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DECOMPOSITION OF HIGHER ORDER TANGENT FIELDS AND CALCULUS OF VARIATIONS

R. J. ALONSO

ABSTRACT. Given a linear connection on a differentiable manifold M, we obtain a method for decomposing higher order tangent fields (in $\mathcal{T}^\infty$) by means of ordinary tangent fields and differential operators. As a consequence, we can find primitives for the result of the action of $\mathcal{T}^\infty$ on the module of differential forms on M of maximum degree. With certain restrictions, the same model applies to forms with values in a $\mathcal{C}^\infty(M)$-module. When the aforementioned module is the contact module of a fibered manifold, we can apply this result to the calculus of variations and then we get the Poincaré-Cartan operators associated to the decomposition obtained. This method clarifies the role played by the linear connection in the higher order variational theory. Finally we give an outline of the application to the variational bicomplex.

1. Introduction

In this paper we give a new approach to the Poincaré-Cartan forms in the calculus of variations. There are many references about first order variational calculus, let us cite a few of the pioneering works [2, 4, 5, 7, 13]. The situation becomes more complicated in the higher order case (see, for instance, [3, 6, 8, 10, 11, 16, 19]). As proved in [12], the Poincaré-Cartan operator, which is central to the variational calculus, is not natural in higher order. In other words, its construction requires the use of some additional element.

In our approach we take into account that the main objects in the theory (in particular, the vertical differential of a Lagrangian density) are elements of certain modules over the ring of differential operators $\mathcal{D}$. On the other hand, one of the most important tool of the calculus of variations is the method of integration by parts, hence the necessity of finding such a method. We will solve this problem by means of suitable decompositions of higher order tangent fields on the base manifold. This clarifies where the introduction of an additional object is needed. Furthermore, this technique is available in the whole Euler-Lagrange resolution. This work is partially based in [1], but some improvements have been made.

The paper is organized as follows. In section §2 we recall the basic facts about $\mathcal{D}$-modules. In §3 we will see how to use a decomposition of higher order tangent fields

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to factorize its action on the exterior forms of maximum degree. In §4 and §5 we will show that a linear connection produces a decomposition morphism. In section §6 we will describe the differential calculus with values in the contact module of a given fibered manifold. The aforementioned calculus will allow us in §7 to define the Euler-Lagrange operator. In section §8 we will see how to find the Poincaré-Cartan operators once a decomposition morphism is given. Finally in §9, we will apply this method in order to find generalized Poincaré-Cartan operators when we replace the contact module by its exterior powers.

2. $\mathcal{D}$-modules

Let M be a smooth manifold of dimension n, and let $\mathcal{D}$ be the ring of differential operators on M. We have a filtration by $\mathcal{D}^0 = \mathcal{C}^\infty(M) \subset \mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}$ and direct sums $\mathcal{D}^k = \mathcal{C}^\infty(M) \oplus \mathcal{T}^k$, where $\mathcal{T}^k$ stands for the tangent fields of order k (that is $\mathcal{T}^k = \{P \in \mathcal{D}^k;\ P(1) = 0\}$). Each $\mathcal{D}^k$ may be endowed with two different $\mathcal{C}^\infty(M)$-module structures; the ordinary one is obtained when we consider the product by functions on the left and the second one, $\mathcal{D}^k_{right}$, when we consider the product on the right. We define $\mathcal{D}_{right}$ in the same way.

Because of the non commutativity of $\mathcal{D}$, we will need to distinguish between left and right $\mathcal{D}$-modules. A left $\mathcal{D}$-module structure is the same thing as a $\mathcal{C}^\infty(M)$-module endowed with a flat derivation law. In particular $\mathcal{C}^\infty(M)$ and $\mathcal{D}$ are left $\mathcal{D}$-modules. Examples of right $\mathcal{D}$-modules are $\mathcal{D}_{right}$ by composition on the right and, the most important case, the module of differential forms of maximum degree $\Omega^n = \Omega^n(M)$, by means of minus the Lie derivative.

Moreover, from $\mathcal{D}$-modules we can get new ones by using tensor products (over M) and Leibnitz rules. To sum up, let $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$ be left $\mathcal{D}$-modules and $\mathcal{B}$ a right $\mathcal{D}$-module. Then $\mathcal{A}_1 \otimes_M \mathcal{A}_2$ is a left $\mathcal{D}$-module, if $a_i \in \mathcal{A}_i$, $i = 1, 2$ and $D \in \mathcal{T}$ then we set $D\cdot(a_1 \otimes a_2) = D\cdot a_1 \otimes a_2 + a_1 \otimes D\cdot a_2$. Likewise, $\mathcal{A} \otimes_M \mathcal{B}$ is a right $\mathcal{D}$-module, if $a \in \mathcal{A}$ and $b \in \mathcal{B}$ then we set $(a \otimes b)\cdot D = -D\cdot a \otimes b + a \otimes b \cdot D$. For example, $\mathcal{D} \otimes_M \mathcal{D}$ is a left $\mathcal{D}$-module, and $\mathcal{D} \otimes_M \Omega^n$ is a right $\mathcal{D}$-module. In particular we have a functor $\{\text{Left } \mathcal{D}\text{-modules}\} \longrightarrow \{\text{Right } \mathcal{D}\text{-modules}\}$ which sends a left $\mathcal{D}$-module $\mathcal{A}$ to $\mathcal{A} \otimes_M \Omega^n$. See [21] for further information about $\mathcal{D}$-modules.

3. Decomposition morphisms

The Lie derivative of a n-form $\omega \in \Omega^n$, with respect to a tangent field $D \in \mathcal{T}$, is the exact form $L_D\omega = d i_D\omega$. In fact, we have a primitive: $i_D\omega$. We would like to generalize this property to higher order tangent fields.

Let $P \in \mathcal{T}^\infty$ and ω as before. Let us suppose that P is written as $\sum Q \circ D$, where $Q \in \mathcal{D}$ and $D \in \mathcal{T}$. Then we have $\omega \cdot P = \sum(\omega \cdot Q)\cdot D = -\sum L_D(\omega \cdot Q) = d(-\sum i_D(\omega \cdot Q))$. Here we have the problem of decomposing each P as $\sum Q \circ D$ or something equivalent to this for our purposes.

Let us consider the inverse process, which consists of composing differential operators with tangent fields:

$$G\colon \overline{\mathcal{D}_{right} \otimes_M \mathcal{T}} \longrightarrow \mathcal{T}^\infty, \qquad Q \otimes D \longrightarrow Q \circ D$$

where the overline means that on the tensor product we consider the structure of $\mathcal{C}^\infty(M)$-module (resp. left $\mathcal{D}$-module) given by $f * (P \otimes D) = fP \otimes D$ (resp. $Q * (P \otimes D) = Q \circ P \otimes D$), for every $P, Q \in \mathcal{D}$, $D \in \mathcal{T}$ and $f \in \mathcal{C}^\infty(M)$. With this structure G is $\mathcal{C}^\infty(M)$ and $\mathcal{D}$ linear.

Definition 3.1. G will be called the composition morphism.

Remark. We can consider a finite order analogous to G, $G_k : \overline{\mathcal{D}^{k-1}_{right} \otimes_M \mathcal{T}} \longrightarrow \mathcal{T}^k$. In this case the $\mathcal{D}$ linearity has non sense.

Let us consider the injective map $g : \Lambda^2\mathcal{T} \longrightarrow \overline{\mathcal{D}^1_{right} \otimes_M \mathcal{T}}$ which sends $D_1 \wedge D_2$ to $\frac{1}{2}(D_1 \otimes D_2 - D_2 \otimes D_1 - 1 \otimes [D_1, D_2])$. It is easy to check that G_2 vanishes on the image of g. Let us denote by $\mathcal{D}^k * \Lambda^2\mathcal{T}$ the result of the action of $\mathcal{D}^k$ on $g(\Lambda^2\mathcal{T}) \subset \overline{\mathcal{D}_{right} \otimes_M \mathcal{T}}$.

Proposition 3.2.

1. *For each integer k, G_k is surjective.*
2. *$Ker(G_1) = 0$ and $Ker(G_k) = \mathcal{D}^{k-2} * \Lambda^2\mathcal{T}$ for $k \geq 2$.*

Proof.

1. Locally every higher order tangent field can be written as a composition of differential operators with ordinary tangent fields. By using partitions of the unity argument the result follows.
2. G_1 is the identity map on $\mathcal{T}$. For $k \geq 2$ we use that each element in $Ker(G_k)$ may be reduced (for example, by using local charts) to a sum of terms like $R \circ D_1 \otimes D_2 - R \circ D_2 \otimes D_1 - R \otimes [D_1, D_2] = R * g(D_1 \wedge D_2) \in \mathcal{D}^{k-2} * \Lambda^2\mathcal{T}$. $\square$

Corollary 3.3. *G_k is an isomorphism if and only if $k = 1$ or $n = 1$.*

Definition 3.4. A section F of G will be called a decomposition morphism. A section F_k of G_k will be called a decomposition morphism of order k.

Corollary 3.5.

1. *The set of decomposition morphisms is identified with $F_k + Hom_M(\mathcal{T}^k, \mathcal{D}^{k-2} * \Lambda^2\mathcal{T})$, where F_k is a particular one.*
2. *For $k = 1$ or $n = 1$, the decomposition morphisms are unique.*

Proposition 3.6. *There exists a canonical decomposition morphism of order 2.*

Proof. We define $F_2(D_1 \circ D_2) = \frac{1}{2}(D_1 \otimes D_2 + D_2 \otimes D_1 + 1 \otimes [D_1, D_2])$, for each pair of tangent fields $D_1, D_2 \in \mathcal{T}$. $\square$

Now we can solve the problem settled at the beginning of this section. Let F be a given decomposition morphism and let us consider the following sequence of maps

$$\mathcal{T}^\infty \otimes_M \Omega^n \xrightarrow{F \otimes id} \overline{\mathcal{D}_{right} \otimes_M \mathcal{T}} \otimes_M \Omega^n \xrightarrow{\phi} \Omega^{n-1}$$

where $\phi((Q \otimes D) \otimes \omega) = -i_D(\omega \cdot Q)$. Let us denote $\phi \circ (F \otimes \mathrm{Id})$ by ϕ_F. We see that ϕ_F recovers in a rigorous way the calculations done at the beginning of this section. Thus

Theorem 3.7. *For each decomposition morphism F, the following diagram is commutative*

$$\begin{tikzcd} \mathcal{T}^\infty \otimes_M \Omega^n \arrow[rr, "L"] \arrow[dr, "\phi_F"'] & & \Omega^n \\ & \Omega^{n-1} \arrow[ur, "d"'] & \end{tikzcd}$$

where L is defined as $L(P \otimes \omega) = \omega \cdot P$, for $P \otimes \omega \in \mathcal{T}^\infty \otimes_M \Omega^n$.

Proof. If $F(P) = \sum Q \otimes D$ then $\sum Q \circ D = P$, therefore $L(P \otimes \omega) = \omega \cdot P = d(-\sum i_D(\omega \cdot Q)) = d\phi_F(P \otimes \omega)$. □

There is no change when we consider $\mathcal{A} \otimes_M \Omega^n$ instead of Ω^n, where $\mathcal{A}$ is a left $\mathcal{D}$-module. We define ϕ_F in the same way as before.

Theorem 3.8. *For each decomposition morphism F, there exists a map ϕ_F such that the following diagram is commutative*

$$\begin{tikzcd} \mathcal{T}^\infty \otimes_M (\mathcal{A} \otimes_M \Omega^n) \arrow[rr, "L"] \arrow[dr, "\phi_F"'] & & \mathcal{A} \otimes_M \Omega^n \\ & \mathcal{A} \otimes_M \Omega^{n-1} \arrow[ur, "d_{cov}"'] & \end{tikzcd}$$

where L is defined from the right $\mathcal{D}$-module structure of $\mathcal{A} \otimes_M \Omega^n$, and d_{cov} is the covariant exterior differential associated with the covariant derivation law defined on $\mathcal{A}$.

Proof. Note that the action of $\mathcal{D}$ on $\mathcal{A} \otimes_M \Omega^n$ is generated by the following one of $\mathcal{T}$: $B \cdot D = -L_D^{cov} B = -d_{cov} i_D B$, where $B \in \mathcal{A} \otimes_M \Omega^n$, $D \in \mathcal{T}$ and L_D^{cov} is the covariant Lie derivative with respect to D. The rest follows as before. □

4. Gradation associated with a linear connection

Let us fix a linear connection ∇ on M and let us consider the exact sequences

$$0 \longrightarrow \mathcal{T}^{k-1} \longrightarrow \mathcal{T}^k \xrightarrow{\sigma_k} S^k\mathcal{T} \longrightarrow 0$$

where σ_k means taking the symbol on $\mathcal{T}^k \subset \mathcal{D}^k$ (recall that $\mathcal{T}^k/\mathcal{T}^{k-1} \simeq S^k\mathcal{T}$).

From ∇ we will construct sections s_k of σ_k by recurrence:

$$s_1(D) = D$$

$$s_k(D_I) = \frac{1}{k}\sum_{i=1}^{k}\{D_i \circ s_{k-1}(D_{I_i}) - s_{k-1}(D_i^\nabla(D_{I_i}))\}$$

where $D_I = D_1 \cdots D_k$ and $D_{I_i} = D_1 \cdots D_{i-1}D_{i+1} \cdots D_k$ for $D_i \in \mathcal{T}$, $i = 1, \ldots, k$; in addition we have made use of the derivation law induced by ∇ on the symmetric powers of $\mathcal{T}$. It is not hard to check that the s_k are well defined and, by induction, $\sigma_k \circ s_k = \mathrm{Id}_{S^k\mathcal{T}}$. Thus

Proposition 4.1. *Each linear connection ∇ produces a gradation on $\mathcal{T}^\infty$,*

$$\mathcal{T}^\infty \overset{s}{\approx} \bigoplus_k S^k \mathcal{T}$$

where $s = \oplus_k s_k$.

See [9] for another construction of this gradation.

5. Decomposition morphism associated with a linear connection

Let ∇ be as before. The gradation associated with ∇ allows us to work with homogeneous elements in $\mathcal{T}^\infty$.

Theorem 5.1. *Each linear connection ∇ produces a decomposition morphism F_∇.*

Proof. We define $F_\nabla \colon \mathcal{T}^\infty \longrightarrow \overline{\mathcal{D}_{right} \otimes \mathcal{T}}$ by recurrence (notation as before)

$$F_\nabla(s_1(D)) = F_\nabla(D) = 1 \otimes D$$

$$F_\nabla(s_k(D_I)) = \frac{1}{k}\sum_i \{D_i * F_\nabla(s_{k-1}(D_{I_i})) - F_\nabla(s_{k-1}(D_i^\nabla D_{I_i}))\}$$

Performing induction on k and using the $\mathcal{D}$ linearity of G we have

$$G \circ F_\nabla(s_k(D_I)) = \frac{1}{k}\sum_i \{D_i \circ G \circ F_\nabla(s_{k-1}(D_{I_i})) - G \circ F_\nabla(s_{k-1}(D_i^\nabla D_{I_i}))\} =$$

$$= \frac{1}{k}\sum_i \{D_i \circ s_{k-1}(D_{I_i}) - s_{k-1}(D_i^\nabla D_{I_i}))\} = s_k(D_I). \qquad \square$$

Remarks. 1) For finite order we also have decompositions $F_{\nabla,k}$ on each $\mathcal{T}^k$, $k \geq 1$. 2) For $k = 2$ all of these $F_{\nabla,2}$ equal to the decomposition F_2 defined in section §3.

6. $\mathcal{C}$-valued calculus

Let $\pi \colon Y \longrightarrow M$ be a fibered manifold, which in a local fibered chart is written as $\pi \colon (y_j, x_i) \longrightarrow (x_i)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ and let J^k be the space of k-jets of local sections of π. Local fibered coordinates (x_i, y_j) on Y induce fibered coordinates $(x_i, y_{j\alpha})$ on J^k as usual, where α runs over the multiindex of order $|\alpha|$ lower or equal than k.

Let us consider the $\mathcal{C}^\infty(M)$-module $\mathcal{C}_k$ formed by the contact 1-forms on J^k. Locally $\mathcal{C}_k$ is spanned (over $\mathcal{C}^\infty(J^k)$) by the forms $\theta_{j\alpha} = dy_{j\alpha} - y_{j\alpha+1_i} dx_i$, $|\alpha| < k$. By means of pull-back we have inclusions $\mathcal{C}_k \subset \mathcal{C}_{k+1}$.

Definition 6.1. $\mathcal{C} = \operatorname{inj}\lim \mathcal{C}_k$ will be called the contact module of π.

Each tangent field on M induces derivations

$$\widehat{D} \colon \mathcal{C}^\infty(J^k) \longrightarrow \mathcal{C}^\infty(J^{k+1})$$

which are defined in the following way. Let $\mathfrak{p}^{k+1}$ a jet in J^{k+1} which projects on x in M, and f an smooth function on J^k; then we put $(\widehat{D}f)(\mathfrak{p}^{k+1}) = D_x(\mathfrak{p}^{k,1} f)$, where $\mathfrak{p}^{k,1} \colon \mathcal{C}^\infty(J^k) \longrightarrow \mathcal{C}^\infty(M)/\mathfrak{m}_x^2$ is the image of $\mathfrak{p}^{k+1}$ by the immersion $J^{k+1} \subset J^1 J^k$ (see [17] for the definition of jets of sections as morphisms).

Definition 6.2. $\widehat{D}$ will be called the total or formal derivative associated with D.

The assignment $D \longrightarrow \widehat{D}$ produces a section of $(TJ^k)_{J^{k+1}} \longrightarrow (TM)_{J^{k+1}}$ whose associated retract θ^{k+1} (with values in the vertical tangent bundle with respect to the projection $J^k \longrightarrow M$) is the so called canonical 1-form on J^{k+1}. Locally $\theta^{k+1} = \sum \theta_{j\alpha} \otimes \frac{\partial}{\partial y_{j\alpha}}$.

On the other hand, we can think of $\widehat{D}$ as a map $J^{k+1} \longrightarrow TJ^k$ along the projection $J^{k+1} \longrightarrow J^k$. In local fibered coordinates, we have $\widehat{\frac{\partial}{\partial x_i}} = \frac{\partial}{\partial x_i} + y_{j\alpha+1_i} \frac{\partial}{\partial y_{j\alpha}}$. Now we will use the concept of infinitesimal variation in the sense of Kumpera (see [15, 18] or also [19]) in order to define from $\widehat{D}$ a Lie derivative $L_{\widehat{D}} \colon \Omega^p(J^k) \longrightarrow \Omega^p(J^{k+1})$, $p \geq 0$, which fulfills Cartan formula $L_{\widehat{D}} = i_{\widehat{D}} d + d i_{\widehat{D}}$ (where $i_{\widehat{D}}$ stands for the formal inner product).

Proposition 6.3.

1. $L_{\widehat{D}} \mathcal{C}_k \subset \mathcal{C}_{k+1}$.
2. *The rule* $D \rightsquigarrow L_{\widehat{D}}$ *on* $\mathcal{C}$ *is a flat derivation law.*

Proof.

1. Locally, $L_{\widehat{\frac{\partial}{\partial x_i}}} \theta_{j\alpha} = \theta_{j\alpha+1_i}$ (see [15, 18] or [19]).
2. It follows from 1) and $[\widehat{D}_1, \widehat{D}_2] = \widehat{[D_1, D_2]}$ for $D_1, D_2 \in \mathcal{T}$. □

Corollary 6.4. *$\mathcal{C}$ is a left $\mathcal{D}$-module.*

Definition 6.5. The above mentioned derivation law on $\mathcal{C}$ will be called the formal derivation law.

In this way, we have obtained a differential calculus on $\Omega^{\cdot}(M)$ with values in $\mathcal{C}$. In particular, we get a covariant exterior differential $\widehat{d}$ and a covariant Lie derivative $\widehat{L}_D$ related by $\widehat{L}_D = i_D \widehat{d} + \widehat{d} i_D$.

Let us define $\overline{\mathcal{C}}_1 \subset \mathcal{C}$ as $\mathcal{C}_1 \otimes_M \mathcal{F}$, where $\mathcal{F} = \operatorname{inj\,lim} C^\infty(J^k)$. By means of local coordinates it can be proved the following result.

Proposition 6.6. *The map* $L \colon \mathcal{D}_{right} \otimes_M \overline{\mathcal{C}}_1 \longrightarrow \mathcal{C}$, *sending* $P \otimes \sigma \in \mathcal{D}_{right} \otimes_M \overline{\mathcal{C}}_1$ *to* $P \cdot \sigma$ *(where the dot stands for the action of* $\mathcal{D}$ *on* $\mathcal{C}$*) is an isomorphism. In other words, all the contact forms can be obtained from the first order ones through formal Lie derivatives.*

7. Euler-Lagrange operators

Let us consider the morphism of left $\mathcal{D}$-modules $\overline{\overline{\mu}} \colon \mathcal{D} \longrightarrow \mathcal{D} \otimes_M \mathcal{D}$ sending 1 to $1 \otimes 1$. Taking into account the isomorphism $\mathcal{C} \simeq \mathcal{D}_{right} \otimes_M \overline{\mathcal{C}}_1$, we will get $\overline{\mu} \colon \mathcal{C} \longrightarrow \mathcal{D} \otimes_M \mathcal{C}$. Finally we put $\mu = \overline{\mu} \otimes \mathrm{Id}_{\Omega^n}$ to obtain a right $\mathcal{D}$-modules morphism

$$\mu \colon \mathcal{C} \otimes_M \Omega^n \longrightarrow \mathcal{D} \otimes_M \mathcal{C} \otimes_M \Omega^n$$

whose local expression is $\mu(\theta_{j\alpha} \otimes \omega) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} (\frac{\partial}{\partial x})^\beta \otimes \theta_{j\gamma} \otimes \omega$ where $\omega \in \Omega^n$.

Remark. $\overline{\overline{\mu}}$ is a reflection on $\mathcal{D}$ of the product of Taylor expansions. Besides, μ is closely related to the so called vertical lift.

On the other hand, the right structure of $\mathcal{C} \otimes_M \Omega^n$ produces a map

$$L\colon \mathcal{D} \otimes_M \mathcal{C} \otimes_M \Omega^n \longrightarrow \mathcal{C} \otimes_M \Omega^n,$$

sending $P \otimes \sigma \otimes \omega \in \mathcal{D} \otimes_M \mathcal{C} \otimes_M \Omega^n$ to $(\sigma \otimes \omega) \cdot P$.

Lemma 7.1. *L vanishes on each $A \cdot D$, where $D \in \mathcal{T}$ and $A \in \mathcal{D} \otimes_M \mathcal{C} \otimes_M \Omega^n$.*

Proof. If $A = P \otimes B$, where $B \in \mathcal{C} \otimes_M \Omega^n$ and $P \in \mathcal{D}$, then $A \cdot D = -D \circ P \otimes B + P \otimes B \cdot D$ (see §2). Thus, $L(A \cdot D) = -B \cdot (D \circ P) + (B \cdot D) \cdot P = 0$. □

Theorem 7.2.

1. *$\tau = L \circ \mu$ vanishes on the subspace $\widehat{d}(\mathcal{C} \otimes_M \Omega^{n-1})$ of $\mathcal{C} \otimes_M \Omega^n$.*
2. *τ is a projector.*
3. $\mathcal{C} \otimes_M \Omega^n \simeq (\overline{\mathcal{C}}_1 \otimes_M \Omega^n) \oplus \widehat{d}(\mathcal{C} \otimes_M \Omega^{n-1})$.

Proof. Let us denote the result of the action of $\mathcal{T}$ on $\mathcal{C} \otimes_M \Omega^n$ by $(\mathcal{C} \otimes_M \Omega^n) \cdot \mathcal{T}$ (and analogously for $\mathcal{T}^\infty$).

1. It is not hard to check that $\widehat{d}(\mathcal{C} \otimes_M \Omega^{n-1}) \simeq (\mathcal{C} \otimes_M \Omega^n) \cdot \mathcal{T}$ (for example by using local charts). Then we apply $\mathcal{D}$-linearity and the lemma above.
2. For each $A \in \mathcal{C} \otimes_M \Omega^n$ we have $A - \tau A = (\mathrm{Id} - \tau)A = L \circ (1 \otimes \mathrm{Id} - \mu)A$. Now, $\mathrm{Im}(1 \otimes \mathrm{Id} - \mu) \subseteq \mathcal{T}^\infty \otimes_M \mathcal{C} \otimes_M \Omega^n$, therefore $\mathrm{Im}(\mathrm{Id} - \tau) \subseteq (\mathcal{C} \otimes_M \Omega^n) \cdot \mathcal{T}^\infty = (\mathcal{C} \otimes_M \Omega^n) \cdot \mathcal{T} = \widehat{d}(\mathcal{C} \otimes_M \Omega^n)$. Thus, $A - \tau A \in \widehat{d}(\mathcal{C} \otimes_M \Omega^n)$, and then, from 1), we deduce that $\tau A - \tau^2 A = 0$.
3. It suffices to check that $\mathrm{Im}(\tau) = \overline{\mathcal{C}}_1 \otimes_M \Omega^n$. On $\overline{\mathcal{C}}_1 \otimes_M \Omega^n$, τ is the identity, thus $\overline{\mathcal{C}}_1 \otimes_M \Omega^n \subseteq \mathrm{Im}(\tau)$. Moreover, for each A in $\mathcal{C} \otimes_M \Omega^n$ which is locally $\sum f_{j\alpha} \sigma_{j\alpha} \otimes \omega$, where $\omega = dx_1 \wedge \cdots \wedge dx_n$, we have $\tau A = \sum (-1)^{|\alpha|} (\widehat{\frac{\partial}{\partial x}})^\alpha (f_{j\alpha}) \theta_j \otimes \omega \in \overline{\mathcal{C}}_1 \otimes_M \Omega^n$. □

A Lagrangian density of order k on the fibered manifold π is an element λ in $(\Omega^n)_{J^k} \subset (\Omega^n)_{J^\infty}$. If ω is a volume element (local or global) we have $\lambda = \mathcal{L}\omega$ for some function $\mathcal{L} \in \mathcal{C}^\infty(J^k)$. We define a map δ from the space of Lagrangian densities $(\Omega^n)_{J^\infty}$ to $\mathcal{C} \otimes_M \Omega^n$ by making the derivative with respect to the canonical 1-form θ^k. Locally we have

$$\delta\lambda = \sum \frac{\partial \mathcal{L}}{\partial y_{j\alpha}} \theta_{j\alpha} \otimes \omega$$

Definition 7.3. We will call $\mathcal{E} = \tau \circ \delta\colon (\Omega^n)_{J^\infty} \longrightarrow \overline{\mathcal{C}}_1 \otimes_M \Omega^n$ the Euler-Lagrange operator.

Corollary 7.4. *For each Lagrangian density we have $\delta\lambda = \mathcal{E}(\lambda) + H(\lambda)$, where $H(\lambda) \in \widehat{d}(\mathcal{C} \otimes_M \Omega^{n-1})$.*

8. Poincaré-Cartan operators

Let us fix a decomposition morphism F. The map $L\colon \mathcal{T}^\infty \otimes_M \mathcal{C} \otimes_M \Omega^n \longrightarrow \mathcal{C} \otimes_M \Omega^n$ factorizes as $L = \widehat{d} \circ \phi_F$ (see section §3). Thus,

Theorem 8.1. *For each decomposition morphism F there exists an operator*

$$\Theta_F : \mathcal{C} \otimes_M \Omega^n \longrightarrow \mathcal{C} \otimes_M \Omega^{n-1}$$

such that $A = \tau A + \widehat{d}\Theta_F(A)$, for any $A \in \mathcal{C} \otimes_M \Omega^n$. That is, we can recover a $\widehat{d}$-primitive of the exact part of $A \in \mathcal{C} \otimes \Omega^n$.

Proof. We have $A = \tau A + L \circ (1 \otimes \mathrm{Id} - \mu)A$, then it suffices to put $\Theta_F(A) = \phi_F \circ (1 \otimes \mathrm{Id} - \mu)A$. □

Definition 8.2. The operator Θ_F will be called the Poincaré-Cartan operator associated with F.

Remark. There exists finite order analogous $\Theta_{F_k} : \mathcal{C}_k \otimes_M \Omega^n \longrightarrow \mathcal{C}_{2k-1} \otimes_M \Omega^n$. Taking into account the properties of F_k we deduce that $\Theta_1 = \Theta_{F_1}$ are not depending of F and, for $n = 1$, also Θ_F does not. Finally, for $k = 2$, there exists a canonical Θ_{F_2}.

Corollary 8.3. *With the above notation, for each Lagrangian density λ we have a decomposition*

$$\delta\lambda = \mathcal{E}(\lambda) + \widehat{d}\Theta_F(\delta\lambda)$$

where $\mathcal{E}(\lambda) = \overline{\mathcal{C}}_1 \otimes_M \Omega^n$ and $\Theta_F(\delta\lambda) \in \mathcal{C} \otimes_M \Omega^n$.

By means of the alternation map $\mathcal{C} \otimes \Omega^n \longrightarrow \mathcal{C} \wedge \Omega^n$, we can move the above formula from the formal to the ordinary differential calculus. The differential $\widehat{d}$ becomes the so called horizontal differential. As a result we get a version of the first variation formula. This allows us to find out the equations for the critical sections of a given variational problem (see [10, 11]).

9. Generalized Poincaré-Cartan operators

The method used above also works in the case of exterior powers of $\mathcal{C}$ case. In this way we will get analogous higher degree operators to the Poincaré-Cartan. Therefore we may apply the preceding techniques to the variational bicomplex (see [2, 22, 23]).

First, we note that the formal covariant derivative on $\mathcal{C}$ extends naturally to the exterior powers $\Lambda^{\cdot}\mathcal{C} = \oplus_r \Lambda^r\mathcal{C}$. Thus, we have a formal differential calculus on $\Lambda^{\cdot}\mathcal{C} \otimes_M \Omega^{\cdot}$; in particular we have the corresponding $\widehat{d}$ and $\widehat{L}_D$ derivatives ($D \in \mathcal{T}$). Moreover, $\Lambda^{\cdot}\mathcal{C}$ is also a left $\mathcal{D}$-module. Second, we span $\overline{\mu} : \mathcal{C} \longrightarrow \mathcal{D} \otimes_M \mathcal{C}$ to $\overline{\mu}_r : \Lambda^r\mathcal{C} \longrightarrow \mathcal{D} \otimes_M \Lambda^r\mathcal{C}$ by deriving the alternate product. Therefore, we can define $\mu_r : \Lambda^r\mathcal{C} \otimes_M \Omega^n \longrightarrow \mathcal{D} \otimes_M \Lambda^r\mathcal{C} \otimes_M \Omega^n$. Now, it is easy to check that μ_r is $\mathcal{D}$-linear. In third place, we keep on denoting L the action map

$$L : \mathcal{D} \otimes_M \Lambda^{\cdot}\mathcal{C} \otimes_M \Omega^n \longrightarrow \Lambda^{\cdot}\mathcal{C} \otimes_M \Omega^n$$

induced by the right $\mathcal{D}$-module structure of $\Lambda^{\cdot}\mathcal{C} \otimes_M \Omega^n$. Let us consider also the derivative $\delta : \Lambda^r\mathcal{C} \otimes_M \Omega^q \longrightarrow \Lambda^{r+1}\mathcal{C} \otimes_M \Omega^q$ generated by the δ in the above sections (this is the vertical differential in the formal calculus). As in section §7, we have

Lemma 9.1.

1. *L vanishes on each $A \cdot D$, where $D \in \mathcal{T}$ and $A \in \mathcal{D} \otimes_M \Lambda^{\cdot}\mathcal{C} \otimes_M \Omega^n$.*

2. $\overline{\tau}_r = L \circ \mu_r$ *vanishes on the subspace* $\widehat{d}(\Lambda^{\cdot}\mathcal{C} \otimes_M \Omega^{n-1})$ *of* $\Lambda^{\cdot}\mathcal{C} \otimes_M \Omega^n$.

Theorem 9.2.

1. $\tau_r = \frac{1}{r}\overline{\tau}_r$ *is a projector on* $\Lambda^r\mathcal{C} \otimes_M \Omega^n$.
2. $\Lambda^r\mathcal{C} \otimes_M \Omega^n \simeq \mathrm{Im}(\tau_r) \oplus \widehat{d}(\Lambda^r\mathcal{C} \otimes_M \Omega^n)$.

Proof.

1. Let $A = \sigma_1 \wedge \cdots \wedge \sigma_r \otimes \omega \in \Lambda^r\mathcal{C} \otimes_M \Omega^n$ and $\beta(\sigma) = \sigma - \overline{\mu}(\sigma) \in \mathcal{T}^\infty \otimes_M \mathcal{C}$ for each $\sigma \in \mathcal{C}$. Then
$$\overline{\tau}_r(A) = (L \circ \mu_r)A = L(\sum_i \sigma_1 \wedge \cdots \wedge \overline{\mu}(\sigma_i) \wedge \cdots \wedge \sigma_r \otimes \omega) =$$
$$= r(\sigma_1 \wedge \cdots \wedge \sigma_r \otimes \omega) + L(\sum_i \sigma_1 \wedge \cdots \wedge \beta(\sigma_i) \wedge \cdots \wedge \sigma_r \otimes \omega).$$
Therefore $\overline{\tau}_r(A) = rA + B$ where $B \in (\Lambda^r\mathcal{C} \otimes_M \Omega^n) \cdot \mathcal{T}^\infty = \widehat{d}(\Lambda^r\mathcal{C} \otimes_M \Omega^{n-1})$. From the above lemma, we deduce that $\overline{\tau}_r^2 A = r\overline{\tau}_r A$.
2. It suffices to proof that $Ker(\tau_r) = \widehat{d}(\Lambda^r\mathcal{C} \otimes_M \Omega^{n-1})$. By the above lemma we deduce that $\widehat{d}(\Lambda^r\mathcal{C} \otimes_M \Omega^{n-1}) \subseteq Ker(\tau_r)$. Now, let $A \in Ker(\tau_r)$, then $0 = \tau_r A = A + B$ where $B \in \widehat{d}(\Lambda^r\mathcal{C} \otimes_M \Omega^{n-1})$. Therefore $Ker(\tau_r) \subseteq \widehat{d}(\Lambda^r\mathcal{C} \otimes_M \Omega^{n-1})$. □

Remark. The spaces $\Lambda^p\mathcal{C} \otimes_M \Omega^q$ endowed with the horizontal and vertical differentials define the variational bicomplex in the infinite formulation (see the above mentioned references). For the formulation on jets spaces of finite order we refer to [14].

We will denote $A_0 = (\Omega^n)_{J^\infty}$, and $A_r = \mathrm{Im}(\tau_r)$ for $r > 0$.

Definition 9.3. The operators $\mathcal{E}_i : A_r \longrightarrow A_{r+1}$ given by $\mathcal{E}_r = \tau_{r+1} \circ \delta$ will be called the higher degree Euler-Lagrange operators. $\mathcal{E}_0$ is the ordinary Euler-Lagrange operator and $\mathcal{E}_1$ is known as the Helmholtz-Sonin operator.

Remark. From $(A_r, \mathcal{E}_r)$ it may be constructed the Euler-Lagrange resolution (see [22], where the τ_r operators are defined by means of local charts).

Finally, we will construct primitives for the $\widehat{d}$-exact terms in the direct sum of the above theorem.

Theorem 9.4. *Given a decomposition morphism* F, *there exists operators*
$$\Theta_F^r : \Lambda^r\mathcal{C} \otimes_M \Omega^n \longrightarrow \Lambda^r\mathcal{C} \otimes_M \Omega^{n-1}$$
such that $A = \tau_r(A) + \widehat{d}\Theta_F^r(A)$ *for each* $A \in \Lambda^r\mathcal{C} \otimes_M \Omega^n$. *The* Θ_F^r *will be called the Poincaré-Cartan operator of degree* r.

Proof. It suffices to take $\Theta_F^r = \phi_F \circ (1 \otimes \mathrm{Id} - \mu_r)$. □

Remarks. 1) When $n = 1$, the Θ_F^r are not depending on F. 2) There are finite order analogous by restricting to $\Lambda^r\mathcal{C}_k \otimes_M \Omega^n$ and taking a decomposition F_k. For example, the $\Theta_{F_1}^r$ are uniquely defined and a $\Theta_{F_2}^r$ may be canonically selected.

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Departamento de Matemáticas, Universidad de Salamanca
Plaza de la Merced 1-4, E-37008 Salamanca, SPAIN
E-mail: ricardo@gugu.usal.es