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TOPOLOGY AND CHARACTERISTIC CLOSURES OF K–CONTACT MANIFOLDS

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ABSTRACT. The dimension of a characteristic closure on a closed K-contact 2n+1-manifold M is at most the smaller of n+1 and 2n+1 minus the rank of the vector space of harmonic vector fields on M. Moreover, there are at least n+1 closed characteristics on M and if their number is finite, then each of the Betti numbers of M is at most equal to 1. If in addition, M is simply connected, then it is homeomorphic to a sphere.

1. Introduction

A contact form on a 2n+1-dimensional manifold M is a 1-form α such that the identity

$$\alpha \wedge (d\alpha)^n \neq 0$$

holds everywhere on M. Given such a 1-form α, there is always a unique vector field ξ satisfying $\alpha(\xi) = 1$ and $i_\xi d\alpha = 0$. The vector field ξ is called the *characteristic vector field* of the contact manifold (M, α) and the corresponding 1-dimensional foliation is called a contact flow.

The 2n-dimensional distribution $D(x) = \{v \in T_xM/\alpha(x)(v) = 0\}$, which is left invariant by ξ, is called the *contact distribution*. It carries a 1-1 tensor field J such that $J^2 = -I_{2n}$, where I_{2n} is the identity 2n by 2n matrix. The tensor field J extends to all of TM by requiring $J\xi = 0$.

Also, the contact manifold (M, α) carries a nonunique riemannian metric g adapted to α and J in the sense that the identities

$$d\alpha(X, Y) = g(X, JY) \text{ and } \alpha(X) = g(\xi, X)$$

are satisfied for any vector field X and Y on M. Such a metric g is called a *contact metric*. When the vector field ξ is Killing relative to the contact metric g, the triple (M, α, g) is called a K-contact manifold. K-contact structures have been widely studied in [BLA]. One of their characteristic identities is

$$(\nabla_X J)Y = R(\xi, X)Y \, , \tag{1}$$

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where R is the riemannian curvature tensor ([BLA]).

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,.$$

A flow on a manifold M is said to be almost regular ([THO]) if each point on M belongs to a flow box pierced by the flow at most a finite number of times. As an easy consequence of a theorem of Wadsley ([WAD]), any almost regular contact flow on a compact manifold is riemannian. Therefore, by Proposition 1 in [RU5], any almost regular compact contact manifold is K-contact.

Weinstein Conjecture ([WEI]) states that any compact contact manifold should carry at least one closed characteristic. This conjecture has been proved in some special cases, among others, it is proved in [VIT] for compact hypersurfaces of contact type in R^{2n} and in [RU2] for compact K-contact manifolds.

Many properties of a flow can be derived from those of the algebra of foliate vector fields; for example the leaf closures of an isometric flow on a compact manifold are orbits of an abelian Lie algebra of Killing foliate vector fields ([MOS]). For K-contact flows, we point out the fact that the leaf closures can have dimension at most $n+1$ when the manifold is $2n+1$-dimensional.

In section 6, the lower bound for the dimension of leaf closures of K-contact flows will be made more precise using the first Betti number of the manifold.

The existence of at least 2 closed characteristics has been proven for circle invariant contact forms in [BA1]. Using a result of Weinstein ([WEI]) about critical manifolds of a circle invariant function, we will provide sharper lower bounds for the number of closed characteristics for the class of contact flows mentioned above.

The k^{th} Betti numbers of a compact sasakian $2n+1$-dimensional manifold are zero or even for odd k smaller or equal to n ([BLT]). In particular, the first Betti number of such a manifold is zero or even ([TAC]). A classical topological application of Morse Theory states that a closed n-dimensional manifold admitting a Morse function with only 2 critical points is homeomorphic to the sphere. Our Theorem 8.1 could be seen as a generalization of this result to Morse-Bott Theory on contact manifolds.

2. Killing foliate vector fields

Let M be a $2n+1$ dimensional manifold with a contact form α on it. Sections of the contact distribution $D = ker\ \alpha$ are called *horizontal* vector fields, whereas any vector field proportional to ξ is called a *vertical* vector field.

The 2-form $d\alpha$ induces a symplectic structure on the distribution D; in other words, D is a symplectic vector bundle over M. When the contact flow is regular ([BLA]), M is the total space of a fibre bundle

$$\pi: M \to B$$

over the symplectic manifold B, the quotient of M under the $\mathbf{R}$ action induced by ξ.

Definition 2.1. A vector field X is said to be *foliate* if its Lie bracket with ξ is a vertical vector field. X will be called *transverse* if it is foliate and horizontal.

Definition 2.2. A differential form β will be said to be *basic* if $i_\xi \beta = 0$ and $L_\xi \beta = 0$.

We will consider a particular type of foliate vector fields, mainly those which are infinitesimal isometries. They arise naturally in the setting of isometric flows ([CAR]) where they describe the leaf closures.

Let M be a $2n+1$-dimensional compact K-contact manifold. Since ξ is Killing with respect to some K-contact metric, as we have already mentionned before, the closures of the characteristics are orbits of an abelian Lie algebra $\mathcal{G}$ of Killing vector fields ([MOS]).

Definition 2.3. A vector field X on M is said to be a *contact vector field* if the identity

$$L_X \alpha = h\alpha$$

holds for some function h on M. When $h \equiv 0$, that is $L_X \alpha = 0$, X is said to be strict contact.

The function $\alpha(X)$ is called the *contact hamiltonian* function of the contact vector field X and is related to h by the formula

$$d(\alpha(X))(\xi) = h \tag{2}$$

where ξ is the characteristic vector field of the contact form α.

Theorem 2.1. *Let M be a K-contact manifold with K-contact metric g. Then any Killing foliate vector field on M is a strict contact vector field with basic contact hamiltonian.*

Proof. Let g be a fixed K-contact metric and let X be a Killing foliate vector field on M. Since α is the metric dual of ξ [RU2], one has, for any horizontal vector field A:

$$\begin{aligned} L_X \alpha(A) &= X\alpha(A) - \alpha([X, A]) \\ &= Xg(\xi, A) - g(\xi, [X, A]) \\ &= Xg(\xi, A) - g(\xi, [X, A]) - g([X, \xi], A) \\ &= L_X g(\xi, A) = 0\,. \end{aligned}$$

Therefore

$$L_X \alpha = h\alpha \tag{3}$$

for some function h, where $h = d(\alpha(X))(\xi) = i_\xi d i_X \alpha$ by formula (2). Note that

$$\begin{aligned} \alpha([X, \xi]) &= g(\xi, [X, \xi]) \\ &= Xg(\xi, \xi) - L_X g(\xi, \xi) - g(\xi, [X, \xi]) \\ &= -\alpha([X, \xi])\,; \end{aligned}$$

hence $\alpha([X, \xi]) = 0$ which implies that

$$[X, \xi] = 0\,, \tag{4}$$

since $[X,\xi]$ is a vertical vector field. Using identities (3) and (4), we see that

$$h = L_X\alpha(\xi) = X\alpha(\xi) - \alpha([X,\xi]) = 0\,.$$

We shall denote by $\mathcal{L}$ the Lie algebra of Killing foliate vector fields. Any element X in $\mathcal{L}$ decomposes uniquely as

$$X = \bar{X} + \alpha(X)\xi\,.$$

The fact that $\alpha(X)$ is basic, implies that $\bar{X}$ is a transverse vector field. Also the identity

$$i_{\bar{X}}d\alpha = -di_X\alpha \tag{5}$$

holds.

Remark 2.1. On a compact manifold M with an isometric flow, the Lie algebra $\mathcal{G}$ which describes the leaf closures is a subalgebra of $\mathcal{L}$.

Proposition 2.1. *Let M be a K-contact compact manifold with Lie algebra of Killing foliate vector fields $\mathcal{L}$. If X, $Y \in \mathcal{L}$ and $[X,Y]=0$, then $[\bar{X},\bar{Y}]=0$.*

Proof. The vanishing of $[X,Y]$ implies that

$$0 = [\bar{X}+\alpha(X)\xi, \bar{Y}+\alpha(Y)\xi] = [\bar{X},\bar{Y}] + [\bar{X}(\alpha(Y)) - \bar{Y}(\alpha(X))]\xi\,.$$

So $[\bar{X},\bar{Y}]$ is a vertical vector field and hence, applying identity (5),

$$\begin{aligned}0 = i_{[\bar{X},\bar{Y}]}d\alpha &= L_{\bar{X}}i_{\bar{Y}}d\alpha - i_{\bar{Y}}L_{\bar{X}}d\alpha\\ &= di_{\bar{X}}i_{\bar{Y}}d\alpha + i_{\bar{X}}di_{\bar{Y}}d\alpha - i_{\bar{Y}}i_{\bar{X}}d^2\alpha - i_{\bar{Y}}di_{\bar{X}}d\alpha\\ &= d(d\alpha(\bar{Y},\bar{X}))\,,\end{aligned}$$

thus $d\alpha(\bar{X},\bar{Y})$ is a constant function. But since $d\alpha(\bar{X},\bar{Y}) = -d(\alpha(X))(\bar{Y})$ must vanish at some point on the compact manifold M, we deduce that $d\alpha(\bar{X},\bar{Y})=0$, which implies that

$$\alpha([\bar{X},\bar{Y}]) = -d\alpha(\bar{X},\bar{Y}) = 0\,,$$

and therefore $[\bar{X},\bar{Y}]=0$ since $[\bar{X},\bar{Y}]$ is a vertical vector field.

Before we state our next theorem, we define the rank of a set of vector fields on a given manifold to be the maximum dimension of tangent planes spanned by elements of the set at any point on the manifold.

Theorem 2.2. *Let M be a K-contact compact $2n+1$-dimensional manifold and $\mathcal{L}_0$ a maximal abelian Lie subalgebra of Killing vector fields containing the characteristic vector field. Then the rank of $\mathcal{L}_0$ is at most equal to $n+1$.*

Proof. Each set of k linearly independent local commuting Killing foliate vector fields determines a local $k-1$ dimensional horizontal integrable distribution by Proposition 2.1, and it is well known that the maximum dimension of an integral submanifold of the contact distribution is n, so $k-1\le n$ and hence $k \le n+1$.

Corollary 2.1. *A leaf closure of a K-contact $2n+1$-dimensional compact manifold has at most dimension $n+1$.*

Proof. Corollary 2.1 is an immediate consequence of the fact that the leaf closures are orbits of $\mathcal{G}$, a subalgebra of some $\mathcal{L}_0$.

3. Harmonic vector fields on K-contact manifolds

Starting with a contact metric structure on M with structure tensors J and g, one defines an almost complex operator $\tilde{J}$ on $M \times \mathbf{R}$ by

$$\tilde{J}(X + f\frac{d}{dt}) = JX - \alpha(X)\frac{d}{dt} + f\xi \,.$$

If the almost complex structure defined by $\tilde{J}$ is integrable ([BLA]), then the original contact metric structure is called *sasakian*. In that case, the two equivalent identities

$$L_\xi g = 0; \; L_\xi J = 0 \tag{6}$$

are satisfied. A contact metric structure satisfying any of the above equivalent identities is called *K-contact*. We refer to [BLA] for more properties of these structures. Let us point out however that on 3-dimensional compact manifolds, K-contact flows are sasakian.

The basic first cohomology of a K-contact flow on a compact manifold is known to be isomorphic to its first de Rham cohomology ([RU4]). It is therefore expected that the de Rham cohomology will affect the topology of those flows. If the flow is sasakian, we will show how the first Betti number of the manifold puts some restrictions on the maximum dimension of leaf closures.

In [RU3], examples of non-almost regular sasakian flows on the 3-dimensional sphere $\mathbf{S}^3$ were provided. As a consequence of our next theorem, such non-almost regular K-contact flows cannot be found on a 3-dimensional compact manifold with nonvanishing first Betti number.

We first prove the following elementary lemma.

Lemma 3.1. *Let Z be a harmonic vector field on a $2n+1$-dimensional K-contact manifold M with structure tensors J and g. Then Z is a transverse vector field.*

Proof. Since $\beta = g(Z, .)$ is a harmonic 1-form, it is basic ([RU3]); in particular the identity $L_\xi \beta = 0$ holds. Therefore, for any vector field Y on M, one has

$$\begin{aligned} g([\xi, Z], Y) &= \xi g(Z, Y) - L_\xi g(Z, Y) - g(Z, Y) - g(Z, [\xi, Y]) = \xi\beta(Y) - \beta([\xi, Y]) \\ &= L_\xi \beta(Y) = 0 \,. \end{aligned}$$

Thus

$$[\xi, Z] = 0 \,.$$

Denoting by $r_1(M)$ the largest dimension of tangent planes spanned by harmonic vector fields on M and by $b_1(M)$ the first Betti number of M; then $r_1(M)$ is at most equal to $b_1(M)$ and we have the following theorem:

Theorem 3.1. *Let M be a compact K-contact $2n+1$-dimensional manifold with structure tensors J and g. Then a leaf closure of the K-contact flow has dimension at most* $\inf\{n+1, \; 2n+1-r_1(M)\}$.

Proof. Let $\{Z_k\}, \; k \in \{1, 2, ..., b_1(M)\}$ be linearly independant harmonic vector fields whose rank is $r_1(M)$ at some point $p \in M$, hence on an open set O containing all such points $p \in M$. Recall that orbit closures on K-contact flows on a compact manifold are orbits of a compact abelian group G of isometries, hence a torus. More precisely, denoting by φ_t the 1-parameter group of isometries generated by ξ and by $\mathcal{I}(M)$ the group of isometries of M, then G is the closure of φ_t in $\mathcal{I}(M)$. Keeping the same notation as in the previous section, let $\mathcal{G}$ denote the Lie algebra of G, so that elements in $\mathcal{G}$ are commuting Killing vector fields on M. Let $V \in \mathcal{G}$, then V commutes with ξ. On one hand the flow is known to have at least 2 closed characteristics ([RU4]) and V is proportional to ξ along each of them, hence $g(V, Z_k) = 0$ along those closed orbits. On the other hand, due to the fact that V is Killing and Z_k are harmonic, all $g(V, Z_k)$ are constants. It follows that one has $g(V, Z_k) = 0$ everywhere. Consequently, in the open set O, the dimension of orbits of $\mathcal{G}$ is at most $2n + 1 - r_1(M)$. By Lemma 3.1, all Z_k commute with ξ. As a consequence of this, the open set O contains all orbits of its points. Therefore, thanks to the theory of compact transformation groups ([DIE] page 43), the open set O contains an orbit whose closure has maximum dimension; thus the maximum dimension of a leaf closure is $2n + 1 - r_1(M)$. This combined with Corollary 2.1 completes the proof of Theorem 3.1.

Corollary 3.1. *Let M be a K-contact compact 3-dimensional manifold with nonzero first Betti number. Then the contact flow on M is almost regular.*

Proof. In dimension 3, K-contact compact manifolds are sasakian. Thus, if there is a nonzero harmonic 1-form on M, then $r_1(M) = 2$ and by Theorem 3.1, a leaf closure has dimension at most equal to $\inf\{2,\ 1\} = 1$. Hence all characteristics are closed.

4. Critical manifolds of circle invariant functions

Let M be a compact manifold with an effective circle action on it. In general, the quotient Σ of M by the cirlce action is not a manifold in the ordinary sense, it is a V-manifold ([WE1]) in the sense of Satake ([SAT]).

Definition 4.1. The Lusternik-Schnirelman category of a space M, cat(M), is the minimum number of contractible open sets that can cover M.

Definition 4.2. The cuplength of a space M, denoted cuplength(M), is the maximum number m of positive degree cohomology classes $[\omega_1]$, $[\omega_2]$,...,$[\omega_m]$ such that $\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_m \neq 0$ on M.

By the Lusternik-Schnirelman theory ([SCH]), one has

$$cat(M) \geq cuplength(M) + 1\,. \tag{7}$$

The following proposition has been taken from [WE1]:

Proposition 4.1. *Let M be a compact manifold with an effective circle action and let Σ be the corresponding quotient V-manifold. Then any circle invariant function f on M has at least $cat(\Sigma)$ critical circles.*

Proof. Let $L_1, \ldots, L_n$ be the critical circles of f. Choose circle invariant tubular neighborhoods of L_i, $i = 1, 2, \ldots, n$, each contractible to L_i. The gradiant vector field $grad\ f$ is circle invariant and thus induces a vector field $\bar{grad}\ f$ on Σ.

Let Ψ_t denote the 1-parameter group generated by $\bar{gradf}$. Each of the tubular neighborhoods of L_i projects to a contractible neighborhood U_i of $\bar{L}_i$ in Σ. Let $V_i = \cup_{t\geq 0}\Psi_t(U_i)$. Each V_i is open, contractible to U_i and hence to $\bar{L}_i$ in Σ.

Since for each $x \in \Sigma$, $\lim_{t\to-\infty} \Psi_t(x) = \bar{L}_i$ for some i, we have $\Sigma \subset \cup_{i=1}^n V_i$. This proves that $n \geq cat\Sigma$.

5. Circle invariant contact forms

Let M be a 2n+1 dimensional manifold with a circle invariant contact form α. If Z denotes the infinitesimal generator of the circle action, then it is shown in [BA1] that critical circles of the circle invariant function $\alpha(Z)$ are closed characteristics of α. For completeness, we briefly present the proof of this fact.

Let us denote by $\mathbf{S}$ the function $-\alpha(Z) = -i_Z\alpha$. Then, taking into account that the vector field Z leaves α invariant, we observes that

$$d\mathbf{S} = -di_Z\alpha = -L_Z\alpha + i_Z d\alpha = i_Z d\alpha\,.$$

Hence, a point $p \in M$ is critical if and only if Z and ξ are proportional at p. This implies in turn that the Z and ξ orbits through p are the same since Z and ξ commute.

When the restriction of α to each orbit of the circle action is nonsingular, a sharper lower bound to the number of those closed characteristics can be obtained. This is the case, as we shall see later, when α is a K-contact form.

Theorem 5.1. *Let M be a 2n+1 dimensional compact manifold with a circle invariant contact form α. If Z is the infinitesimal generator of the circle action and $\alpha(Z) \neq 0$ everywhere on M, then α has at least n+1 closed characteristics.*

Proof. As already mentionned above, critical circles of the circle invariant function $\mathbf{S} = \alpha(Z)$ are closed characteristics of α ([BA1]). Let $\beta = \frac{1}{\alpha(Z)}\alpha$. Then β is also a contact form. Since

$$d\beta = (-\frac{1}{(\alpha(Z))^2})di_Z\alpha\wedge\alpha + \frac{1}{\alpha(Z)}d\alpha \tag{8}$$

and $L_Z\alpha = 0$, we see that

$$i_Z d\beta = -\frac{Z\alpha(Z)}{(\alpha(Z))^2}\alpha + \frac{1}{\alpha(Z)}(di_Z\alpha + i_Z d\alpha) = 0\,, \tag{9}$$

that is, Z is the characteristic vector field of β. As a consequence, the quotient V-manifold Σ of M by the circle action generated by Z carries a symplectic form and since $(d\beta)^n \neq 0$, one sees that $cuplength(\Sigma) \geq n$. Now, by Proposition 4.1 and inequality (7), the function $\mathbf{S} = \alpha(Z)$ has at least $n+1$ critical circles, which are also closed characteristics of α.

Suppose now that M is a 2n+1-dimensional K-contact manifold with K-contact form α, characteristic vector field ξ and K-contact metric g. Since M is compact, its isometry group $\mathcal{I}(M)$ is a compact Lie group by a classical theorem of Myers and Steenrod ([MYS]). Let φ_t denote the real 1-parameter group of isometries generated by ξ. The closure of φ_t in $\mathcal{I}(M)$ is a torus group, hence one can find arbitrary close to ξ, a periodic Killing vector field Z which commutes with ξ and such that $\alpha(Z) \neq 0$ everywhere on M. A straightforward calculation shows that Z leaves α invariant. This shows also that K-contact forms satisfies the hypotheses of Theorem 5.1 and we have the following corollary:

Corollary 5.1. *Let M be a compact K-contact $2n+1$-dimensional manifold. Then the characteristic flow has at least $n+1$ closed leaves.*

Examples of K-contact flows with exactly $n+1$ closed characteristics are known on the $2n+1$-dimensional sphere ([RU2]). We ask ourselves the question: If a closed K-contact $2n+1$-dimensional manifold M has a finite number of closed characteristics, is M necessarily a homology sphere? The affirmative answer to this question in dimension 3 follows trivially from Corollary 3.1.

Using the material in the next section, we will prove the following theorem:

Theorem 5.2. *Let M be a closed K-contact manifold with a finite number of closed characteristics. Then each of the Betti numbers of M is equal to at most 1.*

In particular, if M is a sasakian closed 2n+1 dimensional manifold, then the i^{th} and $(2n+1-i)^{th}$ Betti numbers of M are zero for odd $i \leq n$ ([BLT]).

6. Morse theory on K-contact manifolds

In this section, we will apply elementary Morse theory to the function that generates closed leaves of K-contact flows. We begin the section with some terminology, referring to [BOT] for more details on Morse theory.

A connected submanifold $N \subset M$ is called a nondegenerate critical manifold of a function f on M if the following conditions are satisfied:

(1) Each point $p \in N$ is a critical point of f.
(2) The hessian of f, $Hess_f$, is nondegenerate in directions normal to N.

The normal bundle νN of a nondegenerate critical manifold N is decomposed into a positive and negative part

$$\nu N = \nu^+ N \oplus \nu^- N$$

where $\nu_p^+ N$ and $\nu_p^- N$ are the positive and negative eigenspaces of the hessian of f, $Hess_f$. The fiber dimension of $\nu^- N$, denoted by λ_N, is referred to as the index of N relative to f. A function f all of whose critical manifolds are nondegenerate is said to be a *clean function.* The following proposition is due to [GST].

Proposition 6.1. *Let $f: X \to \mathbf{R}$ be a clean function. Suppose that the indices of all critical manifolds of f are even. Then f has a unique local maximum.*

Proof. Let $C_1, \ldots, C_k$ be the critical manifolds of local maxima and $C_{k+1}, \ldots, C_n$ the remaining critical manifolds. Then $f = constant\ a_i$ on C_i, where $a_1, \ldots, a_k$ are the local maxima of f. Let Ψ_t be the 1-parameter group generated by the gradient of f; and let

$$W_i = \{x \in X,\ \Psi_t(x) \to C_i \text{ as } t \to +\infty\}.$$

One of the fundamental theorems of Morse theory is that $dim\ W_i =\ index\ C_i + dim\ C_i$ and $X = \cup W_i$ (disjoint union) is a cell decomposition. So

$$X = W_1 \cup \ldots. \cup W_k \cup W_{k+1} \cup \cdots \cup W_n$$

where W_i is open for $i \leq k$ and $codim\ W_i \geq 2$ for $i > k$. But, if W_i is of codimension ≥ 2, it cannot disconnect X, so $X - \cup_{i>k} W_i$ is connected, and hence $k = 1$.

Let θ^- denote the orientation bundle of $\nu^- N$ and

$$P_t(M; \mathbf{R}) = \sum t^k dim H^k(M; \mathbf{R})$$

the Poincaré series of M. If one defines the Morse series ([BOT]) of f relative to the coefficient field $\mathbf{R}$ by

$$\mathcal{M}_t(f) = \sum t^{\lambda_N} P_t(N; \theta^- \otimes \mathbf{R}) \tag{10}$$

where the sum runs over all critical manifolds of f, then the Morse inequalities hold:

$$\mathcal{M}_t(f) \geq P_t(M; \mathbf{R})\,. \tag{11}$$

The inequalities (11) implies that $\mathcal{M}_t(f)$ majorizes $P_t(M; \mathbf{R})$ coefficient by coefficient.

Let M be a closed K-contact manifold with characteristic vector field ξ. We choose a vector field Z as in Theorem 5.1 so that closed characteristics are critical manifolds of the function

$$\mathbf{S} = \alpha(Z)\,.$$

Those critical manifolds admit a more specific description as is contained in the following proposition ([RU3]):

Proposition 6.2. *Let (M, α) be a closed K-contact manifold with characteristic vector field ξ. Then the set F_ξ of periodic points of ξ is a nonempty union $F_\xi = \cup V_i$ where each V_i is a closed totally geodesic submanifold of odd dimension.*

Proof. Let G be the closure of the 1-parameter group φ_t of isometries generated by ξ. The orbits of G are the closures of the ξ-orbits. Let $H_1,\ldots,H_k \subset G$ be the isotropy subgroups, so that each closed orbit is a set of fixed points of $H_1,\ldots$ or H_k. Let $\mathcal{H}_i, \ldots, \mathcal{H}_k$ be the corresponding Lie algebras of Killing vector fields. Each periodic point of ξ is a zero of either $\mathcal{H}_1,\ldots$or $\mathcal{H}_k$. But zeros of Lie algebras of Killing vector fields are known to satisfy the conclusion of Proposition 6.2 ([KOB]).

Each connected component of F_ξ is in fact an invariant regular K-contact submanifold of M. Here, invariant should be understood in the sense of ([KEM], page 372).

We now proceed to compute the hessian $Hess_{\mathbf{S}}$ of $\mathbf{S} = \alpha(Z)$ in directions perpendicular to a connected component N of F_ξ. Let $p \in N$ and v, w be tangent vectors perpendicular to N at p. We extend v and w into local vector fields V and W by parallel translation along geodesics emanating from p. In particular, the identities $(\nabla V)(p) = 0 = (\nabla W)(p)$ is valid and will be used repeatedely as well as the identity $\nabla_X \xi = -JX$ valid on any K-contact manifold ([BLA]) and the fact that Z is a Killing vector field commuting with ξ.

$$\begin{aligned}
Hess_{\mathbf{S}}(p)(v,w) &= (V(Wg(\xi,Z)))(p) \\
&= (V(-g(JW,Z) + g(\xi, \nabla_W Z)))(p) \\
&= (-g(\nabla_V JW, Z) - g(JW, \nabla_V Z) - Vg(\nabla_\xi Z, W))(p) \\
&= (-g(\nabla_V JW, Z) - g(JW, \nabla_V Z) + Vg(JZ, W))(p) \\
&= (-g(\nabla_V JW, Z) - g(JW, \nabla_V Z) + g(\nabla_V JZ, W) \\
&\quad + g(JZ, \nabla_V W))(p) \\
&= (-g((\nabla_V J)W, Z) - g(J\nabla_V W, Z) - g(JW, \nabla_V Z) \\
&\quad + g((\nabla_V J)Z, W) + g(J\nabla_V Z, W) + g(JZ, \nabla_V W))(p) \\
&= (-g(R(\xi, V)W, Z) + 2g(\nabla_V W, JZ) + 2g(W, J\nabla_V Z) \\
&\quad + g(R(\xi, V)Z, W))(p) \ \textit{by identity } (1) \\
&= (2g(R(\xi, V)Z, W) + 2g(W, J\nabla_V Z))(p)\,.
\end{aligned}$$

Evaluating at p, we have obtained the identity

$$Hess_{\mathbf{S}}(p)(v,w) = 2g(R(\xi, v)Z, w) + 2g(w, J\nabla_v Z)\,. \tag{12}$$

Let $\alpha(Z)(p) = k$ and put $Z = k\xi + \delta$ where δ is a Killing vector field vanishing all along N. Then points at which δ is zero lie on periodic orbits of ξ and one has the identity:

$$\nabla_v Z = -kJv + \nabla_v \delta\,. \tag{13}$$

Lemma 6.1.

(1) *The tangent vector $\nabla_v \delta$ is nonzero and perpendicular to N.*
(2) *The hessian $Hess_{\mathbf{S}}$ of $\alpha(Z)$ along N satisfies*

$$Hess_{\mathbf{S}}(p)(v, J\nabla_v \delta) = 2g(\nabla_v \delta, \nabla_v \delta)$$

and is therefore nondegenerate in directions perpendicular to N.

Proof. Our proof of the first part of assertion (1) was inspired from the work of Kobayashi ([KOB]). If $\nabla_v \delta$ were zero, then, since $\delta(p) = 0$, the vector field δ would be zero along the geodesic γ tangent to v at p, hence every orbit that intersects γ would be closed, contradicting the fact that v is perpendicular to N. Let η be an arbitrary vector field tangent to N in a neighborhood of p. Then

$$\begin{aligned}
g(\nabla_v \delta, \eta) &= g(\nabla_v Z, \eta) - kg(\nabla_v \xi, \eta) \\
&= -g(v, \nabla_\eta Z) + kg(Jv, \eta) = 0\,.
\end{aligned}$$

The first and second terms in the last identity are zero because N is totally geodesic and Jv is perpendicular to N. This proves that $\nabla_v\delta$ is perpendicular to N and completes the proof of assertion (1). Now combining identities (12) and (13) and exploiting the tensor nature of R as well as the identity ([BLA], page 65)

$$R(\xi, v)\xi = -v\,,$$

we see that

$$\begin{aligned} Hess_{\mathbf{S}}(p)(v, J\nabla_v\delta) &= 2g(R(\xi, v)Z, J\nabla_v\delta) + 2g(J\nabla_v Z, J\nabla_v\delta) \\ &= 2kg(R(\xi, v)\xi, J\nabla_v\delta) + 2g(J\nabla_v Z, J\nabla_v\delta) \\ &= -2kg(v, J\nabla_v\delta) + 2g(J\nabla_v Z, J\nabla_v\delta) \\ &= 2kg(Jv, \nabla_v\delta) + 2g(\nabla_v\delta - kJv, \nabla_v\delta) \\ &= 2kg(Jv, \nabla_v\delta) + 2g(\nabla_v\delta, \nabla_v\delta) \\ &\;\; - 2kg(Jv, \nabla_v\delta) = 2g(\nabla_v\delta, \nabla_v\delta) \neq 0\,. \end{aligned}$$

It follows from $Hess_S(p)(v, J\nabla_v\delta) \neq 0$ that $Hess_S(p)$ is nondegenerate in directions perpendicular to N, which completes the proof of Lemma 6.1.

Proposition 6.3. *The function $\mathbf{S} = \alpha(Z)$ is clean and each of its critical manifolds has even index.*

Proof. By Lemma 6.1, each critical submanifold of $\alpha(Z)$ is nondegenerate. It remains to prove the assertion about even index. To that end, let v be any direction perpendicular to N at p. Then, using identity (12),

$$\begin{aligned} Hess_{\mathbf{S}}(p)(Jv, Jv) &= 2g(R(\xi, Jv)Z, Jv) + g(J\nabla_{Jv}Z, Jv) \\ &= -2kg(Jv, Jv) - 2g(Jv, \nabla_v Z) \\ &= -2kg(v, v) + 2g(v, J\nabla_v Z) \\ &= 2g(R(\xi, v)Z, v) + 2g(v, J\nabla_v Z) = Hess_{\mathbf{S}}(p)(v, v)\,. \end{aligned}$$

This clearly establishes the fact that if $Hess_{\mathbf{S}}$ is negative definite in the direction v, it is also negative definite in the direction Jv, hence the indices are all even.

Assume now that the closed K-contact $2n+1$-dimensional manifold M with a function $\alpha(Z)$ as in Proposition 6.3, has a finite number $n+1$ of closed characteristics, one for each even index 0, 2,...,$2n$. Let's denote the critical circles by N_i,

$$\mathcal{M}_t(\alpha(Z)) = \sum_{i=0}^{n} t^{\lambda_{N_i}} P_t(N_i; \theta_i^- \otimes R)$$

where $\lambda_{N_i} = 2i$ are even integers. But since for $i = 0, 1, 2, ..., n$, $\nu^- N_i$ is an orientable vector bundle over S^1, one has:

$$P_t(N_i; \theta_i^- \otimes R) = 1 + t\,.$$

Hence, using the notation $b_i(M)$ for the dimension of $H^i(M; R)$, the Morse inequalities (11) can be written as:

$$\sum_{i=0}^{n} t^{\lambda_{N_i}}(1+t) \geq \sum_{i=0}^{2n+1} b_i(M) t^i\,.$$

As a result, one has the following theorem:

Theorem 6.1. *Let M^{2n+1} be a closed K-contact manifold with $n+1$ closed characteristics. Then each of the real Betti numbers of M^{2n+1} is less or equal to 1.*

7. Stratification for K-contact manifolds

Given a K-contact manifold M^{2n+1} with $n+1$ closed characteristics, N_0, N_1,..., N_n which are critical circles of a function $\alpha(Z)$ with index $0, 2, ..., 2n$ respectively, we shall denote by M^{2k+1}, $0 \leq k \leq n$, the closures of the sets $\{x \in M,\ \lim_{t\to\infty} \Psi_t(x) \in N_k\}$, where Ψ_t is the gradient flow for the function $\alpha(Z)$. For an arbitrary vector field Y on M^{2n+1}, one has

$$d(\alpha(Z))(Y) = di_Z\alpha(Y) = g(JZ, Y)\,.$$

Therefore, Ψ_t is generated by the vector field JZ. For the argument to follow, we point out that ξ, Z and JZ are commuting vector fields. The subsets M^{2k+1} constitute a stratification of M^{2n+1} as follows:

$$M^1 \subset M^3 \subset M^5 \subset \cdots \subset M^{2n-1} \subset M^{2n+1}$$

and each M^{2k+1} is a closed $2k+1$-dimensional submanifold with $k+1$ nondegenerate critical circles for the restriction of $\alpha(Z)$.

Proposition 7.1. *M^3 is a minimal submanifold, which is diffeomorphic to a quotient of the standard S^3 by an abelian finite group of isometries.*

Proof. For $x \in N_0$ or $x \in N_1$; if $v \in T_xM^3$ and $\alpha(v) = 0$, then the fact that $Jv \in T_xM^3$ follows from J-invariance of the splitting $\nu_{N_i} = \nu^-_{N_i} \oplus \nu^+_{N_i}$ for the normal bundle of N_i, $i = 0, 1$ (see Proposition 6.3). For $x \in M^3 - (N_0 \cup N_1)$, the tangent space T_xM^3 is spanned by the vectors JZ, ξ and $Z - \alpha(Z)\xi$ and is therefore J-invariant. This implies that M^3 is a K-contact invariant submanifold and it is known that contact invariant submanifolds are minimal ([END]).

Since M^3 supports an isometric flow with exactly two closed orbits, it follows from Carrière's classification of riemannian flows on compact 3-manifolds ([CAR]), that M^3 is diffeomorphic to either a quotient of S^3 under an abelian finite group of isometries or $S^2 \times S^1$. But $H^1(M^3) = 0$ ([RU1]) and hence M^3 must be diffeomorphic to a lens space.

Remark 7.1. It is possible that each M^{2k+1} is a contact invariant, hence minimal submanifold of M^{2n+1}.

8. Characterization of the sphere

The subsets M^{2k-1} and N_k are disjoint closed submanifolds of M^{2k+1}, so each has a tubular neighborhood in M^{2k+1}. In fact, one can find open neighborhoods U of M^{2k-1} and V of N_k such that M^{2k-1}, $N_k \simeq S^1$ and $S^{2k-1} \times S^1$ are deformation retracts of U, V and $U \cap V$ respectively. The Mayer-Vietoris sequence for the homology yields the following long exact sequence:

$$\cdots \to H_*(S^{2k-1} \times S^1) \to H_*(M^{2k-1}) \oplus H_*(S^1) \to H_*(M^{2k+1}) \to H_{*-1}(S^{2k-1} \times S^1) \to \ldots \tag{14}$$

Lemma 8.1. *If $H_1(M^{2k+1}) = 0$, $k \geq 2$, then $H_1(M^{2k-1}) = 0$ and $H_2(M^{2k-1}) \simeq H_2(M^{2k+1})$; $k \geq 2$*

Proof. Writing the exact sequence (14) from $* = 2$ down to $* = 1$ yields:

$$0 \to H_2(M^{2k-1}) \oplus 0 \to H_2(M^{2k+1}) \to Z \to H_1(M^{2k-1}) \oplus Z \to 0 \to \dots$$

Lemma 8.1 follows from exactness of the above sequence.

Lemma 8.2. *If $H_1(M^{2k+1}) = 0$ and M^{2k-1} is a homology sphere, then M^{2k+1} is also a homology sphere.*

Proof. We need only to prove $H_i(M^{2k+1}) = 0$ for $2 < i \leq k$, the case $i = 2$ being covered by Lemma 8.1 and the cases $i > k$ will follow from duality.

Writing the sequence (14) for $2 < i \leq k$ yields the following:

$$\dots \to 0 \to 0 \to 0 \oplus 0 \to H_i(M^{2k+1}) \to 0 \to \dots$$

Therefore $H_i(M^{2k+1}) = 0$ for $2 < i \leq k$. Now by duality, $H_{2k+1-i}(M^{2k+1}) = 0$ for $2 \leq i \leq n$. This completes the proof of Lemma 8.2.

Theorem 8.1. [RU6] *Every closed simply connected K-contact manifold M^{2n+1} with a finite number $n+1$ of closed characteristics is homeomorphic to the sphere S^{2n+1}.*

Proof. If $\pi_1(M^{2n+1}) = 0$, then $H_1(M^{2n+1}) = 0$ and by Lemma 8.1, $H_1(M^{2k+1}) = 0$ for $1 \leq k \leq n$. Since M^3 is a homology sphere, it follows from Lemma 8.2 and induction on k that M^{2n+1} is a homology sphere and thus, by Smale's solution to the generalized Poincaré conjecture ([SMA]), M^{2n+1} is homeomorphic to S^{2n+1} for $n \geq 2$. The case $n = 1$ is covered by Proposition 7.1.

Remark 8.1. We have actually proved that if M^{2n+1} is simply connected, then each of M^{2k+1}, $1 \leq k \leq n$ is homeomorphic to S^{2k+1}.

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