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GAUGE INVARIANT VARIATIONALLY TRIVIAL $U(1)$-PROBLEMS

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Abstract. A geometrical characterization of gauge invariant variationally trivial lagrangians on the bundle of connections of $U(1)$-principal bundles is given.

1. Introduction

The mathematical notion of connection can be seen from a physical point of view as the potential of a given field theory. Many of those theories come from variational principles over a principal fiber bundle $\pi : P \to M$, and therefore the bundle of connections $\mathcal{C}P$ arises as the natural place in which the action functional is defined. More precisely, if one is dealing with first order variational problems over a ground manifold M with fixed volume form v, a Lagrangian is a mapping from the 1-jet prolongation of the bundle $\mathcal{C}P$ to the real numbers.

On the other hand, the notion of gauge invariance plays an important role in field theories (in particular the theory of electromagnetism, with the structure group $U(1)$). The invariance of a variational functional means that for any gauge transformation $\Phi : P \to P$, the Euler-Lagrange form $\mathcal{E}(\mathcal{L})$ verifies (*cf.* [6])

$$\left(\Phi_{\mathcal{C}}^{(2)}\right)^* \mathcal{E}(\mathcal{L}) = \mathcal{E}((\Phi_{\mathcal{C}}^{(1)})^*\mathcal{L}) = \mathcal{E}(\mathcal{L}) \, ,$$

where $\Phi_{\mathcal{C}}^{(2)}$ stands for the 2-jet prolongation of the induced mapping $\Phi_{\mathcal{C}} : \mathcal{C}P \to \mathcal{C}P$ (see §2.2 below). The most usual way for obtaining this invariance is taking a Lagrangian such that $(\Phi_{\mathcal{C}})^*\mathcal{L} = \mathcal{L}$, or equivalently

$$L_{X_{\mathcal{C}}^{(1)}}\mathcal{L} = X_{\mathcal{C}}^{(1)}(\mathcal{L}) = 0 \, ,$$

for any gauge vector field X. An infinitesimal gauge transformation is a G-invariant vector field (that is, an infinitesimal automorphism on P) which is π-vertical. The set of all gauge vector fields is denoted by gau$P$. In fact, the flow of a vector field in gau$P$ is an one-parameter family of automorphisms of P inducing the identity over M. Thus, a Lagrangian density $\mathcal{L}$ is said to be gau (resp. aut) invariant if the Lie derivative of $\mathcal{L}v$ by the fundamental representation (see §2.2 below) of all gau (resp. aut) vector fields vanishes.

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One of the classical problems in the core of the variational calculus is that determining whether two Lagrangians $\mathcal{L}$, $\mathcal{L}'$ are variationally equivalent; that is, $\mathcal{L}$, and $\mathcal{L}'$ produce the same variational problem. This is equivalent to study the variational triviality of the difference $\mathcal{L} - \mathcal{L}'$. Although general conditions for this problem are known (see, for example [7]), the goal of this paper deals with the important subset of gauge invariant lagrangians over the bundle of connections of any $U(1)$-principal bundle. In this case we have a nice geometrical characterization, namely, one can establish an one-to-one correspondence between the set of gauge invariant varationally trivial Lagrangians and the set of 'divergence free' multivector fields, that is, sections of the bundle $\bigwedge^\bullet TM$ such that the Lie derivative of the volume form vanishes.

Notice that, when the bundle P is trivial (*i.e.*, $P = M \times U(1)$), the bundle of connections is canonically isomorphic to the cotangent bundle, thus characterizing all gauge invariant Lagrangians on T^*M which are variationally trivial.

2. Preliminaries

2.1. The bundle of connections. Let $\pi : P \to M$ a principal bundle with structure group G. The action of G on P induces a natural action on TP such that the quotient TP/G has the structure of vector bundle over M. The Atiyah sequence (see for example [1]) is the exact sequence of vector bundles over M

$$0 \to \mathrm{ad}P \to TP/G \to TM \to 0\,,$$

where $\mathrm{ad}P$ stands for the adjoint bundle $\mathrm{ad}P$; that is, the bundle associated to P by the adjoint representation of G on its Lie algebra $\mathfrak{g}$. It is well known that there exists a one-to-one correspondence between connections on P and splittings of the Atiyah sequence. Accordingly, the bundle of connections $p : \mathcal{C}P \to M$ is defined as the subbundle of $\mathrm{Hom}(TM, TP/G)$ determined by all $\mathbb{R}$-linear mappings $\lambda : T_xM \to (TP/G)_x$ such that $\pi_x \circ \lambda = \mathrm{Id}_{T_xM}$ (*c.f.* [4]). Connections on P can thus be identified to global sections of $p : \mathcal{C}P \to M$. Is not difficult to prove that $\mathcal{C}P$ is an affine bundle modelled over the vector bundle $\mathrm{Hom}(TM, \mathrm{ad}P) \cong T^*M \otimes \mathrm{ad}P$. A different approach for $\mathcal{C}P$ is the following (see for example [2]). Let $\pi_1 : J^1P \to M$ be the 1-jet bundle of local sections of π. The group G acts on J^1P by $j^1_x s \cdot g = j^1_x(R_g \circ s)$, where s is a section of π defined on a neighbourhood of $x \in M$, $g \in G$ and stands for the right translation. the quotient $(J^1P)/G$ exists as a fibered manifold over M and can be identified to the bundle of connections $\mathcal{C}P$.

2.2. The fundamental representation. An automorphism of a principal G-bundle is a diffeomorphism $\Phi : P \to P$ such that $\Phi \circ R_g = R_g \circ \Phi$ $\forall g \in G$, where R_g stands for the right translation by g on P. Every automorphism induces a unique diffeomorphism $\phi : M \to M$ such that $\pi \circ \Phi = \phi \circ \pi$. If ϕ is the identity map, then Φ is said to be a gauge transformation.

It is straightforward to check that a vector field on P is G-invariant if and only if its flow is a uniparametric group of aumorphisms (that is, it is an infinitesimal automorphism). The algebra of all G-invariant vector fields is denoted by $\mathrm{aut}P$. Infinitesimal gauge transformations are those G-invariant vector fields which are π_*-vertical, and accordingly, they are denoted by $\mathrm{gau}P$.

Each automorphism Φ acts on connections as follows. Let Γ be a connection on P and let ω_Γ be its connection form. Then, $\Gamma' = \Phi(\Gamma)$ is the connection corresponding to the connection form $\omega_{\Gamma'} = (\Phi^{-1})^* \omega_\Gamma$. For each Φ, this action induces a unique diffeomorphism

$$\Phi_{\mathcal{C}} : \mathcal{C}P \to \mathcal{C}P$$

such that $\Phi_{\mathcal{C}} \circ \sigma_\Gamma = \sigma_{\Phi(\Gamma)}$, where σ_Γ stands for the global section of $\mathcal{C}P$ induced by any connection Γ. In other words, we have the natural representation

$$\mathrm{Aut}P \to \mathrm{Diff}(\mathcal{C}P)$$

$$\Phi \mapsto \Phi_{\mathcal{C}} \, .$$

The derived representation of the algebra $\mathrm{aut}P$ of infinitesimal automorphisms is denoted by

$$\mathrm{aut}P \to \mathfrak{X}(\mathcal{C}P)$$

$$X \mapsto X_{\mathcal{C}} \, ,$$

and is called the 'fundamental representation'.

2.3. Coordinates. From now, consider that the structure group G is the abelian group $U(1)$.

Let $(V; q^1, ..., q^n)$ a coordinate open domain in M such that π is trivializable over it, that is, $\pi^{-1}(V) \cong V \times U(1)$. A vector field X on $\pi^{-1}(V)$ is $U(1)$-invariant if and only if it can be written as

$$X = f^i\left(q^1, \dots, q^n\right) \frac{\partial}{\partial q^i} + g\left(q^1, \dots, q^n\right) \cdot \xi, \quad f^i, g \in C^\infty(V)\, , \tag{1}$$

where ξ is the fundamental vector field on P defined by $i \in \mathfrak{u}(1) \cong i\mathbb{R}$. Note that each $U(1)$-invariant vector field X on P is π-projectable and its projection is given by

$$\pi_* X = f^i\left(q^1, \dots, q^n\right) \frac{\partial}{\partial q^i} \, .$$

In particular, X is π-vertical (and hence it is a gauge vector field) if and only if $f^i = 0$.

If $p : \mathcal{C}P \to M$ is the bundle of connections of P, a trivialization $\pi^{-1}(V) \cong V \times U(1)$ induces a coordinate system in $p^{-1}(V) \cong \mathcal{C}(V \times U(1))$ as follows. Every connection form on $V \times U(1)$ can be uniquely written as $\omega_\Gamma = (d\theta + \pi^*\omega) \otimes i$, where ω is an arbitrary 1-form on V and θ is the angle coordinate on $U(1)$. Therefore the bundle of connections $\mathcal{C}(V \times U(1))$ can be identified to T^*V and a natural coordinate system $(p^{-1}(V); q^1, \dots, q^n, p_1, \dots, p_n)$ is induced by taking $\omega = p_i(\omega)\left(\mathrm{d}q^i\right)_x$, for every covector $\omega \in T^*_x V$.

The fundamental representation is then given by ([5, III.B]):

$$X_{\mathcal{C}} = f^i \frac{\partial}{\partial q^i} - \left(\frac{\partial g}{\partial q^i} + \frac{\partial f^h}{\partial q^i} p_h \right) \frac{\partial}{\partial p_i} \, , \tag{2}$$

where the local expression of $X \in \mathrm{aut}P$ is given by the formula (1). In particular, for gauge vector fields we have

$$X_{\mathcal{C}} = -\frac{\partial g}{\partial q^i} \frac{\partial}{\partial p_i}, \quad X \in \mathrm{gau}P \, . \tag{3}$$

2.4. Jet prolongation. Let $p_1\colon J^1(\mathcal{C}P) \to M$ be the first jet prolongation of the bundle of connections. Each coordinate open domain $(V; q^1, \dots, q^n)$ in M induces a system of coordinates $(q^i, p_i; p_{i,j})$, $i, j = 1, \dots, n$, in $p_1^{-1}(V) \subseteq J^1(\mathcal{C}P)$, which are defined by

$$p_{i,j}\left(j_x^1\omega\right) = \frac{\partial\left(p_i \circ \omega\right)}{\partial q^j}(x),\ i, j = 1, \dots, n, \tag{4}$$

where $(p^{-1}(V);\, q^i, p_i)$ is the coordinate system on the bundle of connections introduced in §2.3. Given the coordinate expression (formula (2)) of the fundamental representation, from general formulas for jet prolongation we obtain

$$\begin{aligned} X_{\mathcal{C}}^{(1)} = {} & f^i \frac{\partial}{\partial q^i} - \left(\frac{\partial g}{\partial q^i} + \frac{\partial f^h}{\partial q^i} p_h\right) \frac{\partial}{\partial p_i} \\ & - \left(\frac{\partial^2 g}{\partial q^i \partial q^j} + \frac{\partial f^h}{\partial q^i} p_{h,j} + \frac{\partial f^h}{\partial q^j} p_{i,h} + \frac{\partial^2 f^h}{\partial q^i \partial q^j} p_h\right) \frac{\partial}{\partial p_{i,j}}, \end{aligned} \tag{5}$$

which is the jet prolongation of the algebra of infinitesimal automorphisms.

2.5. Lie derivative with respect to a multi-vector field. We recall (*cf.* [3, p.79]) that a decomposable multi-vector field $\chi = X_1 \wedge \ldots \wedge X_k \in \bigwedge^k \mathfrak{X}(M)$, induces a $C^\infty(M)$-linear graded endomorphism $i_\chi\colon\ \Omega^\bullet(M) \to \Omega^\bullet(M)$ of degree $-k$ by setting

$$i_\chi \omega_r = i_{X_1} \circ \ldots \circ i_{X_k} \omega_r \in \Omega^{r-k}(M), \quad \omega_r \in \Omega^r(M). \tag{6}$$

For an arbitrary multi-vector field χ we can define i_χ by extending the above formula by linearity. Then the Lie derivative $\mathrm{L}_\chi\colon\ \Omega^\bullet(M) \to \Omega^\bullet(M)$ with respect to $\chi \in \bigwedge^k \mathfrak{X}(M)$ is given by the formula

$$\mathrm{L}_\chi = i_\chi \circ \mathrm{d} - (-1)^k \mathrm{d} \circ i_\chi = [i_\chi, \mathrm{d}]. \tag{7}$$

We remark that L_χ is a graded operator of degree $-k+1$.

3. Invariant Lagrangians

3.1. Gauge invariance. A Lagrangian function $\mathcal{L}\colon J^1(\mathcal{C}P) \to \mathbb{R}$ is said to be gauge invariant if

$$\tilde{X}^{(1)}(\mathcal{L}) = 0,\ \forall X \in \mathrm{gau}P.$$

As we are dealing with the abelian structure group $U(1)$, the Utiyama theorem states that a Lagrangian is gauge invariant if and only if it can be written as

$$\mathcal{L} = \overline{\mathcal{L}} \circ \Omega,$$

where $\Omega\colon J^1(\mathcal{C}P) \to \bigwedge^2 T^*M \otimes \mathrm{ad}P$ is the mapping which assigns to each 1-jet of connections its curvature as a two form on M; *i.e.*, $\Omega\left(j_x^1\omega_\Gamma\right) = \left(\Omega_\Gamma\right)_x$ and $\overline{\mathcal{L}}\colon \bigwedge^2 T^*M \otimes \mathrm{ad}P \to \mathbb{R}$ is an arbitrary differentiable function (*cf.* [4]). Remark that, as $U(1)$ is abelian, the bundle $\mathrm{ad}P$ is canonicaly isomorphic to $M \times \mathfrak{u}(1)$; if we identify $\mathfrak{u}(1) \cong i\mathbb{R}$, then $\bigwedge^2 T^*M \otimes \mathrm{ad}P \cong \bigwedge^2 T^*M \otimes \mathbb{R} = \bigwedge^2 T^*M$ and we can simplify the statement of the theorem. Let us introduce coordinates on the bundle $p_2\colon \bigwedge^2 T^*M \to M$. For every 2-covector $w_2 \in \bigwedge^2 T_x^*M$ we set

$$w_2 = p_{ij}(w_2)\left(\mathrm{d}q^i \wedge \mathrm{d}q^j\right)_x, \quad i < j.$$

Then, (q^i, p_{ij}) is a coordinate system on $p_2^{-1}(V)$. We set $p_{ij} = -p_{ji}$ for $i \geq j$. With respect to these coordinates and those defined in (4), the equations of Ω are

$$p_{ij} \circ \Omega = p_{i,j} - p_{j,i} \,. \tag{8}$$

Hence gauge invariant Lagrangians can be locally written as

$$\mathcal{L} = \overline{\mathcal{L}}\left(q^h, p_{i,j} - p_{j,i}\right) \,.$$

3.2. aut$P$**-invariance.** Let us asume that M is oriented by a volume form v_n. A Lagrangian density $\mathcal{L}v_n$, $\mathcal{L} \in C^\infty(J^1(\mathcal{C}P))$, is said to be invariant under the Lie algebra of all infinitesimal automorphisms of P (in short, aut$P$-invariant) if

$$L_{X_{\mathcal{C}}^{(1)}}(\mathcal{L}v_n) = 0\,,\ \forall X \in \mathrm{aut}P\,.$$

As $L_{X_{\mathcal{C}}^{(1)}}(\mathcal{L}v_n) = \left(X_{\mathcal{C}}^{(1)}(\mathcal{L})\right)v_n + \mathcal{L}\,\mathrm{div}\,(\pi_*X)\,v_n = 0$, where the divergence is taken with respect to v_n, the invariance condition is equivalent to the following:

$$X_{\mathcal{C}}^{(1)}(\mathcal{L}) + \mathcal{L}\,\mathrm{div}\,(\pi_*X) = 0, \forall X \in \mathrm{aut}P\,. \tag{9}$$

4. Variationally trivial Lagrangians

A Lagrangian density $\mathcal{L}v_n$, $\mathcal{L} \in C^\infty(J^1(\mathcal{C}P))$, is said to be variationally trivial if every diferential is a solution to the Euler-Lagrange equations of the given Lagrangian; that is, for every connection ω defined on a neighbourhood of $\pi^{-1}(x)$, $x \in M$ we have

$$\frac{\partial\mathcal{L}}{\partial p_i}(j_x^1\omega) - \frac{\partial}{\partial q^k}\left(\frac{\partial\mathcal{L}}{\partial p_{i,k}} \circ j^1\omega\right)(x) = 0\,, \quad \text{identically.} \tag{10}$$

Lemma 1. *A Lagrangian density $\mathcal{L}v_n$ on $\mathcal{C}P$ is variationally trivial if and only if locally the following partial differential equations are satisfied:*

$$\frac{\partial^2\mathcal{L}}{\partial p_{i,h}\partial p_{j,k}} + \frac{\partial^2\mathcal{L}}{\partial p_{i,k}\partial p_{j,h}} = 0,\ \forall h,i,j,k = 1,\dots,n. \tag{11}$$

$$\frac{\partial\mathcal{L}}{\partial p_i} = \frac{\partial^2\mathcal{L}}{\partial q^k\partial p_{i,k}} + \frac{\partial^2\mathcal{L}}{\partial p_j\partial p_{i,k}}p_{j,k},\ \forall i = 1,\dots,n. \tag{12}$$

Proof. Expanding the equation (10) we obtain

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial p_i}(j_x^1\omega) &= \frac{\partial^2\mathcal{L}}{\partial q^k\partial p_{i,k}}(j_x^1\omega) + \frac{\partial^2\mathcal{L}}{\partial p_j\partial p_{i,k}}(j_x^1\omega)\frac{\partial f_j}{\partial q^k}(x) + \\ &\quad \frac{\partial^2\mathcal{L}}{\partial p_{j,h}\partial p_{i,k}}(j_x^1\omega)\frac{\partial^2 f_j}{\partial q^h\partial q^k}(x)\,,\end{aligned}$$

where $\omega = f_j \mathrm{d}q^j$. Once the 1-jet of ω at x has been fixed, the second partial derivatives $\left(\partial^2 f_j/\partial q^h\partial q^k\right)(x)$ can be arbitrarily taken. Hence equation (11) follows and also equation (12). $\square$

Notation: Given a multi-vector field $\chi_k \in \Gamma(M, \bigwedge^{2k}TM)$ of even degree $2k$, we denote by $\bar{\chi}_k \colon \bigwedge^2 T^*M \to \mathbb{R}$ the mapping given by

$$\bar{\chi}_k(w_2) = i_{\chi_k}(w_2 \wedge \overset{(k}{\ldots} \wedge w_2),\ w_2 \in \bigwedge^2 T^*M\,,$$

where i_χ is the total insertion operator (*cf.* §2.5).

The geometrical characterization of gauge invariant, variationally trivial Lagrangians is stated in the following

Theorem 1. *Let $\pi : P \to M$ be a $U(1)$-principal fiber bundle over an orientable n-dimensional manifold M with fixed volume form v_n. A first order gauge invariant Lagrangian density $\mathcal{L}v_n$ on $\mathcal{C}P$ is variationally trivial if and only $\mathcal{L}$ can be written as follows:*

$$\mathcal{L} = \bar{\chi} \circ \Omega ,$$

where $\Omega\colon J^1(\mathcal{C}P) \to \bigwedge^2 T^*M \otimes \mathrm{ad}P \cong \bigwedge^2 T^*M$ *is the curvature mapping* (*see* §3.1) *and*

$$\bar{\chi} = \sum_{k=0}^{[n/2]} \bar{\chi}_k ,$$

each χ_k being a multi-vector field of degree $2k$, such that (cf. §2.5)

$$\mathrm{L}_{\chi_k} v_n = 0 .$$

Proof. As $\mathcal{L}v_n$ is gauge invariant according to Utiyama's theorem (§3.1) we have

$$\mathcal{L} = \overline{\mathcal{L}} \circ \Omega ,$$

where $\overline{\mathcal{L}}$ is an arbitrary differentiable function on $\bigwedge^2 T^*M$. The variational triviality of $\mathcal{L}$ tells us that the above composition should satisfy locally the equations (11)–(12). Taking into account the equations of Ω (see formula (8)) we have

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial p_{ih} \partial p_{jk}} \circ \Omega + \frac{\partial^2 \overline{\mathcal{L}}}{\partial p_{ik} \partial p_{jh}} \circ \Omega = 0, \ \forall h, i, j, k = 1, \dots , n,$$

and

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial q^k \partial p_{ik}} \circ \Omega = 0, \ \forall i = 1, \dots , n,$$

as $\partial \mathcal{L}/\partial p_i$, $\partial^2 \mathcal{L}/\partial p_j \partial p_{i,k}$ vanish for $\mathcal{L}$ does not depend on p_i. Since Ω is a surjective mapping we obtain the corresponding equations for $\overline{\mathcal{L}}$ on the bundle $\bigwedge^2 T^*M$; *i.e.*,

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial p_{ih} \partial p_{jk}} + \frac{\partial^2 \overline{\mathcal{L}}}{\partial p_{ik} \partial p_{jh}} = 0, \ \forall h, i, j, k = 1, \dots , n . \tag{13}$$

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial q^k \partial p_{ik}} = 0, \ \forall i = 1, \dots , n . \tag{14}$$

From (13) we obtain $\partial^2 \overline{\mathcal{L}}/\partial p_{ih}^2 = 0$. Hence for each variable p_{ih} we have $\overline{\mathcal{L}} = a + b p_{ih}$, where a, b are differentiable functions depending on q^j and the rest of the variables p_{kl}. By recurrence we conclude that $\overline{\mathcal{L}}$ is a polynomial in the variables p_{ih} with coefficients in $C^\infty(M)$; that is,

$$\overline{\mathcal{L}} = \sum_k f^{i_1 \dots i_{2k}} p_{i_1 i_2} \cdots p_{i_{2k-1} i_{2k}}, \ f^{i_1 \dots i_{2k}} \in C^\infty(M) . \tag{15}$$

For $k = h$ in (13) we conclude that the functions $f^{i_1 \dots i_{2k}}$ with a repeated index, must vanish. For arbitrary indices, again equation (13) tells us that $f^{i_1, \dots, j, \dots, l, \dots, i_{2k}} + f^{i_1, \dots, l, \dots, j, \dots, i_{2k}} = 0$. Since transpositions generate the group of permutations, we obtain

$$f^{i_{\pi(1)}, \dots, i_{\pi(2k)}} = (\mathrm{sgn}\, \pi)\, f^{i_1, \dots, i_{2k}} ,$$

for every permutation $\pi \in S_{2k}$. Therefore,

$$\overline{\mathcal{L}} = \sum_{k=0}^{[n/2]} \sum_{i_1<\ldots<i_{2k}} \sum_{\pi\in S_{2k}} (\mathrm{sgn}\pi)\, f^{i_1,\cdots,i_{2k}} p_{i_{\pi(1)}i_{\pi(2)}} \cdots p_{i_{\pi(2k-1)}i_{\pi(2k)}} .$$

The, it is straightforward to check that $\overline{\mathcal{L}} = \bar{\chi} = \sum_{k=0}^{[n/2]} \bar{\chi}_k$, where

$$\chi_k = (-1)^{\binom{2k}{2}} 2^k \sum_{i_1<\ldots<i_{2k}} f^{i_1,\cdots,i_{2k}} \frac{\partial}{\partial q^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial q^{i_{2k}}} .$$

Let us check that the equation (14) implies that each multi-vector field χ_k is divergence free. Let $\left(V; q^1, \ldots, q^n\right)$ be a coordinate open domain such that $v_n = \mathrm{d}q^1 \wedge \ldots \wedge \mathrm{d}q^n$. The Lie derivative of the volume form with respect to the multi-vector field χ_k is

$$\begin{aligned} (\mathrm{L}_{\chi_k} v_n) &= c_k \cdot (-1)^{k+1} \sum_{i_1<\ldots<i_{2k}} \mathrm{d}\left(f^{i_1,\cdots,i_{2k}} i_{\frac{\partial}{\partial q^{i_1}}\wedge\ldots\wedge\frac{\partial}{\partial q^{i_{2k}}}} \left(\mathrm{d}q^1 \wedge \ldots \wedge \mathrm{d}q^n\right)\right) \\ &= -c_k \sum_{i_1<\ldots<i_{2k}} (-1)^{i_1+\cdots+i_{2k}} \mathrm{d}\left(f^{i_1,\cdots,i_{2k}} \mathrm{d}q^{j_1} \wedge \ldots \wedge \mathrm{d}q^{j_{n-2k}}\right) , \end{aligned}$$

where $c_k = (-1)^{\binom{2k}{2}} 2^k$ and $j_1 < \ldots < j_{n-2k}$ is the complementary subset of $i_1 < \ldots < i_{2k}$ in $\{1, \ldots, n\}$. Hence

$$c_k^{-1} (\mathrm{L}_{\chi_k} v_n) = -\sum_{i_1<\ldots<i_{2k}} (-1)^{i_1+\cdots+i_{2k}} \sum_{\alpha} \frac{\partial f^{i_1,\cdots,i_{2k}}}{\partial q^\alpha} \mathrm{d}q^\alpha \wedge \mathrm{d}q^{j_1} \wedge \ldots \wedge \mathrm{d}q^{j_{n-2k}} .$$

If $j_0 < \ldots < j_{n-2k}$ is the complementary subset of $i_2 < \ldots < i_{2k}$ in $\{1, \ldots, n\}$, then the coefficient of $\mathrm{d}q^{j_0} \wedge \mathrm{d}q^{j_1} \wedge \ldots \wedge \mathrm{d}q^{j_{n-2k}}$ in the above formula is

$$(-1)^{i_2+\cdots+i_{2k}} \sum_{\alpha} \frac{\partial f^{\alpha,i_2,\cdots,i_{2k}}}{\partial q^\alpha} ,$$

where for (not necessarily increasing) pairwise different indices $i_1, \ldots, i_{2k}$ we have set $f^{i_1,\cdots,i_{2k}} = (\mathrm{sgn}\pi) f^{\pi(i_1),\cdots,\pi(i_{2k})}$, π being the unique permutation of the set $\{i_1, \ldots, i_{2k}\}$ such that $\pi(i_1) < \ldots < \pi(i_{2k})$. Accordingly, $\mathrm{L}_{\chi_k} v_n = 0$ if and only if

$$\sum_{\alpha} \frac{\partial f^{\alpha,i_2\cdots,i_{2k}}}{\partial q^\alpha} = 0, \quad i_2, ..., i_{2k} \text{ fixed.} \tag{16}$$

In the other hand, from the formula (15) we have

$$\begin{aligned} \frac{\partial^2 \overline{\mathcal{L}}}{\partial q^\alpha \partial p_{j\alpha}} &= \sum_{k=0}^{[n/2]} \frac{\partial f^{i_1\ldots i_{2k}}}{\partial q^\alpha} \frac{\partial p_{i_1 i_2} \cdots p_{i_{2k-1}i_{2k}}}{\partial p_{j\alpha}} \\ &= \sum_{k=0}^{[n/2]} k \frac{\partial f^{j,\alpha,i_3\ldots i_{2k}}}{\partial q^\alpha} p_{i_3 i_4} \cdots p_{i_{2k-1}i_{2k}} . \end{aligned}$$

The above expression vanishes if and only if

$$\sum_{\alpha} \frac{\partial f^{j,\alpha,i_3\ldots i_{2k}}}{\partial q^\alpha} = 0 ,$$

which is equivalent to (16), thus finishing the proof. $\square$

Remark. Given a variationally trivial Lagrangian $\mathcal{L}$, the action

$$\int_M L(j_x^1\omega)v_n$$

does not depend on the choice of the connection ω taken. Hence, variationally trivial Lagrangians produce Gauss-Bonnet type theorems.

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