DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 419–427

METHODS OF NON–ASSOCIATIVE ALGEBRA IN DIFFERENTIAL GEOMETRY

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Abstract. A new field at the junction of algebra and geometry, so called loopuscular geometry, is surveyed. Open problems and methods of their solutions are discussed.

0. Introduction

In our report we intend to present a new vision of geometry via Non-associative Algebra. This vision is based on the work of mine and my school during the last twenty six years. See Bibliography and our monograph "Smooth quasigroups and Loops" Kluwer academic publishers 1999 (to appear).

Recent interest of Physicists in nonassociative addition of velocities in Special Relativity has been resulted in the concept of Gyrogroup [A. Ungar 94a, 94 b] (which appeared to be a Left Bol loop with the left Bruck identity) and following understanding that this is in close relation with Hyperbolic geometry (Lobachevski spaces). But it is quite remarkable that any Affinely connected (in particular, Riemannian) space may be treated in a similar way in the language of Loop theory which suggests a new instrument for General Relativity and Mathematical Physics.

In the first part we survey the foundations from a geometric point of view with emphasis on the smooth quasigroup theory.

In the second part we discuss the construction of loop action theory and promises for its applications to the different geometric matters.

I. Loopuscular Geometry

1.1 . How loops appear in Geometry? Let (M, ∇) be a smooth affinely connected manifold (for example, a Lobachevski space). Having given three points a, x, y from a normal neighbourhood of a point a, one may define an operation $x \underset{a}{*} y$ (binary if a is fixed) by means of the parallel translation of the geodesic arc $\smile{ay}$

1991 *Mathematics Subject Classification.* 20N05, 53C35.
Key words and phrases. Symmetric spaces, loops, quasigroups, geoodules, affine connections, loop actions.
This paper is in final form and no version of it will be submitted for publication elsewhere.

along the geodesic arc $\breve{ax}$, that is the arc $x(\breve{x \underset{a}{*} y})$ should be parallel to the arc $\breve{ay}$ along the arc $\breve{ax}$.

The same may be presented as

$$x \underset{a}{*} y = \mathrm{Exp}_x \tau_x^a \mathrm{Exp}_a^{-1} y\,, \tag{1}$$

where Exp_c is the exponential map and τ_x^a is the parallel translation along the geodesic arc $\breve{ax}$.

The system $\langle M, \underset{a}{*}, a\rangle$ is a *loop*, that is, $p \underset{a}{*} y = q$ and $x \underset{a}{*} p = q$ are uniquely solvable and a is the two-sided *neutral*: $x \underset{a}{*} a = a \underset{a}{*} x = x$.

Thus any affinely connected (M, ∇) can be equipped with a loop structure in a neighbourhood of an arbitrary point a. (Everywhere for Lobachevski space.)

As well, one can introduce some system of unary operations

$$\omega_t^a (= t_a) \colon x \mapsto \mathrm{Exp}_a t\, \mathrm{Exp}_a^{-1} \quad (t \in \mathbf{R})\,, \tag{2}$$

and consider the system $\mathcal{M}_a = \langle M, \underset{a}{*}, a, (\omega_t^a)_{t\in\mathbf{R}}\rangle$ which, as it is easily seen, satisfies the additional properties

$$\begin{gathered} (t+u)_a x = t_a x \underset{a}{*} u_a x, \quad (tu)_a x = t_a(u_a x)\,, \\ 1_a x = x. \quad (t, u \in \mathbf{R},\ x \in M)\,. \end{gathered} \tag{3}$$

The above system is called an *odule* (*geodesic odule*).

See [L. V. Sabinin 77, 81], [L. V. Sabinin, P. O. Miheev 90].

1.2. Remark. Any odule is some deformation of a vector space (loop composition instead of vectors addition and a deformed multiplication by scalars). One ought to note that, generally speaking,

$$t_a(x \underset{a}{*} y) \neq t_a x \underset{a}{*} t_a y\,.$$

In the smooth case it is easy to show that if

$$t_a(x \underset{a}{*} y) = t_a x \underset{a}{*} t_a y \tag{4}$$

then the above odule turns out to be a vector space.

1.3. In order to get an adequate description of an affinely connected space (M, ∇) by means of its geodesic odules one needs additional identities, which establish the relations between odules centered at different points:

$$L_{t_a x}^{u_a x} \circ L_{u_a x}^a = L_{t_a x}^a \tag{5}$$

(1-st Geoodular Identity)

and

$$L_x^a \circ t_a \circ (L_x^a)^{-1} = t_x \tag{6}$$
(2-nd Geoodular Identity).

(Here $L_b^a c \stackrel{\text{def}}{=} b \underset{a}{*} c$.)

The system of odules $\mathcal{M}_a = \langle M, \underset{a}{*}, a, (\omega_t^a)_{t \in \mathbf{R}} \rangle$ satisfying (5) and (6) defines so called *geoodular structure* on a manifold M.

See [L. V. Sabinin 77, 81].

1.4. As a result, any affinely connected manifold (M, ∇) generates its *natural geoodular structure*. See [L. V. Sabinin 77, 81], [L. V. Sabinin, P. O. Miheev 90].

1.5. Conversely, given an arbitrary smooth geoodular manifold M generates an *canonical affine connection* ∇ by

$$\begin{cases} \nabla_{X_a} Y = \dfrac{d}{dt} \{ \, (\, L^a_{\gamma\,(t)} \,)^{-1}_{*,\,a} \, Y_{\gamma\,(t)} \, \}_{t=0} \,, \\ \qquad\qquad \gamma(0) = a \,, \quad \dot{\gamma}(0) = X_a \,. \end{cases} \tag{7}$$

See [L. V. Sabinin 77, 81], [L. V. Sabinin, P. O. Miheev 90].

1.6. Both above constructions are mutually invertible, that is

(1) The canonical affine connection to the natural geoodular structure of an affinely connected space coincides with the affine connection of the last one.

That is (M, ∇) generates the natural geoodular structure. The natural geoodular structure generates the canonical affine connection and (M, ∇) coincides with the canonical affine connection.

(2) The natural geoodular structure of the canonical affine connection to a given geoodular structure coincides with the last one.

That is, a geoodular structure generates the canonical affine connection, the canonical affine connection generates the natural geoodular structure and this natural geoodular structure coincides with the initially given.

See [L. V. Sabinin 77, 81], [L. V. Sabinin, P. O. Miheev 90].

In the language of Category theory one may say that *the Category of Affinely connected manifolds is equivalent to the Category of Smooth Geoodular manifolds.*

1.7. Now we would like to clarify, how so called reductive, $\nabla T = 0$, $\nabla R = 0$, and symmetric, $T = 0$, $\nabla R = 0$, manifolds (T being torsion, R being curvature) can be characterized by the properties of their natural geoodular structures.

Let us introduce

$$L(x, z, y) \stackrel{\text{def}}{=} x \underset{z}{*} y \,. \tag{8}$$

Then

$$L_b^a L(x, z, y) = L(L_b^a x, L_b^a z, L_b^a y) \tag{9}$$

is called *the Reductive identity* (see [L.V. Sabinin 77, 81], [L.V. Sabinin, P.O. Miheev 90]) and

$$(-1)_a L(x,z,y) = L((-1)_a x, (-1)_a z, (-1)_a y) \tag{10}$$

is called *the Symmetric identity* (see [L.V. Sabinin 77, 81], [L.V. Sabinin, P.O. Miheev 90]).

It turns out to be that

(1) (M,∇) *is reductive if and only if its natural geoodular structure is reductive* ([L.V. Sabinin 77, 81], [L.V. Sabinin, P.O. Miheev 90]).

(2) (M,∇) *is symmetric if and only if its natural geoodular structure is symmetric* ([L.V. Sabinin 77, 81], [L.V. Sabinin, P.O. Miheev 90]).

Note that $(-1)_a = s_a$ is nothing but usual symmetry of E. Cartan.

1.8. Elementary holonomy. It is of value to define

$$(L_c^a)^{-1} \circ L_c^b \circ L_b^a = h^a(b,c)\,, \tag{11}$$

(the operator of) *elementary holonomy*, which characterizes the parallel displacement of an arc along the geodesic triangle with the vertices a, b, c.

(See [L.V. Sabinin 77, 81], [L.V. Sabinin, P.O. Miheev 90]).

Intuitively it is evident that an elementary holonomy describes the presence of curvature and the following definition is quite natural.

A geoodular space is of *trivial curvature* (*with absolute parallelism*) if $h^a(b,c) = \mathrm{id}$ $(\forall\, a,b,c)$. (See [L.V. Sabinin 77, 81], [L.V. Sabinin, P.O. Miheev 90].)

It turns out to be that *an affinely connected space is with absolute parallelism ($R=0$) if and only if its natural geoodular structure is with absolute parallelism.* ([L.V. Sabinin 77, 81], [L.V. Sabinin, P.O. Miheev 90].)

1.9. A geoodular space is called *flat* if

(a) it is of absolute parallelism,

(b) all its geodesic odules are vector spaces.

It turns out to be that

(a) If an affinely connected space is flat ($T=0, R=0$) then its natural geoodular structure is flat.

(b) If any geodesic odule of a smooth geoodular structure is a vector space then its canonical affine connection is flat ($T=0, R=0$). See [L.V. Sabinin 77, 81], [L.V. Sabinin, P.O. Miheev 90].

1.10. Remark. Thus a number of important geometric properties of an affinely connected manifold can be formulated in purely algebraic way without Analysis.

1.11. The above presented demonstrates that relations between different odules are due to elementary holonomies. By this reason, if we wish to concentrate all information about geoodular space, (structure) into one fixed geodesic odule, we need additional operations.

Let $\langle M, \underset{\varepsilon}{*}, \varepsilon, (t_\varepsilon)_{t\in\mathbf{R}}\rangle$ be a fixed odule centered at a point ε. For the sake of simplicity we write $*,\ t,\ h(b,c)$ instead of $\underset{\varepsilon}{*},\ t_\varepsilon,\ h^\varepsilon(b,c)$, etc.

We introduce one additional ternary operation $h\colon (b,c,x) \mapsto h(b,c)x = h(b,c;x)$ and consider our odule together with this operation, that is $\langle M, *, \varepsilon, (t)_{t\in\mathbf{R}}, h\rangle$. It is possible to express the geoodular identities, equivalently, in the language of just introduced algebra, namely, we get:

(12) $$h(a,b)tx = th(a,b)x \quad \text{(homogeneity identity)}$$

(13) $$\begin{cases} h(a, a*b)tb = \ell(a,b)tb \quad \text{(joint identity)},\\ \qquad \text{where } \ell(a,b) = L^{-1}_{a\cdot b} \circ L_a \circ L_b\,, \end{cases}$$

(14) $$\begin{cases} h(c*ta, c*ua) \circ h(c, c*ta = h(c, *ua)\\ \qquad (h\text{-identity}), \end{cases}$$

(15) $$h(\varepsilon, q) = \mathrm{id} \quad (\varepsilon\text{-identity}).$$

Such an odule with the additional ternary operation h satisfying (12)–(15) is called a *holonomial odule* (*h-odule*).

(See [L.V. Sabinin 77, 81], [L.V. Sabinin, P.O. Miheev 90].)

And the earlier considered means that *any geoodular structure is equivalent to its holonomial odule centered at arbitrary fixed point (at least locally).*

1.12. It is of importance to show, how reductive and symmetric spaces (or geoodular structures) can be described by the properties of their holonomial odules.

Thus in smooth case,

(16) $$\left\{\begin{array}{c} \ell(a,b)(x*y) = \ell(a,b)x * \ell(a,b)y\\ \text{(left } A\text{-property)},\\ ta*(ua*b) = (t+u)a*b\\ \text{(left monoalternativity)},\\ h(a,b) = \ell(a, L_a^{-1}b) \end{array}\right\}$$

if and only if our space (geoodular structure) is reductive. ([L.V. Sabinin 77, 81], [L.V. Sabinin, P.O. Miheev 90].)

(17) $$\left\{\begin{array}{c} L_{(a*b)^2} = L_a \circ L_b \circ L_b \circ L_a \quad \text{(symmetric property)}\\ \text{(being equivalent to } L_a \circ L_b \circ L_a = L_{a\cdot(b\cdot a)}\,\text{, the left Bol property,}\\ \text{with } \ (a*b)^2 = a*(b*(b*a))\,\text{, the left Bruck property)},\\ h(a,b) = \ell(a, L_a^{-1}b) \end{array}\right\}$$

if and only if our space (geoodular structure) is symmetric. ([L.V. Sabinin 81], [L.V. Sabinin, P.O. Miheev 90].)

1.13. It follows from (12), (13) that for any holonomial odule

$$t\ell(a,b)b = \ell(a,b)tb \quad \text{(geometricity identity)}. \tag{18}$$

One may raise a question, if a smooth odule $\langle M, *, \varepsilon, (t)_{t\in\mathbf{R}}\rangle$ possesses the property of geometricity, can one find such a ternary operation h that the above odule together with this operation is a holonomial odule?

If so, is such h unique?

The answer is positive, but h is not unique.

Thus the property of geometricity is necessary and sufficient for a smooth odule to be a geodesic odule of some geoodular structure (affinely connected space).

(See [L.V. Sabinin 81], [L.V. Sabinin, P.O. Miheev 90].)

1.14. Remark. There exist some generalizations of the above theory. Instead of an odule one can consider a *diodule* adding one more operation

$$x \underset{a}{+} y \overset{\text{def}}{=} \operatorname{Exp}_a(\operatorname{Exp}_a^{-1} x + \operatorname{Exp}_a^{-1} y)\,. \tag{19}$$

([L.V. Sabinin 81], [L.V. Sabinin, P.O. Miheev 90].) Using this one can formulate the notion of a *metric geoodular space* which is equivalent to a *metric affinely connected space* (M, ∇, g) (g is twice covariant symmetric nondegenerate tensor field, $\nabla g = 0$).

In particular, in these frames one may treat any Riemannian (Pseudorieman-nian) space.

1.15. Conclusion. The above presented gives us a purely algebraic approach to Differential Geometry, introducing the fundamental notion of Holonomial odule (loop with supplementary structure), which is a far going generalization of Gyrogroup, with promissing applications to Mathematical Physics, Algebra and Geometry.

Such structures possess an algebraic value by itself, without references to Analysis, constituting the new branch of Mathematical research— Nonlinear Geometric Algebra.

2. Smooth loops actions

2.1. Definition. Let M be a smooth manifold and $\{\langle Q, \overset{x}{\cdot}, \varepsilon\rangle\}_{x\in M}$ be a system of local loops.

If $b \mapsto f_b$ $(Q \to \operatorname{Diff} M)$ is defined for b near ε and

$$x f_a f_b = x f_{(a \overset{x}{\cdot} b)} \text{ (near } \varepsilon), \;\; x f_\varepsilon = x\,, \tag{20}$$

then we say that a *smooth right action* $Q \xrightarrow{f} \operatorname{Diff} M$ is given.

Analogously one may define a *smooth left action.*

2.2. Example. The right natural action R induced by a loop $\langle Q, \cdot, \varepsilon\rangle$, $a \overset{R}{\mapsto} R_a$ $(x \cdot a = xR_a)$.

Evidently,

$$xR_aR_b = xR_{(a \overset{x}{\cdot} b)}\,, \quad a \overset{x}{\cdot} b = a \cdot L_x^{-1}L_{x\cdot a}\, b\,. \tag{21}$$

2.3. Further we use the notations $xf_a = xa$. Then (20) takes the form

$$(xa)b = x(a \overset{x}{\cdot} b)\,. \tag{22}$$

2.4. Remark. If $a \overset{x}{\cdot} b = a \cdot b$ (does not depend on x) then $(xa)b = x(a \cdot b)$ and for a faithful action $a \overset{f}{\mapsto} f_a$ we get the group action.

2.5. The above construction is rather flexible, since, for the same $a \overset{f}{\mapsto} f_a$ there may exist a number of loop systems $\{\langle Q, \overset{x}{\cdot}, \varepsilon\rangle\}_{x\in M}$. It motivates a more rigid construction.

Due to (22) we get $[(xa)b]c = [x(a \overset{x}{\cdot} b)]c$ and, after routine calculations,

$$x[(a \overset{x}{\cdot} b) \overset{x}{\cdot} c] = x[a \overset{x}{\cdot} (b \overset{xa}{\cdot} c)]\,. \tag{23}$$

It does not mean that

$$(a \overset{x}{\cdot} b) \overset{x}{\cdot} c = a \overset{x}{\cdot} (b \overset{xa}{\cdot} c)\,, \tag{24}$$

but we may assume (24) as an additional requirement coming to the notion of a *strict action.*

2.6. The differential equations of a strict action. In a suitable chart, differentiating (22) and (24) by c at $c = \varepsilon$ we get

$$\begin{cases} \overset{x}{A}{}^{\beta}_{\alpha}(a)\dfrac{\partial (xa)^i}{\partial a^{\beta}} = L^i_{\alpha}(xa), \quad x\varepsilon = x, \\ [\dfrac{\partial (b \overset{x}{\cdot} a)^{\alpha}}{\partial a^{\nu}}]\overset{xb}{A}{}^{\nu}_{\beta}(a) = \overset{x}{A}{}^{\alpha}_{\beta}(b \overset{x}{\cdot} a), \quad b \overset{x}{\cdot} \varepsilon = b\,, \end{cases} \tag{25}$$

where $\alpha, \beta, \nu = 1, \ldots, m = \dim Q, \quad i = 1, \ldots, n = \dim M$,

$$L^i_{\alpha}(y) = \frac{\partial (y\cdot c)^i}{\partial c^{\alpha}}|_{c=\varepsilon}$$

(the fundamental basic fields of action) ,

$$\overset{x}{A}{}^{\beta}_{\alpha}(a) = \frac{\partial (a \overset{x}{\cdot} b)^{\beta}}{\partial b^{\alpha}}|_{b=\varepsilon}$$

(the left fundamental basic fields of the loop $a \overset{x}{\cdot} b$) .

2.7. Involving into consideration the integrability conditions for (25) we get

$$\text{(26)} \qquad \begin{cases} [L_\sigma, L_\alpha]^i(x) = c^\mu_{\sigma\alpha}(x) L^i_\mu(x)\,, \\ \dfrac{\partial \overset{x}{B}{}^\lambda_\mu(a)}{\partial a^\nu} - \dfrac{\partial \overset{x}{B}{}^\lambda_\nu(a)}{\partial a^\mu} = -c^\lambda_{\sigma\tau}(xa) \overset{x}{B}{}^\sigma_\nu(a) \overset{x}{B}{}^\tau_\mu(a)\,, \end{cases}$$

where $\overset{x}{B}{}^\lambda_\sigma$ is the inverse matrix to $\overset{x}{A}{}^\mu_\nu$.

The system (25), (26) is closed, which means that its integrability conditions are consequences of the system itself.

In the contrast with the Lie group theory we can not regard here the structure functions $c^\lambda_{\mu\nu}(x)$ as known, since, due to the Jacobi identities, there are some nontrivial relations of nonalgebraic nature between $L^i_\alpha(x)$ and $c^\lambda_{\mu\nu}(x)$.

But we may regard $L^i_\alpha(x)$ and $c^\lambda_{\mu\nu}(x)$ as known and satisfying the nonlinear bracket relations

$$\text{(27)} \qquad [L_\alpha, L_\beta](x) = c^\lambda_{\alpha\beta}(x) L_\lambda(x)\,,$$

(see the first line of (26)). Then one may regard (26), (27) as a complitely intergrable system with unknowns

$$\varphi^\alpha(a) = (b \overset{x}{\cdot} a)^\alpha, \quad \overset{x}{A}{}^\lambda_\mu(a)\,, \quad \psi^i(a) = (xa)^i\,.$$

It means that by nonlinear Lie brackets system one may (locally) restore the strict action.

The above said is only one fragment of the complete theory (similar to the restoration of a Lie group by its algebra Lie of vector fields).

Further we need some analogue of the third Lie theorem. That is not done yet (because nobody suggested the proper infinitesimal object).

2.8. Remark. In our opinion it is evident that the loop action theory gives an adequate approach to non-linear Poisson and Lie brackets and should replace such a monstrous constructions as differentiable groupoids. See [M. V. Karasev, V. P. Maslov 91].

2.9. The described above is only one possible approach to the loop action theory. Indeed, in the Bol loops theory it is quite natural to consider so called Nono systems $(((xa)b)a = x[(a \cdot b) \cdot a])$ as Bol actions, since for a right Bol loop $R_a \circ R_b \circ R_a = R_{(a \cdot b) \cdot a}$.

The theory of Bol action may be easily constructed in an analogy with the Bol loops theory.

After that one may generalize the situation and consider the families $\{f_a : x \to x f_a\}_{a \in Q,\, x \in M}$ of bijections with the property

$$\text{(28)} \qquad \begin{cases} x f_a f_b f_a = x f_{F(x;a,b)}\,, \\ f_\varepsilon = \mathrm{id}, \quad F(x,\varepsilon,b) = b\,. \end{cases}$$

The infinitesimal theory for such a generalized Bol actions has not been considered yet.

References

M. V. Karasev, V. P. Maslov

[91] *Nonlinear Poisson brackets. Geometry and Quantization*, (Russian), "Nauka" press, Moscow, 1991. There is English translation.

L.V. Sabinin

[77] *Odules as a new approach to a geometry with a connection*, (Russian), Reports of Ac. of Sci of the USSR (Math.) **233** no. 5 (1977), 800–803; English translation: Soviet Math. Dokl. **18** no. 2 (1977), 515–518, Amer. Math. Soc. **MR**57(1979) # 1340.

[81] *Methods of Nonassociative Algebra in Differential Geometry*, (Russian), Supplement to Russian translation of S. Kobayashi and K. Nomizu "Foundations of Differential Geometry" vol. 1, Nauka press, Moscow, **MR**84b: 53002.

[99] *Smooth Quasigroups and Loops*, Monograph, Kluwer acadenic publishers, 1999, (to appear). (English)

L.V. Sabinin, P.O. Miheev

[91] *Quasigroups and Differential Geometry*, Quasigroups and Loops: Theory and Applications, Collective monograph, (O. Chein, H. Pflugfelder and J.D.H Smith, eds.), Heldermann Verlag, Berlin, 1990, Ch. XII, pp. 357–430. (English) **MR** 93g:20133.

A. Ungar

[94a] *The abstract complex Lorentz transformation groups with real metric. 1. Special relativity formalism to deal with the holomorphic automorphism group of the unit ball in any complex Hilbert space*, J. Math. Phys. **35(3)** (March 1994), 1408–1426.

[94b] *The holomorphic automorphism group of the complex disk*, Aequationes Mathematicae **17** no. 2 (1994), 240–254.

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