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THE ROZANSKY-WITTEN INVARIANTS OF HYPERKÄHLER MANIFOLDS

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Abstract. We investigate invariants of hyperkähler manifolds introduced by Rozansky and Witten. For each tri-valent graph with $2n$ vertices we get an invariant of a hyperkähler manifold of dimension $4n$. The invariants are the same for cohomologous graphs (graphs equivalent under the IHX relations). This allows us to use hyperkähler manifolds to define elements in the dual of the graph cohomology. Conversely, we regard elements in the dual of the graph cohomology (such as those arising in Chern-Simons theory) as *virtual* hyperkähler manifolds. Certain combinations of graphs give rise to invariants which can be identified with the Chern numbers, and we use the virtual manifolds to obtain further unexpected relations between the graph invariants and Chern numbers.

1. Introduction. In Rozansky-Witten [10] new invariants of 3-manifolds were constructed from a 3-dimensional topological σ-model whose target space is a (compact or asymptotically flat) hyperkähler manifold X. A Feynman diagram calculation of the partition function shows that it is a finite type invariant of 3-manifolds similar to those appearing in perturbative Chern-Simons theory. In particular, from Chern-Simons theory we get invariants of a 3-manifold Y which look like $\sum c_\Gamma(\mathfrak{g}) I_\Gamma(Y)$, where Γ is a tri-valent graph, $I_\Gamma(Y)$ is the standard integral of the product of linking forms, and the *weights* $c_\Gamma(\mathfrak{g})$ depend on the Lie algebra $\mathfrak{g}$ of the gauge group (see [8]). The new approach gives us a system of weights $b_\Gamma(X)$ depending on the hyperkähler manifold X such that the 3-manifold invariant has the above form.

We will call these weights *graph invariants*, and in fact they depend only on the cohomology class of the graph Γ. Hence we can think of hyperkähler manifolds as being elements of the dual of the graph cohomology. Conversely, we can regard elements of the dual of the graph cohomology as being *virtual* hyperkähler manifolds. For example, the system of weights occurring in perturbative Chern-Simons

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theory gives us elements of the dual of the graph cohomology and we will look quite closely at the virtual hyperkähler manifolds arising when the gauge group is SU(2). In particular, by calculating the graph invariants on these virtual manifolds we arrive at a formula relating a certain graph invariant, called Θ^n, to the Chern numbers (in general, the Chern numbers of a hyperkähler manifold can be identified with certain linear combinations of graph invariants).

This formula holds for the virtual manifolds arising in SU(2) Chern-Simons theory, but there is strong evidence to suggest that it holds for all hyperkähler manifolds. For example, when X is the Hilbert scheme of points on a K3 surface, we can calculate the value of Θ^n by following Kapranov's approach [7], which liberates the graph invariants from their metric dependence by reformulating the calculation in terms of the Atiyah class. In low dimensions, the Chern numbers of the Hilbert schemes have been calculated by Göttsche [4], and it is then a simple matter to verify the formula in this case.

For compact X, the invariant Θ^n can also be expressed in terms of the L^2-norm of the curvature and the volume of X. Hence, if the formula were to hold in general, we would obtain an expression for the L^2-norm of the curvature of a compact hyperkähler manifold in terms of characteristic numbers and the volume.

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2. Definition of the graph invariants. Let X be a hyperkähler manifold of real dimension $4n$, with tangent bundle $T = TX$. Then X has a Riemannian metric such that the holonomy of the Levi-Civita connection lies in an $\mathrm{Sp}(n)$ subgroup of $\mathrm{SO}(4n)$. The Riemann curvature tensor is a $(1,1)$-form $R_{k\bar{l}}$ taking values in $\mathrm{End}(T)$, which we can identify with the Lie algebra of $\mathrm{Sp}(n)$. Using the holomorphic symplectic form $\omega_{ij} \in \Gamma(X, \bigwedge^2 T^*) = \Omega^{2,0}(X)$ to identify T and the cotangent bundle T^*, we can rewrite the curvature as

$$\Phi_{ijk\bar{l}} = \omega_{im} R^m{}_{jk\bar{l}},$$

where the m and j coordinates on R refer to $\mathrm{End}(T)$. By the above comments, Φ must be symmetric in i and j. It is also symmetric in j and k as the curvature is torsion-free, and hence $\Phi \in \Omega^{0,1}(S^3 T^*)$.

Let Γ be an oriented tri-valent graph with $2n$ vertices. The orientation means that we associate a direction to each edge and we order the vertices (up to an even number of changes). We take a copy of $\Phi_{ijk\bar{l}}$ for each vertex of Γ and associate i, j, and k to the edges attached to the vertex. Using the inverse of the holomorphic symplectic form, $\omega^{ij} = \omega^{-1} \in \Gamma(X, \bigwedge^2 T)$, we contract indices along the edges of Γ. The orientation of Γ determines the order in which we multiply the Φ's associated to the vertices, and the order of the indices on the ω^{-1}'s associated to the edges. The result is $\Gamma(\Phi) \in \Omega^{0,2n}(X)$. Finally, multiplying this by $\omega^n \in \Omega^{2n,0}(X)$ and integrating over X gives us the graph invariant

$$\Gamma(X) = \int_X \Gamma(\Phi)\omega^n \in \mathbb{C}.$$

The simplest example is when $n = 1$ and Γ is the unique tri-valent graph Θ with two vertices. Then

$$\Theta(\Phi)_{\bar{l}_1\bar{l}_2} = \omega^{i_1i_2}\omega^{j_1j_2}\omega^{k_1k_2}\Phi_{i_1j_1k_1\bar{l}_1}\Phi_{i_2j_2k_2\bar{l}_2} \in \Omega^{0,2}(X)$$

and

$$\Theta(X) = \int_X \Theta(\Phi)_{\bar{l}_1\bar{l}_2}\omega_{m_1m_2}e^{m_1} \wedge e^{m_2} \wedge e^{\bar{l}_1} \wedge e^{\bar{l}_2},$$

where e^m and $e^{\bar{l}}$ are local bases for $\Omega^{1,0}$ and $\Omega^{0,1}$ respectively.

The original description by Rozansky and Witten (in [10]) of how to construct the invariant $\Gamma(X)$ is slightly different. The complexified tangent bundle $TX_{\mathbb{C}}$ decomposes as $V \otimes S$, where V is a rank $2n$ complex vector bundle with structure group $\mathrm{Sp}(n)$ and S is a trivializable rank two bundle. When $n = 1$, for example, X is four-dimensional and V and S are just the spinor bundles. The Levi-Civita connection on $TX_{\mathbb{C}}$ reduces to an $\mathrm{Sp}(n)$-connection on V (times the trivial connection on S), and they proceed with the curvature of this connection. Once one has chosen a hyperkähler metric on X, the compatible complex structures lie on a sphere corresponding to the trivializations of the bundle S. So by their approach it is clear that $\Gamma(X)$ does not depend on the choice of compatible complex structure, since no trivialization of S is assumed.

In the next section we will describe an alternative way of constructing $\Gamma(X)$ from which it will be clear that there is no metric dependence either, and hence that $\Gamma(X)$ is invariant under arbitrary deformations of the hyperkähler metric.

Reversing the orientation of Γ changes the sign of the invariant. Less obvious is the fact that if three graphs Γ_I, Γ_H, and Γ_X are related by the IHX relation (see Figure 1), then the corresponding invariants are likewise related. This result is a consequence of the Bianchi identity for the curvature, which says $DR_{k\bar{l}} = 0$ (where D is the exterior derivative defined using the Levi-Civita connection). We can use this identity to show that the integrand

$$\Gamma_I(\Phi) - \Gamma_H(\Phi) + \Gamma_X(\Phi)$$

is a total derivative, from which the result follows. Thus the graph invariants depend only on the cohomology class of Γ in the *graph cohomology* (equivalence classes of oriented tri-valent graphs under the IHX and orientation reversing relations). Hence we can think of hyperkähler manifolds as elements of the dual of the graph cohomology.

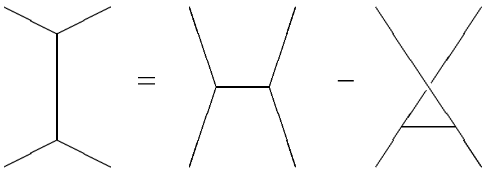


Figure 1: The IHX relation

On a hyperkähler manifold X of real dimension $4n$ we shall be particularly interested in the invariant corresponding to the tri-valent graph Θ^n (ie. the disjoint union of n copies of Θ). For compact X, it can be expressed as

$$\Theta^n(X) \quad = \quad \lambda_n \|R\|_2^{2n}/(\mathrm{vol}(X))^{n-1}, \tag{1}$$

where $\|R\|_2$ is the L^2-norm of the curvature of X, $\mathrm{vol}(X)$ is the volume of X, and λ_n is some universal constant. Also important is the following lemma:

Lemma 1. The invariant $\Theta^n/n!$ is multiplicative.

Proof We can use the canonical identification $T(X \times Y) = TX \times TY$ to write the curvature of $X \times Y$, which lies in $\Omega^{1,1}(\mathrm{End}(T(X \times Y)))$, in block from. The result easily follows. □

3. Kapranov's approach via the Atiyah class. The Bianchi identity implies that $\Phi \in \Omega^{0,1}(S^3T^*)$ is $\bar{\partial}$-closed, and hence it can be regarded as the representative of a Dolbeault cohomology class. In fact, $\alpha = [\Phi] \in \mathrm{H}^1(X, S^3T^*)$ is the Atiyah class of X. The graph invariants defined above depend only on the cohomology class of Φ. More explicitly, we can define $\Gamma(\alpha) \in \mathrm{H}^{2n}(X, \mathcal{O})$ by taking a copy of α for each vertex of Γ and contracting indices along edges, like before. Since X is hyperkähler, $\mathrm{H}^{2n}(X, \mathcal{O})$ is one-dimensional with a canonical basis. Hence $\Gamma(\alpha)$ can be thought of as a number, which is precisely $\Gamma(X)$.

The beauty of this approach is that we no longer require an explicit hyperkähler metric. In fact, X does not have to be hyperkähler at all; the above construction works whenever X is complex and holomorphic symplectic. All we need is a holomorphic symplectic form and some representation of the Atiyah class. In general, the Atiyah class α_E of a bundle E on X ($\alpha = \alpha_T$ is the Atiyah class of the tangent bundle) is the cohomological obstruction to the existence of a global holomorphic connection on E, and can be described by Čech cohomology, Dolbeault cohomology, or even as the residue of a meromorphic connection with simple poles along a divisor of X.

Using this latter description, we are able to calculate the value of the invariant Θ^n on the Hilbert scheme $X^{[n]}$ of n points on a K3 surface X.

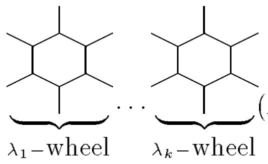
Proposition 2.

$$\Theta^n(X^{[n]}) = (2\pi i)^{2n} 12^n (n+3)^n .$$

4. Chern numbers. The Chern numbers of X can also be written in terms of integrals of the curvature, and hence there are relations between them and the graph invariants. By Chern-Weil theory, the Chern character of X can be written

$$\mathrm{ch}(T) = \sum_{j=0}^{2n} \frac{1}{j!}\mathrm{Tr}\big((\frac{R}{2\pi i})^j\big)\,,$$

where the product R^j is in $\Omega^{j,j}(\mathrm{End}(T))$, and taking the trace in $\mathrm{End}(T)$ reduces this to a (j,j)-form. We can use this to show that if $(\lambda_1, \ldots, \lambda_k)$ is a partition of $2n$, then

$$(-1)^k(2\pi i)^{2n} s_{\lambda_1}\cdots s_{\lambda_k}(X) = \sum$$


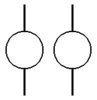
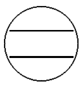
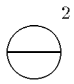
(X)

where $s_\lambda = \lambda!\mathrm{ch}_\lambda$ is a rescaling, and the sum is taken over all different ways of joining the spokes of the wheels (and thus obtaining a tri-valent graph).

It is interesting to note here that the Atiyah class was first introduced in order to provide a holomorphic realization of the Chern classes, contrasting with the use of connections in Chern-Weil theory (see [1]). The Atiyah class is the cohomological obstruction to the existence of a (global) holomorphic connection.

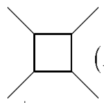
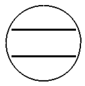
For $n = 1, 2$, and 3 we can invert the above relations, and hence it is possible to write all of the graph invariants in terms of the Chern numbers. For $n \geq 4$ this is no longer the case as the dimension of the connected component of the graph cohomology becomes greater than one (although the general pattern is unknown, for small n this dimension behaves like 1, 1, 1, 2, 2, 3, 4, 5, 6, 8, ...). In other words, there are more graph invariants than Chern numbers when $n \geq 4$.

As an example, consider the above relations when $n = 2$. We find

$$(2\pi i)^4 s_2^2(X) = \sum$$

$(X) = 2$

$(X) +$


(X)

and

$$-(2\pi i)^4 s_4(X) = \sum$$

$(X) = \frac{5}{2}$

$(X)\,,$

where we have used the IHX relation to rewrite certain graphs. It follows that

$$\begin{aligned}\Theta^2(X) &= (2\pi i)^4(s_2^2(X) + \frac{4}{5}s_4(X)) \\ &= (2\pi i)^4\frac{4}{5}(7c_2^2(X) - 4c_4(X))\,.\end{aligned}$$

Recall that for compact X we have equation (1):

$$\Theta^2(X) = \lambda_2 ||R||_2^4/\mathrm{vol}(X)\,,$$

for some constant λ_2, which happens to be positive. Hence $\Theta^2(X)$ must be positive, and thus

$$7c_2^2(X) - 4c_4(X) > 0\,.$$

Together with the fact that the Todd-genus, $\mathrm{Td} = (3c_2^2 - c_4)/720$, of an 8-real-dimensional hyperkähler manifold must equal three, this gives an upper bound for the Euler characteristic of X:

$$c_4(X) \quad < \quad 3024. \tag{2}$$

5. Chern-Simons theory. We also get invariants associated to graph cohomology classes in perturbative Chern-Simons theory. Instead of the curvature of a hyperkähler manifold, we use the structure constants of $\mathfrak{g}$ (the Lie algebra of the gauge group G), and instead of the holomorphic symplectic form we use a multiple of the Killing form to contract along edges. We take an orientation of Γ to mean an ordering of the three edges attached to each vertex (again up to an even number of changes), but in fact this is equivalent to our earlier notion of orientation, as can be shown by passing via the first homology group of the graph (Kapranov [7] gives a very clear account). These invariants clearly change sign under orientation reversal. We can also show that the IHX relation follows from the Jacobi identity, and hence these invariants descend to invariants of the graph cohomology. Thus we get elements of the dual of the graph cohomology, which we can think of as *virtual* hyperkähler manifolds.

For $G = \mathrm{SU}(2)$, we can calculate the numerical values of the powers of Θ and the various sums of wheels, and compare them (recall that the sums of wheels are proportional to the Chern numbers). The virtual hyperkähler manifolds that arise form a basic sequence in Hirzebruch's terminology (see [6]), and following his work on multiplicative sequences we find that a generating sequence for $\Theta^n/n!$ is $\mathrm{Td}^{1/2\dagger}$, where Td is the Todd genus. Thus we find

$$\begin{aligned}
\Theta &= -(2\pi i)^2 s_2\,,\\
\Theta^2 &= (2\pi i)^4 (s_2^2 + \frac{4}{5} s_4)\,,\\
\Theta^3 &= -(2\pi i)^6 (s_2^3 + \frac{12}{5} s_2 s_4 + \frac{64}{35} s_6)\,,\\
\Theta^4 &= (2\pi i)^8 (s_2^4 + \frac{24}{5} s_2^2 s_4 + \frac{48}{25} s_4^2 + \frac{256}{35} s_2 s_6 + \frac{1152}{175} s_8)\\
\Theta^5 &= -(2\pi i)^{10} (s_2^5 + 8 s_2^3 s_4 + \frac{48}{5} s_2 s_4^2 + \frac{128}{7} s_2^2 s_6 + \frac{1152}{35} s_2 s_8 + \frac{512}{35} s_4 s_6 + \frac{12288}{385} s_{10})
\end{aligned}$$

etc. What this means is that these relations hold for disjoint sums and products of the virtual hyperkähler manifolds associated to SU(2) Chern-Simons theory.

On the other hand, we already know that for $n = 1, 2$, and 3 we can write all of the graph invariants in terms of the Chern numbers (this applies to *all* hyperkähler manifolds). Indeed if we do this, we get precisely the above relations for Θ, Θ^2, and Θ^3. What is surprising is that we can show that the fourth relation also holds for all hyperkähler manifolds (by rewriting the right hand side in terms of sums of wheels, then simplifying the resulting graphs using the IHX relation).

†At least up to some factors. The formulae that follow incorporate these adjustments.

Furthermore, we can use Proposition 2 to check these relations for the Hilbert scheme $X^{[n]}$ of n points on a K3 surface X. The Chern numbers of these manifolds are known for $n \leq 6$ (due to Göttsche, [4]), and it is a simple matter to verify the relations are satisfied in these cases. Thus there is strong evidence to suggest the following:

Conjecture 3. For all hyperkähler manifolds, a generating sequence for $\Theta^n/n!$ is $\mathrm{Td}^{1/2}$.

One consequence of the conjecture, together with equation (1), is that the L^2−norm of the curvature of a compact hyperkähler manifold can be expressed in terms of characteristic numbers and the volume. We also obtain inequalities involving the Chern numbers, in a similar vein to the inequality (2).

The other well-known family of hyperkähler manifolds is the generalized Kummer varieties, $T^{[[n]]}$. However, even less is known about their Chern numbers: for $n \leq 3$ it is possible to calculate them from the Hirzebruch χ_y-genus (known for all n; see [5]) using the Riemann-Roch formula, but for greater n this method breaks down. Thus while it seems likely that we should be able to calculate the value of the invariant Θ^n on $T^{[[n]]}$ using Kapranov's approach, at this stage we would still be unable to verify Conjecture 3 for these manifolds beyond $n = 3$ (and of course we already know that for $n \leq 4$ it is true for *all* hyperkähler manifolds).

It should be noted that Conjecture 3 can be interpreted at two different levels. As stated above, it gives a sequence of relations involving graph invariants of hyperkähler manifolds (bear in mind that the Chern numbers occurring in $\mathrm{Td}^{1/2}$ can all be written in terms of graph invariants). A stronger result would be a sequence of relations involving the underlying graph cohomology classes from which the graph invariants are constructed.

6. Final comments. One obvious question when dealing with *virtual* hyperkähler manifolds is: when are they realized by *actual* hyperkähler manifolds? Given that so few compact examples are known, being able to associate actual manifolds to the virtual ones would be a major breakthrough, albeit a somewhat optimistic aim. One of the first obstacles is the integrality of the characteristic numbers. For example, we can check that the Chern numbers of the virtual manifolds arising in SU(2) Chern-Simons theory are integral. Even so, this still leaves us a long way from finding an actual realization.

We used the example of SU(2) because it made the calculations involved tractable, but there is no reason to restrict our attention to this particular gauge group. Further evidence for Proposition 3 could be obtained by duplicating the calculations of Section 5 for different gauge groups.

Appendix. Since presenting this work at the Brno conference, the author was shown the article [2]. It is fairly easy to show that Conjecture 2 of that article (the wheeling isomorphism) implies the relation in graph cohomology from which our Conjecture 3 follows, after constructing the corresponding graph invariants. In other words, it implies the stronger form of our conjecture. Furthermore, the wheeling isomorphism conjecture is proved on the level of Lie algebras (where it is essentially the Duflo isomorphism [3]). This is the generalization of our perturbative Chern-Simons theory, ie. it involves attaching the structure constants of $\mathfrak{g}$ (the Lie algebra of the gauge group) to the vertices and contracting along edges using the Killing form. It immediately follows that Conjecture 3 holds for the virtual hyperkähler manifolds arising in Chern-Simons theory with *any* gauge group. In particular, this also means that the wheeling isomorphism holds up to degree 12 (where the degree is half the number of vertices, which is n in our case), and hence Conjecture 3 is true for all hyperkähler manifold up to real dimension 48.

Finally, the postscript in [2] suggests that the wheeling isomorphism follows in general from the results in [9]. Of course, it is possible that a simpler proof of Conjecture 3 exists, either on the level of graph invariants or as the more general statement about graph cohomology.

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