subset \operatorname{codom} \beta$,

(2) $\pi(\operatorname{dom} L) = X$,

(3) $\beta^{-1}|_{\operatorname{dom} L}$ is an isomorphism of Poisson manifolds,

(4) $\beta^{-1}(\operatorname{dom} L)$ is a set of Lagrange's extremals.

Let f be a function such that $\operatorname{dom} f \subset TX$. We say that f is a *π-projectable function* if and only if there exists a function $\tilde{f}$ such that $f = \tilde{f} \circ \pi$. The function $\tilde{f}$ is said to be the *projection* of f.

2. Local expressions on dom L

Theorem 1. *Let $W(X)$ be a Poisson manifold of geodesic arcs generated by a Lagrange function L, H the corresponding Hamilton function, Γ the corresponding linear symmetric connection on TX, $n = \dim X$ the natural number, $U \subset \operatorname{dom} L \subset TX$ an open set with standard fiber coordinates*

$$x^1, x^2, \ldots, x^n, v^1, v^2, \ldots, v^n. \tag{5}$$

Then there exist π-projectable functions $\Gamma^i_{jk} = \Gamma^i_{kj}$, $g^{ij}_k = g^{ji}_k$, g_i on the domain U and a constant function c on the domain U such that

$$\{x^i, x^j\} = 0\,, \tag{6}$$

$$\{x^i, v^j\} = g^{ij}_k\, v^k\,, \tag{7}$$

$$\{v^i, v^j\} = \left(g^{ir}_k\, \Gamma^j_{lr} - g^{jr}_k\, \Gamma^i_{lr}\right) v^k\, v^l\,, \tag{8}$$

$$H = g_i\, v^i + c\,, \tag{9}$$

where projections of functions Γ^i_{jk} are components of connection Γ.

Proof. Since the connection Γ is symmetric, $\Gamma^i_{jk} = \Gamma^i_{kj}$. The canonical symplectic structure on U is defined by relations

$$\left\{x^i, x^j\right\} = 0\,, \tag{10}$$

$$\left\{x^i, \frac{\partial L}{\partial v^j}\right\} = \delta^i_j\,, \tag{11}$$

$$\left\{\frac{\partial L}{\partial v^i}, \frac{\partial L}{\partial v^j}\right\} = 0\,, \tag{12}$$

where δ^i_j is the Kronecker symbol. Using (4) we get

$$-\frac{\partial^2 L}{\partial v^i\,\partial v^j}\,\Gamma^j_{kl}\,v^k\,v^l + \frac{\partial^2 L}{\partial v^i\,\partial x^j}\,v^j - \frac{\partial L}{\partial x^i} = 0. \tag{13}$$

By definition,

$$H = v^i\,\frac{\partial L}{\partial v^i} - L. \tag{14}$$

From (10), (11), (12), (13), and (14) it follows that

$$\{x^i, H\} = v^i\,, \tag{15}$$

$$\{v^i, H\} = -\Gamma^i_{jk}\,v^j\,v^k\,. \tag{16}$$

From (1), (3), (10), and [3] we get (6), (7), (8), and $g^{ij}_k = g^{ji}_k$. From (6), (7), (11)

$$g^{ik}_l\,v^l\,\frac{\partial^2 L}{\partial v^k\,\partial v^j} = \delta^i_j\,, \tag{17}$$

and so $\partial^2 L/\partial v^k\,\partial v^j$ are homogeneous functions of degree -1 in v^i. Hence

$$v^i\,\frac{\partial^3 L}{\partial v^i\,\partial v^k\,\partial v^j} + \frac{\partial^2 L}{\partial v^k\,\partial v^j} = 0. \tag{18}$$

From (14), (18) we obtain $\partial^2 H/\partial v^k\,\partial v^j = 0$. This implies (9), where g_i, c are π-projectable functions. Substituting (9) into (16) and comparing the terms of the first degree in v^i on both sides, we get $\{v^i, c\} = 0$. Since c is a π-projectable function, by (6) it follows that $\{x^i, c\} = 0$. Because $\operatorname{dom} L$ is a symplectic manifold, c is a constant function. This completes the proof. $\square$

3. Local expressions on TX

Since formulas (6), (7), (8), (9), (15), (16) are polynomials in the variables v^i, according to (2) they can be extended to every standard fibered coordinate system. Because standard fibered coordinate systems make an atlas on TX, the structure of Poisson manifold and the domain of Hamilton function can be extended from $\operatorname{dom} L$ not only to $\operatorname{codom}\beta$ but also to the whole manifold TX. The manifolds $\operatorname{codom}\beta$ and TX, however, are not symplectic yet. Since according to [3] β is an isomorphism of Poisson manifolds, all coordinate expressions above can be transferred on $W(X)$ as well.

On the other hand, the domain of smooth Lagrange function L can not be extended to $\operatorname{codom}\beta$. Indeed, (17) does not hold on the zero section of TX. However, such an extension is not necessary — if (2) holds, then the Poisson algebras over TX and $W(X)$ are uniquely determined by the Lagrange function L.

4. Local expressions on *W(X)*

Adding a suitable constant to the Lagrange function L, we can obtain

$$c = 0 \tag{19}$$

on TX. We will assume without loss of generality that (19) holds. For simplicity, we will also identify $W(X)$ and $\operatorname{codom}\beta \subset TX$ in such a way that $\beta = \mathrm{id}_{W(X)}$.

Theorem 2. *Let $W(X)$ be a Poisson manifold of geodesic arcs generated by a Lagrange function L, H the corresponding Hamilton function, $\gamma \in W(X)$ a geodesic arc, $n = \dim X$ the natural number. Then there exists an open neighborhood $U \subset W(X)$ of the geodesic arc γ, fiber coordinates*

$$x^1, x^2, \dots, x^n, v_1, v_2, \dots, v_n, \tag{20}$$

on U, and π-projectable functions $g_i^{jk} = g_i^{kj}$, g^i on U such that

$$\{x^i, x^j\} = 0\,, \tag{21}$$

$$\{x^i, v_j\} = g_j^{ik}\, v_k\,, \tag{22}$$

$$\{v_i, v_j\} = \frac{1}{2}\left(\frac{\partial g_i^{kl}}{\partial x^j} - \frac{\partial g_j^{kl}}{\partial x^i}\right) v_k\, v_l\,, \tag{23}$$

$$H = g^i\, v_i\,. \tag{24}$$

Lemma. *Let $V \subset X$ be an open set, $U = \pi^{-1}(V) \cap W(X)$ the open set with standard fiber coordinates* (5)*, g_{ij} π-projectable functions on U such that $\det g_{ij} \neq 0$ for each point of U, $v_i = g_{ij}\, v^j$ functions on U. Then* (20) *are fiber coordinates on U satisfying relations* (21), (24), *and*

$$\{x^i, v_j\} = a_j^{ik}\, v_k\,, \tag{25}$$

$$\{v_i, v_j\} = \tfrac{1}{2}\, b_{ij}^{kl}\, v_k\, v_l\,, \tag{26}$$

where g^i, a_j^{ik}, $b_{ij}^{kl} = b_{ij}^{lk}$ are some π-projectable functions on U.

Proof. Since $\det g_{ij} \neq 0$, functions (20) make a coordinate system on U. If we denote by g^{ij} components of the solution of equations

$$g_{ij}\, g^{jk} = \delta_k^i\,, \tag{27}$$

then from (6), (7), (8), (9), (19) by a direct computation we get (21), (24), (25), (26), where

$$g^i = g^{ij} g_j\,,$$

$$a_j^{ik} = g_r^{il}\, g_{jl}\, g^{rk}\,,$$

$$b_{ij}^{kl} = g_r^{pq}\left[\left(\frac{\partial g_{is}}{\partial x^p} - \Gamma_{ps}^m g_{im}\right) g_{jq} - \left(\frac{\partial g_{js}}{\partial x^p} - \Gamma_{ps}^m g_{jm}\right) g_{iq}\right]\left(g^{rk} g^{sl} + g^{rl} g^{sk}\right).$$

We see that $b_{ij}^{kl} = b_{ij}^{lk}$. Since functions g_i, g_{ij}, g_k^{ij}, Γ_{jk}^i are π-projectable, functions g^i, a_k^{ij}, b_{ij}^{kl} are π-projectable too. This completes the proof. $\square$

Proof of Theorem 2. The proof is in two steps. Step 1: $\gamma \in \operatorname{dom} L$. Let us consider a standard fiber coordinate system on a neighborhood of γ. Since (17) implies $\det(\partial^2 L/\partial v^i\, \partial v^j) \neq 0$ at γ, there exists a chart neighborhood $U \subset \operatorname{dom} L \subset W(X)$ of γ with Legendre coordinates x^i, p_i, where $p_i = \partial L/\partial v^i$. From (10), (11), (12)

$$\{f, g\} = \frac{\partial f}{\partial x^k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial x^k}\frac{\partial f}{\partial p_k} \tag{28}$$

for each pair of smooth functions f, g on U. Further, adding suitable constants to the Legendre coordinates, we can obtain $x^i(\gamma) = 0$, $p_i(\gamma) = 0$. Then the inverse Legendre transformation can be expanded giving $v^i = g^i + g^{ij} p_j + \frac{1}{2}\, g^{ijk}\, p_j\, p_k + \frac{1}{6}\, g^{ijkl}\, p_j\, p_k\, p_l + \dots$, where g^i, g^{ij}, g^{ijk}, g^{ijkl}, ... are π-projectable functions on U and $\det g^{ij} \neq 0$ for each point of U. Denoting

$$v_i = g_{ij}\, v^j, \tag{29}$$

where g_{ij}'s make the solution of equations (27), and using Lemma, we get (21), (24), (25), (26), where

$$v_i = g_i + p_i + \tfrac{1}{2}\, g_i^{jk}\, p_j\, p_k + \tfrac{1}{6}\, g_i^{jkl}\, p_j\, p_k\, p_l + \dots, \tag{30}$$

$g_i^{jk} = g_i^{kj}$. Differentiating relations (25), (26) with respect to p_r, putting all $p_i = 0$, and using (28), (30) we obtain

$$g_j^{ir} = a_j^{ir}, \tag{31}$$

$$\frac{\partial g_i}{\partial x^k}\, g_j^{kr} - \frac{\partial g_j}{\partial x^k}\, g_i^{kr} = b_{ij}^{kr}\, g_k. \tag{32}$$

In the same way, differentiating (25), (26) with respect to p_r and p_s, we get

$$g_j^{irs} = a_j^{ik}\, g_k^{rs}, \tag{33}$$

$$\frac{\partial g_i^{rs}}{\partial x^j} + \frac{\partial g_i}{\partial x^k}\, g_j^{krs} - \frac{\partial g_j^{rs}}{\partial x^i} - \frac{\partial g_j}{\partial x^k}\, g_i^{krs} = b_{ij}^{rs} + b_{ij}^{kl}\, g_k\, g_l^{rs}. \tag{34}$$

Relations (31), (32), (33), (34) imply

$$\frac{\partial g_i^{rs}}{\partial x^j} - \frac{\partial g_j^{rs}}{\partial x^i} = b_{ij}^{rs}, \tag{35}$$

and relations (25), (26), (31), (35) imply (22), (23). This completes Step 1 of the proof.

Step 2: $\gamma \notin \operatorname{dom} L$. Then by (2) there exists a point $\overline{\gamma} \in \operatorname{dom} L \subset W(X)$ such that $\pi(\overline{\gamma}) = \pi(\gamma)$ and by Step 1 there exists an open neighborhood $\overline{U} \subset W(X)$ of $\overline{\gamma}$, fiber coordinates x^i, v_i on $\overline{U}$, and π-projectable functions $g_i^{jk} = g_i^{kj}$, g^i on $\overline{U}$

such that (21), (22), (23), (29) hold. Now, if we define $U = (\pi^{-1} \circ \pi(\overline{U})) \cap W(X)$, then $\gamma \in U$. Since the domain of standard fiber coordinates x^i, v^i is U and x^i, g_{ij} are π-projectable functions, the domain of coordinates x^i, v_i can be extended by (29) from $\overline{U}$ to the whole neighborhood U. Then by the Lemma (21), (24), (25), (26) hold on U. Since (22), (23) hold on $\overline{U}$ and a_k^{ij}, b_{ij}^{kl} are π-projectable functions, (22), (23) hold on the whole neighborhood U. This completes the proof of Theorem 2. □

5. Local expressions for connection Γ

Local expressions (21), (22), (23) for the Poisson algebra $C^\infty(W(X))$ are independent, in contradistinction of (6), (7), (8), on components of the connection Γ. But these components can be calculated using Hamilton function (24). From (15) we get

$$g^{ij}\, v_j = v^i, \tag{36}$$

where $g^{ij} = g_k^{ij}\, g^k$. Eliminating variables v^i from (16), (36), we obtain

$$\begin{aligned}\left[g_k^{jl}\Big(\frac{\partial g^{im}}{\partial x^j}\, g^k - \frac{\partial g^m}{\partial x^j}\, g^{ik}\Big) + \frac{1}{2}\Big(\frac{\partial g_j^{lm}}{\partial x^k} - \frac{\partial g_k^{lm}}{\partial x^j}\Big)\, g^{ij}\, g^k\right] v_l\, v_m \\ = -\Gamma^i_{jk}\, g^{jl}\, g^{km}\, v_l\, v_m.\end{aligned} \tag{37}$$

Hence, using $g_i^{jk} = g_i^{kj}$, we get

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1}{2}\, g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right) \\ &\quad - \frac{1}{2}\, g^{il}\, g_{jr}\, g_{ks}\left(g^m\, \frac{\partial g_l^{rs}}{\partial x^m} + \frac{\partial g^m}{\partial x^l}\, g_m^{rs} - \frac{\partial g^r}{\partial x^m}\, g_l^{ms} - \frac{\partial g^s}{\partial x^m}\, g_l^{rm}\right),\end{aligned} \tag{38}$$

where g_{ij}'s make the solution of equations (27).

Formula (38) was already proved in the paper [2], but the calculation given here is easier. It is readily seen that if the Lie derivative in the second term equals zero, then Γ is the Levi-Civita connection.

Acknowledgments

This work has been supported by Grant No. 201/98/0853 of the Czech Grant Agency, and Project No. VS 96003 "Global Analysis" of the Ministry of Education, Youth and Sports of the Czech Republic. The author is grateful for this support.

References

[1] Abraham, R., Marsden, J. E., *Foundations of Mechanics*, 2nd Ed., The Benjamin/Cummings Publ. Comp., Reading, 1978.

[2] Klapka, L., *Configuration in-out manifolds in mechanics*, In: Differential Geometry and Its Applications, Proc. Conf. Brno, Czechoslovakia, 1989 (World Scientific, Singapore, 1990) 336-340.

[3] Klapka, L., *Poisson manifolds of geodesic arcs*, In: Differential Geometry and Applications, Proc. Conf. Brno, Czech Republic, 1995 (Masaryk University, Brno, 1996) 603-610. Electronic version in ELibEMS at http://www.emis.de/proceedings/6ICDGA/.

[4] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry, Volume I*, Interscience Publishers, New York, 1963.

[5] Weinstein, A., *The local structure of Poisson manifolds*, J. Diff. Geom. **18** (1983) 523-577.

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