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A REMARK ON FIRST NONLINEAR SPENCER SEQUENCE

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ABSTRACT. We describe an algorithm which allows one to compute the local formulas for first and second operators in the nonlinear first Spencer sequence up to any order. We then indicate how to construct the dual sequence which is related to the original sequence by a commutative diagram where the vertical arrows are induced by the inversion in the jet group. We hope that our approach will uncover the elementary nature of this sequence which has recently shown to be relevant in physics and also has some remarkable relations to connections and make it accessible to a larger audience.

1. Introduction

The principal aim of this article is to develop an algorithm for the local computation of first nonlinear Spencer sequence. Originally, this sequence was formulated at 1960's by D.C. Spencer in his research on the formal theory of systems of PDE. Since then, it has long been studied by various authors from different perspectives and recent works of J.F. Pommaret [3], [4] show that it can be useful in continuum mechanics, relativity and gauge theory. However, it is obvious that the abstract machinery of Spencer sequence appears to be difficult for non-mathematicians and, for example, the method of construction involved in [1] is quite complicated even for specialists. However, in spite of these theoretical difficulties, the fundamental ideas of Spencer sequence are amazingly simple, in fact, so simple that one can get the impression that construction just follows from definitions and elementary calculus. As stated in [1], the first fundamental idea of the sequence is that "The operators involved should measure the deviation of a section in a jet bundle from being holonomic". Although this statement is given for the first linear Spencer sequence, the corresponding first nonlinear complex is nothing but a finite form of it. Secondly, "The basic idea in the construction of the nonlinear complexes consists first in expressing the linear operator D as the adjoint operator D as the adjoint of a certain element with respect to a bracket and then in defining a finite form for this adjoint representation"[1]. Although this second statement seems to be complicated, this difficulty arises from terminology. However, one cannot get a clue for the algorithm of construction from these statements. Such a clue can be found in [2], where the local computations of Spencer sequence is given for the first time up to order 2.

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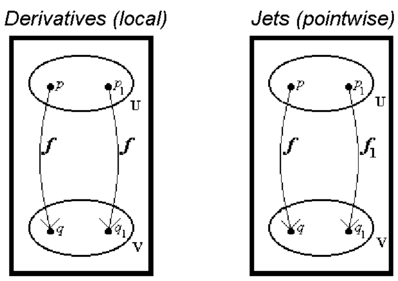
"... induce differential identities on the base manifold by using computations on a convenient principle bundle". However, in spite of the simplicity of the basic ideas, the method for the local computations of Spencer sequence in [2] is not easy, as the author himself remarks in [4]: "Also the reader should convince him/herself that it is almost impossible to write down the preceeding formulas for $q \geq 2$, although they look simple". In this note, we try to develop an algoritm for the local formulas of first nonlinear Spencer sequence at arbitrary orders and try to clarify the origins of the difficulty that makes local computaions almost impossible. Furthermore, our approach allows the formulation of another, 'dual' sequence which is related to the original sequence by inversion in jet group.

2. Preliminary definitions

Let M be an $n-$dimensional manifold, $aut\,(M,M)$ denote the set of all local diffeomorphisms of M. If $\left(x^i,U_p\right)$ and $\left(y^i,U_q\right)$ are two local coordinate neighborhoods of points $p,q\in M$ with $f:U_p\rightarrow U_y$, $q=f\,(p)$, then the point p is called the source and the point q is called the target of f. The $k-jet$ of f, $j_k\,(f)=\left(\partial_\mu f^i\right)$ $1\leq|\mu|\leq k$, is defined as $\partial_\mu f^i\,(x)=\frac{\partial^k f^i}{\partial x^{r_1}\ldots\partial x^{r_k}}$. If $f,g\in aut\,(M,M)$ can be composed, i.e. $y=f\,(x)$, the $k-jet$ of the composition $h=g\circ f$ obeys the chain rule, $j_k\,(h)=j_k\,(g)\circ j_k\,(f)$ which can be written componentwise as $\partial_\mu h^i\,(x)=\Omega^i_\mu\left(\partial_\sigma g^s\,(y),\partial_\rho f^r\,(x)\right)$ $1\leq|\mu|,|\rho|,|\sigma|\leq k$. Here, derivatives of h and f are taken with respect to x and derivatives of g are taken with respect to y. Then, $k-jets$ of elements of $aut\,(M,M)$ forms a transitive Lie pseudogroup with the multiplication law Ω. If the source is identified with target, i.e. $p=q$, then, we obtain a Lie group consisting of $k-jets$ of the elements of $aut\,(M,M)$ which fixes the point p.This Lie group, $k^{th}-order$ jet group, is denoted as $\Pi_k\,(p,p)$ at each point of M. Now, let us replace the parameters depending on *source* coordinates (x^r), by new parameters, , f^r_μ, so-called jet coordinates. Then, $h^i_\mu\,(x)=\Omega^i_\mu\left(\partial_\sigma g^s\,(y),f^r_\rho\,(x)\right)$and the space $\Pi_k\,(M,M)$, defined by jet coordinates, $\left(x^i,f^i_\mu\right)$, is a Lie groupoid of order k on M. Considered as a subbundle of the bundle of $k-jets$ of all maps with source and target on M, $J_k\,(M,M)$, $\Pi_k\,(M,M)$ can also be defined as the bundle of *invertible* $k-jets$.

Although *k-jets* and $k^{\rm th}$*-order derivatives of elements of* $aut\,(M,M)$, which will be denoted shortly as jets and derivatives respectively throughout this note, have the same transformation law under coordinate tranformations on M, the difference between them is one of the fundamental ideas of the first nonlinear Spencer sequence. First of all, a *derivative* is the derivative of a diffeomorhism but jets consists of derivatives of local diffeomorphism in a neighborhood of a point in M. Namely, diffeomorphisms defining a jet at each point of the neighborhood need not be same. In order to clarify this statement we have the following picture



The difference between jets and derivatives

Let us consider $\Pi_1(M,M)$, the bundle of invertible 1-jets. Let $p \in (x^i, U)$ and $F = (x^i, f^i(x), f^i_j(x))$ be a section of $\Pi_1(M,M)$. If F is a derivative, then $f^i_j(x) = \partial_j f^i(x)\ \forall x \in U$. However, if F is a jet, then, $f^i_j(p) = \partial_j f^i(x)\mid_{x=p}$ and $f^i_j(p_1) = \partial_j f^i_1(x)\mid_{x=p_1}$.

So far, we try to clarify the first fundamental idea of Spencer sequence. However, this is not sufficient for developing a local computation algorithm. The crucial point for this step is to "induce differential identities on the base manifold by using computations on a convenient principle bundle".

3. Algorithm

3.1. The operator D. The algorithm formulated in this note is based on a different reading of first nonlinear Spencer sequence. To begin with, consider the first three terms of the sequence

$$0 \longrightarrow aut(M,M) \xrightarrow{j_{k+1}} \Pi_{k+1}(M,M) \xrightarrow{\mathrm{D}} \ldots \tag{1}$$

According to this sequence, the derivatives of elements of M, regarded as sections of $\Pi_{k+1}(M,M)$ satisfy some differential identities with values in some space. Now, the first problem is to determine these differential identities. Since derivatives are regarded as sections of $\Pi_{k+1}(M,M)$, these differential identities must be related with the fundamental property of $\Pi_{k+1}(M,M)$: *The existence of inverse element in* $\Pi_{k+1}(M,M)$.

Let $\left(x^i, f^i(x), f^i_{j_1}, \ldots, f^i_{j_{k+1}\cdots j_1}(x)\right) = (f_n)$, $\quad 0 \le n \le k+1$, $\quad$ a section of $\Pi_{k+1}(M,M)$ and $(g_n) = (f_n)^{-1}$. Throughout this note (g_p) always denotethe inverse of (f_p). Then, we have the following identities

$$\Omega_n(g_m, f_m) - id_n = 0\,, \qquad 0 \le m \le n \le k+1$$

where id_n denote the identity section of $\Pi_n(M,M)$. Let us choose the last two identities, namely, $n = k$ and $n = k+1$.

$$\Omega_k(g_p, f_p) - id_k = 0\,, \qquad 0 \le p \le k \tag{2}$$

$$\Omega_{k+1}(g_q, f_q) - id_{k+1} = 0\,, \qquad 0 \le q \le k+1 \tag{3}$$

Differentiating (2) we obtain

$$\Omega_{k,1}(g_p, \partial(g_p), f_p, \partial(f_p)) - id_{k+1} = 0, 0 \le p \le k \tag{4}$$

where $\partial(g_p)$ denote the derivative of jet section (g_p), and $\Omega_{k,1}$ is the multiplication law of $J_1(\Pi_k(M,M))$. Note that (4) is a differential identity on $\Pi_k(M,M)$ and not on $aut(M,M)$. Hence, it cannot be the differential identity that we seek for. However, if (f_p) and (g_p) are derivatives, we have the following equality

$$\Omega_{k,1}(g_p, \partial(g_p), f_p, \partial(f_p)) - id_{k+1} = \Omega_{k+1}(g_q, f_q) - id_{k+1} = 0 \tag{5}$$

It is obvious that (5) holds only for derivatives and not true for jets in general, since $\partial(g_p) = g_{p+1}$ is valid only for derivatives. Furthermore, the differential identity defined by (5) is a differential identity on the $aut(M,M)$. Using this, we define the image of the derivatives of the elements of $aut(M,M)$ under D as the LHS of (5)

$$\mathrm{D} \circ (j_q(f)) \triangleq \Omega_{k,1}(g_p, \partial(g_p), f_p, \partial(f_p)) - id_{k+1} = 0 \tag{6}$$

However,if one read the remark in [2] mentioned in the introduction of this note carefully, finding a differential identity on the base manifold is only one part of the problem. One has to construct the sequence *by using computations on a convenient space*. Obviously, that space is nothing but $\Pi_k(M,M)$, but then, we have the following question: How to compute the image of D for jets? The answer of this question follows from the distinction between derivatives and jets, in concordance with the statement in [1].

The local coordinate formula for the image of D, can be computed as follows. First, solve g_{p+1} from (3) in terms of (f_{p+1}) and g_1, $0 \leq p \leq k$. Then , replace $\partial(g_p)$ in (4) by this expression. After this replacement, (4) is no longer a differential identity and RHS of (3) may not be zero, and it defines the image of jets under D. In order to clarify these steps let us look at some examples.

$p = 0$

Identity:	$g^i(f(x)) - x^i = 0$
Differentiate:	$g^i_r(f(x))\,\partial_j f^r(x) - \delta^i_j = 0$
Solve for g_1:	Trivial
Replace:	$\chi^i_{,j} = g^i_r(f(x))\,\partial_j f^r(x) - \delta^i_j$

$p = 1$

Identity:	$g^i_{r_1}(f(x))\, f^{r_1}_{j_1}(x) - \delta^i_j = 0$
Differentiate:	$\partial_{r_2} g^i_{r_1}(f(x))\, f^{r_1}_{j_1} \partial_{j_2} f^{r_2}(x) + g^i_{r_1}(f(x))\,\partial_{j_2} f^{r_1}_{j_1}(x) = 0$
Solve for g_2:	$g^i_{r_2 r_1}(f(x)) = -g^i_{s_1}(f(x))\, f^{s_1}_{m_2 m_1}(x)\, g^{m_2}_{r_2}(f(x))\, g^{m_1}_{r_1}(f(x))$
Replace:	$\chi^i_{j_1,j_2} = g^i_r(f(x))\left(\partial_{j_2} f^r_{j_1}(x) - f^r_{j_1 m}(x)\,\chi^m_{,j_2}(x) - f^r_{j_2 j_1}(x)\right)$

$p = k$

Identity:	$\Omega_k(g_q, f_q) - id_k = 0$
Differentiate:	$\partial_{g^n_\nu}\Omega^m_\mu(g_q, f_q)\,\partial_s g^n_\nu(f(x))\,\partial_j f^s(x) + \partial_{f^n_\nu}\Omega^m_\mu(g_q, f_q)\,\partial_j f^n_\nu(x) = 0$
Solve for g_{k+1}:	$\partial_{g^n_\nu}\Omega^m_\mu(g_q, f_q)\, g^n_{s\nu}(f(x)) = -\partial_{f^n_\nu}\Omega^m_\mu(g_q, f_q)\, f^n_{r\nu}(x)\, g^r_s(f(x))$
Replace:	$\chi^m_{\mu,j_{k+1}} = \partial_{f^n_\nu}\Omega^m_\mu(g_q, f_q)\Big(\partial_{j_{k+1}} f^n_\nu(x) - f^n_{r\nu}(x)\,\chi^r_{,j_{k+1}}(x) - f^n_{j_{k+1}\nu}\Big)$

After these local coordinate expressions, we can now formulate the space Im D. As the algorithm suggests,

$$\mathrm{D}\, f^m_{j_{k+1}\mu}(x) = \partial_{f^n_\nu}\Omega^m_\mu(g_q, f_q)\left(\partial_{j_{k+1}} f^n_\nu(x) - f^n_{r\nu}(x)\,\chi^r_{,j_{k+1}}(x) - f^n_{j_{k+1}\nu}\right). \tag{7}$$

Under a coordinate transformation, $\varphi = \psi^{-1}$, on the base manifold, $\chi^m_{\mu,j_{k+1}}(x)$ transform like $\chi^m_{\mu,j_{k+1}}(\varphi(x)) = \partial_{f^n_\nu}\Omega^m_\mu(\partial_\mu\varphi^i, f_q)\,\chi^n_{\nu,r}(x)\,\partial_{j_{k+1}}\psi^r$. It is obvious from this transformation rule that $\mathrm{Im}\,\mathrm{D} \subset T^* \otimes V(\Pi_k(M,M))$. However, this is not the common way of defining Im D. Instead of $T^* \otimes V(\Pi_k(M,M))$, $T^* \otimes J_k(T)$ is preferred. However, this is not a simple choice and stems from the fact that local computations are carried out by the association of $\Pi_{k+1}(M,M)$ with $J_k(T)$. In our opinion, it is that association itself, which is used for local computations, make local computations almost impossible. Neither the fundamental ideas, nor the operators involved in the sequence do not require such a $J_k(T)$ association.

3.2. Integrability Conditions. Now, we are in a position to extend the sequence one step to the right. This extension is possible due to the fact that D is a differential operator on $\Pi_{k+1}(M,M)$ subject to some integrability conditions. Before the general algorithm for arbitrary orders, let us look at some easy examples.

$p=0$

$\mathrm{D}(f_1) = \chi^i_{,j_1}(x) = g^i_{r_1}(f(x))\,\partial_{j_1}f^{r_1}(x) - \delta^i_{j_1}$

$$\partial_{j_1}f^i(x) = f^i_{r_1}(x)\left(\chi^{r_1}_{,j_1}(x) + \delta^{r_1}_{j_1}\right) \tag{8}$$

Differentiate this expression

$$\partial^2_{j_2j_1}f^i(x) = \partial_{j_2}f^i_{r_1}(x)\left(\chi^{r_1}_{,j_1}(x) + \delta^{r_1}_{j_1}\right) + f^i_{r_1}(x)\,\partial_{j_2}\chi^{r_1}_{,j_1}(x) \tag{9}$$

Now we will use

$$\mathrm{D}(f_2) = \chi^i_{j_1,j_2} = g^i_r(f(x))\left(\partial_{j_2}f^r_{j_1}(x) - f^r_{j_1m}(x)\,\chi^m_{,j_2}(x) - f^r_{j_2j_1}(x)\right)$$

in order to obtain

$$\partial_{j_2}f^i_{j_1}(x) = f^i_{r_1}(x)\,\chi^{r_1}_{j_1,j_2} + f^i_{j_1m}(x)\,\chi^m_{,j_2}(x) + f^i_{j_2j_1}(x) \tag{10}$$

Putting (10) into (8),

$$\begin{aligned}\partial^2_{j_2j_1}f^i(x) = \;& f^i_{r_1}(x)\left(\partial_{j_2}\chi^{r_1}_{,j_1}(x) + \chi^{r_1}_{s_1,j_2}\chi^{s_1}_{,j_1}(x) + \chi^{r_1}_{j_1,j_2}\right) + f^i_{j_1m}(x)\,\chi^m_{,j_2}(x)\\ &+ f^i_{j_2r_1}(x) + f^i_{j_2r_1}(x)\,\chi^{r_1}_{,j_1}(x) + f^i_{m_2m_1}(x)\,\chi^{m_2}_{,j_2}(x)\,\chi^{m_1}_{,j_1}(x)\end{aligned}$$

Skew-symmetrizing each side,

$$\partial_{j_2}\chi^{r_1}_{,j_1} - \partial_{j_1}\chi^{r_1}_{,j_2} + \chi^{r_1}_{s_1,j_2}\chi^{s_1}_{,j_1} - \chi^{r_1}_{s_1,j_1}\chi^{s_1}_{,j_2} + \chi^{r_1}_{j_1,j_2} - \chi^{r_1}_{j_2,j_1} = 0 \tag{11}$$

$p=1$

$$\begin{aligned}\chi^i_{j_1,j_2} &= g^i_r(f(x))\left(\partial_{j_2}f^r_{j_1}(x) - f^r_{j_1m}(x)\,\chi^m_{,j_2}(x) - f^r_{j_2j_1}(x)\right)\\ \chi^i_{j_1j_2,j_3} &= g^i_r(f(x))\left(\partial_{j_3}f^r_{j_2j_1}(x) - f^r_{s_1j_2j_1}(x)\,\chi^{s_1}_{,j_3}(x) - f^r_{s_1j_1}(x)\,\chi^{s_1}_{j_2,j_3}(x)\right.\\ &\quad\left. - f^r_{s_1j_2}(x)\,\chi^{s_1}_{j_2,j_3}(x) - f^r_{j_3j_2j_1}(x)\right)\end{aligned}$$

From these equations we obtain,

$$\partial_{j_2}f^i_{j_1}(x) = f^i_r(x)\,\chi^r_{j_1,j_2}(x) + f^i_{rj_1}(x)\,\chi^r_{,j_2}(x) + f^i_{j_2j_1}(x) \tag{12}$$

$$\begin{aligned}\partial_{j_3}f^i_{j_2j_1}(x) \;=\;& f^i_r(x)\,\chi^r_{j_1j_2,j_3}(x) + f^i_{rj_2j_1}(x)\,\chi^r_{,j_3}(x) + f^i_{rj_1}(x)\,\chi^r_{j_2,j_3}(x)\\ &+ f^i_{rj_2}(x)\,\chi^r_{j_1,j_3}(x) + f^i_{j_3j_2j_1}(x)\end{aligned} \tag{13}$$

Differentiating (12) ,using (13)and then skew-symmetrizing last two indexes we obtain a new integrability condition,

(14)

$$\begin{aligned}&\partial_{j_3}\chi^r_{j_1,j_2} - \partial_{j_2}\chi^r_{j_1,j_3} + \chi^r_{s,j_3}\chi^s_{j_1,j_2} - \chi^r_{s,j_2}\chi^s_{j_1,j_3} + \chi^r_{j_1s,j_3}\chi^s_{j_1,j_2} - \chi^r_{j_1s,j_2}\chi^s_{j_1,j_3}\\ &+ \chi^r_{j_1j_2,j_3} - \chi^r_{j_1j_3,j_2} = 0\end{aligned}$$

The general procedure for the calculation of higher order integrability conditions is more complicated but similar. We have,

$$\chi^i_{\mu,j_{k+1}} = \partial_{f^n_\nu}\Omega^i_\mu(g_q,f_q)\mid_{g=f^{-1}}\left(\partial_{j_{k+1}}f^n_\nu - f^n_{\nu r}\chi^r_{,j_{k+1}} - f^n_{j_{k+1}\nu}\right).$$

Now, we try find an equation in the form of $\partial_{j_{k+1}}f^i_\mu = \Gamma^i_{j_{k+1}\mu}\left(\left(f^r_\rho\right),\left(\chi^n_{\nu,s}\right)\right)$. In order to obtain this expression we prove a lemma which is nothing but the application of first fundamental theorem of Lie to $\Pi_k(M,M)$.

Lemma. $\partial_{h^r_\rho}\Omega^i_\mu(f_q,h_q)\mid_{h=id}\partial_{f^n_\nu}\Omega^r_\rho(g_q,f_q)\mid_{g=f^{-1}} = \delta^i_n\delta^\nu_\mu$, *where all the Greek letters are multi-indices and* δ^ν_μ *is zero except the upper and lower multi-indices are same.*

Proof. By the first fundamental theorem of Lie,

$$\partial_{h^r_\rho}\Omega^i_\mu(f_q,h_q)\mid_{h=id}\partial_{f^n_\nu}\Omega^r_\rho(g_q,f_q)\mid_{g=f^{-1}} = \partial_{f^n_\nu}f^i_\mu = \delta^i_n\delta^\nu_\mu,$$

where $\partial_{h^r_\rho}\Omega^i_\mu(f_q,h_q)\mid_{h=id} = \eta^{i\rho}_{r\mu}(f)$ is the infinitesimal generator of the right action of $\Pi_k(M,M)$ on itself. Also note that, $\eta^{i\nu}_{n(r\mu)}(f) = \eta^{i\sigma}_{n\mu}(f)\delta^\nu_{r\sigma} + \partial_{f^s_\sigma}\eta^{i\nu}_{n\mu}(f)f^s_{r\sigma} + f^i_{n\mu}\delta^r_\nu$. □

Then, we obtain the following expression,

$$\partial_{j_{k+1}}f^i_\mu = \eta^{i\nu}_{n\mu}(f)\chi^n_{\nu,j_{k+1}} + f^i_{r\mu}\chi^r_{,j_{k+1}} + f^i_{j_{k+1}\mu} \tag{15}$$

Differentiating and skew-symmetrizing (15) leads to,

$$\begin{aligned}&\eta^{i\nu}_{n\mu}(f)\left(\partial_{j_{k+1}}\chi^n_{\nu,j_k} - \partial_{j_k}\chi^n_{\nu,j_{k+1}} + \chi^n_{vr,j_{k+1}}\chi^r_{,j_k} - \chi^n_{vr,j_k}\chi^r_{,j_{k+1}} + \chi^n_{\nu j_k,j_{k+1}} - \chi^n_{\nu j_{k+},,j_k}\right)\\ &\quad + f^i_{r\mu}\left(\partial_{j_{k+1}}\chi^r_{,j_k} - \partial_{j_k}\chi^r_{,j_{k+1}} + \chi^r_{s,j_{k+1}}\chi^s_{,j_k} - \chi^r_{s,j_k}\chi^s_{,j_{k+1}} + \chi^r_{j_k,j_{k+1}} - \chi^r_{j_{k+1},j_k}\right)\\ &\quad + \left(\partial_{f^r_\rho}\eta^{i\nu}_{n\mu}(f)\eta^{r\sigma}_{s\rho}(f) - \partial_{f^r_\rho}\eta^{i\sigma}_{s\mu}(f)\eta^{r\nu}_{n\rho}(f)\right)\chi^n_{\nu,j_k}\chi^s_{\sigma,j_{k+1}} = 0\end{aligned}$$

Using second fundamental theorem of Lie, we can further simplify the expression above and obtain the highest order integrability condition as

$$\begin{aligned}&\partial_{j_{k+1}}\chi^i_{\mu,j_k} - \partial_{j_k}\chi^i_{\mu,j_{k+1}} + 2C^{i\rho\sigma}_{\mu rs}\chi^r_{\rho,j_k}\chi^s_{\sigma,j_{k+1}} + \chi^i_{\mu s,j_{k+1}}\chi^s_{,j_k} - \chi^i_{\mu s,j_k}\chi^s_{,j_{k+1}}\\ &+\chi^i_{\mu j_k,j_{k+1}} - \chi^i_{\mu j_{k+},,j_k} = 0\end{aligned} \tag{16}$$

where $C^{n\rho\sigma}_{\nu rs}$ are the structure constants of the Lie algebra of the jet group gl_k.The crucial point is that none of the integrability conditions of D depend on $\Pi_{k+1}(M,M)$, i.e.(f_q). This fact allows us to extend the sequence one step to right and add a new space. The space of integrability conditions, as one can easily check, is nothing but $\Lambda^2T^*\otimes V(\Pi_{k-1}(M,M))$. The image of the operator D':$T^*\otimes V(\Pi_k)\to\Lambda^2T^*\otimes V(\Pi_{k-1})$ is defined by the formula

$$\begin{aligned}\mathrm{D}'(\chi) = \widehat{K}^i_{\mu j_k,j_{k+1}} \;=\;& \partial_{j_{k+1}}\chi^i_{\mu,j_k} - \partial_{j_k}\chi^i_{\mu,j_{k+1}} + 2C^{i\rho\sigma}_{\mu rs}\chi^r_{\rho,j_k}\chi^s_{\sigma,j_{k+1}}\\ &+\chi^r_{s,j_{k+1}}\chi^s_{,j_k} - \chi^r_{s,j_k}\chi^s_{,j_{k+1}} + \chi^i_{\mu j_k,j_{k+1}} - \chi^i_{\mu j_{k+},,j_k}\end{aligned} \tag{17}$$

In this way we obtain the exact sequence

$$0\to aut(M,M)\xrightarrow{j_{k+1}}\Pi_{k+1}(M,M)\xrightarrow{\mathrm{D}}T^*\otimes V(\Pi_k)\xrightarrow{\mathrm{D}'}\Lambda^2T^*\otimes V(\Pi_{k-1}). \tag{18}$$

4. Dual Sequence

4.1. Algorithm. The algorithm for the local computations of the dual sequence is quite similar to the original sequence. The idea of dual sequence stems from the fact that, we made a choice in the computation of the image of the operator D without giving any reason. First, we preferred to solve g_{p+1} from (3) instead of

f_{p+1} and then replace $\partial(g_p)$ with this expression. This choice becomes meaningful in the context of writing formulas over source. However,this is not obligatory and we can construct an algorithm for the sequence over target. The algorithm for the dual sequence can be constructed in two steps as in the case of original one. First, in the computation of the local coordinate expression for D, solve f_{p+1} in terms of (g_p) and $f_1$by using (3) and replace $\partial(f_p)$ in (4) by this expression. After similar computations the image of D can be found as

$$\mathrm{D}(g_{p+1}) = \xi^i_{\mu,j_{k+1}} = \partial_{g^n_\nu}\Omega^i_\mu(g_q, f_q)(\partial_r g^n_\nu - g^n_{r\nu}) f^r_{j_{k+1}}\,. \tag{19}$$

The integrability condition of D is also obtained similarly,

$$\begin{gathered}\partial_r\xi^i_{\mu,j_k}\left(\xi^r_{,j_{k+1}} + \delta^r_{j_{k+1}}\right) - \partial_r\xi^i_{\mu,j_{k+1}}\left(\xi^r_{,j_k} + \delta^r_{j_k}\right)\\ +\xi^i_{\mu,r}\left(\xi^r_{j_k,j_{k+1}} - \xi^r_{j_{k+1},j_k}\right) + 2C^{v\rho\sigma}_{\nu rs}\xi^r_{\rho,j_k}\xi^s_{\sigma,j_{k+1}} = 0\,,\end{gathered}$$

$$\begin{aligned}\mathrm{D}'\left(\xi^i_{\mu,j_{k+1}}\right) &= \partial_r\xi^i_{\mu,j_k}\left(\xi^r_{,j_{k+1}} + \delta^r_{j_{k+1}}\right) - \partial_r\xi^i_{\mu,j_{k+1}}\left(\xi^r_{,j_k} + \delta^r_{j_k}\right)\\ &+2C^{v\rho\sigma}_{\nu rs}\xi^r_{\rho,j_k}\xi^s_{\sigma,j_{k+1}} + \xi^i_{\mu,r}\left(\xi^r_{j_k,j_{k+1}} - \xi^r_{j_{k+1},j_k}\right) + \xi^i_{\mu j_k,j_{k+1}} - \xi^i_{\mu j_{k+1},j_k}\,.\end{aligned} \tag{20}$$

The relation of first nonlinear Spencer sequence and its dual.
The computations above provides the following diagram.

$$\begin{array}{ccccccccc} 0 \to & aut(M,M) & \xrightarrow{j_{k+1}} & \Pi_{k+1}(M,M) & \xrightarrow{\mathrm{D}} & T^*\otimes V(\Pi_k) & \xrightarrow{\mathrm{D}'} & \Lambda^2T^*\otimes V(\Pi_{k-1}) \\ & \downarrow inv & & \downarrow inv & & \downarrow I_1 & & \downarrow I_2 \\ 0 \to & aut(M,M) & \xrightarrow{j_{k+1}} & \Pi_{k+1}(M,M) & \xrightarrow{\mathrm{D}} & T^*\otimes V(\Pi_k) & \xrightarrow{\mathrm{D}'} & \Lambda^2T^*\otimes V(\Pi_{k-1}) \end{array}$$

where I_1 and I_2 are defined as

$$\begin{aligned} I_1(\chi) &= \xi^i_{\mu,j_{k+1}} = -\chi^i_{\mu,r}\left(\chi^r_{,j_{k+1}} + \delta^r_{j_{k+1}}\right)^{-1}\,,\\ I_2\left(\widehat{K}\right) &= \widetilde{K}^i_{\mu j_k j_{k+1}} = -\widehat{K}^i_{\mu r_k r_{k+1}}\left(\chi^{r_k}_{,j_k} + \delta^{r_k}_{j_k}\right)^{-1}\left(\chi^{r_{k+1}}_{,j_{k+1}} + \delta^{r_{k+1}}_{j_{k+1}}\right)^{-1}\\ &+ \chi^i_{\mu,s}\left(\chi^s_{,r} + \delta^s_r\right)^{-1}\widehat{K}^r_{r_k r_{k+1}}\left(\chi^{r_k}_{,j_k} + \delta^{r_k}_{j_k}\right)^{-1}\left(\chi^{r_{k+1}}_{,j_{k+1}} + \delta^{r_{k+1}}_{j_{k+1}}\right)^{-1}\,.\end{aligned}$$

These formulas show that first order terms has a special importance. If $\chi^i_{,j} = 0$ then,

$$\xi^i_{\mu,j_{k+1}} = -\chi^i_{\mu,j_{k+1}} \text{ and } \widetilde{K}^i_{\mu j_k j_{k+1}} = -\widehat{K}^i_{\mu j_k j_{k+1}}\,.$$

5. Conclusion

The algorithm developed in this note is not theoretically novel. All of the theoretical machinery has long been well-known. However, this algorithm has two advantages. First, at the practical level, it allows to compute local coordinate expressions of operators in the first nonlinear Spencer sequence. All of the steps involved in the algorithm can be carried out by a computer, and explicit numerical calculations become possible which we believe to be important for applications of this sequence in engineerinng branches. Secondly, this algorithm is consistent with the idea that this sequence is based on a simple differential identity on the base manifold. We believe that studying differential identities on base manifold can provide some insight for other important sequences. It is shown in [5] that using the differential identity provided by associativity property of $\Pi_k(M,M)$, the nonlinear

Janet sequence of connections, global gauge sequence, curvature and torsion tensors of arbitrary order can be computed in local coordinate expressions.

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