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ON SOME DIFFERENTIAL STRUCTURES FOR FUNCTION SPACES

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ABSTRACT. In this paper there are presented and analyzed some *natural* differential structures on the space of smooth mappings. Naturality means that they are strongly related to the structures of spaces between which the smooth mappings considered are defined. This paper is a continuation of the investigations presented in [2] (where one can find the mathematical and the physical motivation).

1. Preliminaries

Let M be a set and $\mathcal{M}$ be a set of real functions defined on M. By $\tau_{\mathcal{M}}$ we denote the weakest topology on M such that all functions of the family $\mathcal{M}$ are continuous. Then, for any subset $A \subset M$, let $\mathcal{M}_A$ be the set of all real functions g on A such that, for any point $m \in M$, there exist an open neighbourhood $U \in \tau_{\mathcal{M}}$ of m and a function $f \in \mathcal{M}$ satisfying the condition $g|A \cap U = f|A \cap U$. By $sc\mathcal{M}$ we shall denote the family of all real functions on M which can be presented as a composition $\omega \circ (f_1, \dots, f_n)$, where $\omega \in C^\infty(\mathbf{R}^n)$, $f_1, \dots, f_n \in \mathcal{M}$, $n = 1, 2, \dots$. A family $\mathcal{M}$ is said to be a *differential structure* on M if $\mathcal{M} = \mathcal{M}_M = sc\mathcal{M}$. For an arbitrary set $\mathcal{M}_0$ of real functions on M, the set $(sc\mathcal{M}_0)_M$ is the smallest differential structure on M containing $\mathcal{M}_0$, and this minimal differential structure is said to be *generated* by $\mathcal{M}_0$, we shall write

$$gen(\mathcal{M}_0) = (sc\mathcal{M}_0)_M \ .$$

One can prove that if $\mathcal{M} = gen(\mathcal{M}_0)$ then $\tau_{\mathcal{M}} = \tau_{\mathcal{M}_0}$.

Let M be a set and $\mathcal{M}$ be a differential structure on M. Then the pair $(M, \mathcal{M})$ is said to be a *differential space* in the sense of Sikorski [1]. For any two differential spaces $(M, \mathcal{M})$ and $(N, \mathcal{N})$, a mapping $s : M \longrightarrow N$ is said to be *smooth* if

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$\beta \circ s \in \mathcal{M}$ for any $\beta \in \mathcal{N}$. For a smooth mapping $s : M \longrightarrow N$ we usually use the notation

$$s : (M, \mathcal{M}) \longrightarrow (N, \mathcal{N}) .$$

For a set M, a differential space $(N, \mathcal{N})$ and a family $\mathcal{S}$ of mappings from M into N we define the differential structure *induced* on M by $\mathcal{S}$

$$ind(\mathcal{S}) = gen(\{\beta \circ s : s \in \mathcal{S} \text{ and } \beta \in \mathcal{N}\})$$

The induced structure $ind(\mathcal{S})$ is the weakest differential structure on the set M with respect to which all mappings of the family $\mathcal{S}$ are smooth. In turn, for a differential space $(M, \mathcal{M})$ and a set N we define the differential structure *coinduced* on N by a family $\mathcal{S}$ of mappings from M into N

$$coind(\mathcal{S}) = \{\beta : \ \beta : N \to \mathbf{R} \text{ and } \beta \circ s \in \mathcal{M}, s \in \mathcal{S}\} .$$

The coinduced structure $coind(\mathcal{S})$ is the strongest differential structure on N with respect to which all mappings of the family $\mathcal{S}$ are smooth. For differential spaces $(M, \mathcal{M})$ and $(N, \mathcal{N})$ we define their *Cartesian product* as a differential space $(M \times N, \mathcal{M} \sqcap \mathcal{N})$ with the *product differential structure* defined as follows

$$\mathcal{M} \sqcap \mathcal{N} = gen(\{\alpha \circ \pi_M : \alpha \in \mathcal{M}\} \cup \{\beta \circ \pi_N : \beta \in \mathcal{N}\}) ,$$

where π_M and π_N denote the natural projections of the Cartesian product $M \times N$ onto M and N, respectively.

The totality of all smooth mappings from one differential space $(M, \mathcal{M})$ into another $(N, \mathcal{N})$ will be denoted by N^M.

For any differential spaces $(K, \mathcal{K})$, $(M, \mathcal{M})$, $(N, \mathcal{N})$ and any smooth mapping $h : (K \times M, \mathcal{K} \sqcap \mathcal{M}) \longrightarrow (N, \mathcal{N})$ we define its *associated map* $\hat{h} : K \longrightarrow N^M$ by the formula

$$\hat{h}(k)(m) = h(k, m) ,$$

for any $k \in K$ and $m \in M$. Obviously, the correspondence $h \mapsto \hat{h}$ is bijective. In order to geometrize the function space N^M one needs to define a differential structure on it with the help of which the family N^M becomes a differential space. There are different posibilities for a suitable differential structure on N^M. An introduction to the program of the geometrization, together with the mathematical and physical motivation, one can find in [2]. Here we only briefly recall the main definitions of the so called proper and admissible differential structures.

For two given differential spaces $(M, \mathcal{M})$, $(N, \mathcal{N})$ a differential structure Λ on N^M is said to be *proper* if for any differential space $(K, \mathcal{K})$ and any smooth mapping $h : (K \times M, \mathcal{K} \sqcap \mathcal{M}) \longrightarrow (N, \mathcal{N})$ also its associated mapping $\hat{h}$ is smooth. Conversely, a differential structure Ξ on N^M is said to be *admissible* if for every smooth $\hat{h}$ its associated mapping h is smooth. This means that the only smooth $\hat{h}'s$ are those which come from smooth $h's$. Some properties od the proper and admissible structures are investigated in [2]. Let us only mention that it is not always so that there exists the proper-admissible structure on N^M, and if it exists then it exists uniquely, which makes it to be a differential structure distinguished from the other ones. The most natural case when there exists the (unique) proper-admissible differential structure is the Euclidean manifold.

Proposition. *Naturally, every Euclidean manifold* $(R^n, C^\infty(\mathbf{R}^n))$ *can be interpreted as the space* $\mathbf{R}^{\{1,\dots,n\}}$ *of all smooth mappings from the discrete space* $\{1,\dots,n\}$ *into the Euclidean line* $(\mathbf{R}, C^\infty(\mathbf{R}))$. *Moreover, one can prove that the Euclidean differential structure* $C^\infty(\mathbf{R}^n)$ *is the (unique) proper-admissible differential structure.*

Proof. The differential structure $C^\infty(\mathbf{R}^n)$ is proper since it is generated by the pointwise evaluations $ev_m : \mathbf{R}^n \ni x \mapsto x(m) \equiv x_m \in \mathbf{R}$. In order to prove the admissibility of $C^\infty(\mathbf{R}^n)$ it is enough to prove the smoothness of the evaluation mapping (see [2])

$$\Omega : \mathbf{R}^n \times \{1,\dots,n\} \ni (x,i) \longmapsto \Omega(x,i) \equiv x(i) \equiv x_i \in \mathbf{R},$$

for $i = 1,\dots n$. But the smoothness of Ω clearly follows from its presentation in the form $\Omega = \omega \circ (\pi_1, \pi_2)$, where $\pi_1 : \mathbf{R}^n \times \{1,\dots,n\} \ni (x,i) \longmapsto x \equiv (x_1,\dots,x_n) \in \mathbf{R}^n$, $\pi_2 : \mathbf{R}^n \times \{1,\dots,n\} \ni (x,i) \longmapsto i \in \{1,\dots,n\}$, and $\omega \in C^\infty(\mathbf{R}^{n+1})$ is defined as

$$\omega(x_1, x_2, \dots, x_n, y) = x_1 \cdot \frac{(y-2)(y-3)\dots(y-n)}{(1-2)(1-3)\dots(1-n)} +$$

$$+ x_2 \cdot \frac{(y-1)(y-3)\dots(y-n)}{(2-1)(2-3)\dots(2-n)} + \dots + x_n \cdot \frac{(y-1)(y-2)\dots(y-(n-1))}{(n-1)(n-2)\dots(n-(n-1))}.$$

From the above example we conclude that the standard differential structure $C^\infty(\mathbf{R}^n)$ is distinguished among all the other structures on $\mathbf{R}^n$. In particular no *exotic* manifold structure on $\mathbf{R}^n$ can be simultaneously proper and admissible.

2. Separation of proper and admissible structures by a group action

Let $(M, \mathcal{M})$ be a differential space and $(G, \mathcal{G})$ be a differential group [4] acting smoothly on M. Denote by ψ the (left) action of G on M, i.e. $\psi : G \times M \longrightarrow M$. The action ψ determines the family $\psi(G, M)$ of smooth mappings

$$\psi_m : G \ni g \longmapsto \psi_m(g) \equiv \psi(g, m) \in M.$$

By definition, $\psi(G, M)$ is a smooth family (see [5] and [3]). Denote by σ the bijection

$$\sigma : M \ni m \longmapsto \sigma(m) \equiv \psi_m \equiv \psi(\cdot, m) \in \psi(G, M) \subset M^G,$$

where M^G denotes the totality of all smooth mappings from G into M. It is interesting to compare the following three (kinds of) structures on $\psi(G, M)$: $\mathcal{F}_\Lambda$, $\mathcal{F}_\sigma$, $\mathcal{F}_\Xi$, for any *proper* structure Λ and any *admissible* structure Ξ on M^G. Namely, by $\mathcal{F}_\sigma$ we denote the differential structure on $\psi(G, M)$ coinduced by σ, i.e.

$$\mathcal{F}_\sigma = coind\{\sigma\} \equiv \{\beta \in \mathbf{R}^{\psi(G,M)} : \beta \circ \sigma \in \mathcal{M}\},$$

in turn $\mathcal{F}_\Lambda$ and $\mathcal{F}_\Xi$ are the localizations of Λ and Ξ, correspondingly, to $\psi(G, M)$, i.e.

$$\mathcal{F}_\Lambda = \Lambda_{\psi(G,M)} \equiv ind\{i\} \equiv gen\{\varphi \circ i : \varphi \in \Lambda\} ,$$

where $i : \psi(G, M) \longrightarrow (M^G, \Lambda)$, and

$$\mathcal{F}_\Xi = (\Xi)_{\psi(G,M)} \equiv ind\{i\} \equiv gen\{\varphi \circ i : \varphi \in \Xi\} ,$$

where $i : \psi(G, M) \longrightarrow (M^G, \Xi)$. In both cases we used the same symbol i which stands for the inclusion mapping of $\psi(G, M)$ into M^G, however the function space M^G is taken with the differential structure Λ and Ξ, correspondingly.

One can prove the following

Proposition.

$$\mathcal{F}_\Lambda \subset \mathcal{F}_\sigma \subset \mathcal{F}_\Xi .$$

Proof. Since the mapping $inv : M \times G \ni (m, g) \longmapsto (g, m) \in G \times M$ is a diffeomorphism, the mapping $\psi_{inv} \equiv \psi \circ inv$ is smooth. Its associated mapping turns out to be $\widehat{\psi_{inv}} \equiv i \circ \sigma : M \longrightarrow M^G$. Since Λ is a proper structure, the mapping $\widehat{\psi_{inv}}$ is smooth with respect to the structure Λ. The last statement means that, for any $\varphi \in \Lambda$, we get $\varphi \circ \widehat{\psi_{inv}} \equiv \varphi \circ i \circ \sigma \in \mathcal{M}$. Hence we conclude that σ is a smooth bijection i.e.

$$\sigma : (M, \mathcal{M}) \longrightarrow (\psi(G, M), \mathcal{F}_\Lambda) .$$

Thus, by definition of the differential structure $\mathcal{F}_\sigma \equiv coind\{\sigma\}$ we obtain

$$\mathcal{F}_\Lambda \subset \mathcal{F}_\sigma .$$

Now, let Ξ be an admissible differential structure on M^G. Then the evaluation mapping $\Omega : M^G \times G \longrightarrow (M, \mathcal{M})$, given by $\Omega(f, g) = f(g)$, is smooth,i.e.

$$\Omega : (M^G \times G, \Xi \sqcap \mathcal{G}) \longrightarrow (M, \mathcal{M}) ,$$

where $\Xi \sqcap \mathcal{G}$ denotes the product differential structure on $M^G \times G$. The smoothness of Ω implies its smoothness with respect to each components. Therefore the mapping $\Omega(\cdot, e) : M^G \ni f \longmapsto \Omega(f, e) \in M$ is smooth, i.e. $\Omega(\cdot, e) : (M^G, \Xi) \longrightarrow (M, \mathcal{M})$, where e denotes the unit element of the group G. Thus, the the restriction of $\Omega(\cdot, e)$ to the subset $\psi(G, M) \subset M^G$ is also smooth, i.e.

$$\Omega(\cdot, e) \circ i : (\psi(G, M), \mathcal{F}_\Xi) \longrightarrow (M, \mathcal{M}) .$$

But

$$\Omega(\cdot, e) \circ i : \psi(G, M) \ni \psi_m \longmapsto \Omega(\psi_m, e) = \psi_m(e) = m \in M ,$$

which means that $\Omega(\cdot, e) \circ i = \sigma^{-1}$. Since the mapping σ is a bijection then

$$\mathcal{F}_\sigma \equiv coind\{\sigma\} = ind\{\sigma^{-1}\} .$$

We have proved that the mapping σ^{-1} is smooth with respect to the differential structure $\mathcal{F}_\Xi$ and, on the other hand, the structure $\mathcal{F}_\sigma = ind\{\sigma^{-1}\}$ is the smallest differential structure on $\psi(G, M)$ with respect to which the mapping σ^{-1} is smooth. Therefore

$$\mathcal{F}_\sigma \subset \mathcal{F}_\Xi ,$$

which ends the proof.

With the help of the bijection σ we get one more elegant comparison. Namely, the structures resulting from Λ and from Ξ can be compared directly with the structure $\mathcal{M}$ on the "basic level" of the space M itself. By using the bijection $\sigma : M \longrightarrow \psi(G, M)$ we can pullback the three structures $\mathcal{F}_\Lambda$, $\mathcal{F}_\sigma$, $\mathcal{F}_\Xi$ onto the set M, which gives us the structures which are diffeomorphic to the previous ones, correspondingly, and they are ordered analogously by the inclusion relation:

$$\mathcal{M}_\Lambda \equiv (\sigma)^* \mathcal{F}_\Lambda \equiv gen\{\lambda \circ \sigma : \lambda \in \mathcal{F}_\Lambda\} ,$$

$$\mathcal{M}_\Xi \equiv (\sigma)^* \mathcal{F}_\Xi \equiv gen\{\xi \circ \sigma : \xi \in \mathcal{F}_\Xi\} ,$$

$$\mathcal{M}_\Lambda \subset \mathcal{M} \subset \mathcal{M}_\Xi .$$

From the last relation one reads clearly that the proper structures are weaker while the admissible are the stronger ones. Additionally, if there is a case that there exists the unique proper-admissible structure then its pullback must coincide with the original structure of the base space $(M, \mathcal{M})$. However, the problem of clear conditions under which there exists the (unique) proper-admissible structure has not been solved as yet. It seems that the difficulty, in general, is caused by the "very bad" partial ordering of the class of all admissible structures.

Especially interesting, within the above framework, is the action of a subgroup of a differential group (or the case when the whole group is acting on itself).

3. Differential structures determined by a measure

Let $(M, \mathcal{M})$ be a differential space with a real measure μ (or a family of measures) defined on it. Then, for a differential space $(N, \mathcal{N})$, we can define a differential structure $\mathcal{F}_\mu$ *induced* by the measure μ on the family N^M.

To illustrate the main idea we shall generalize the proper structure on N^M induced by evaluations [2], i.e.

$$\mathcal{F}_0(N^M) = ind(\{ev_m : m \in M\}) \equiv gen(\{\beta \circ ev_m : \beta \in \mathcal{N}, m \in M\}) ,$$

where, for any $m \in M$,

$$ev_m : N^M \ni f \longmapsto ev_m(f) \equiv f(m) \in N .$$

This structure is natural since it is a generalized version of the finite or countable product differential structure. On the other hand it is in some sense "primitive" since the values $ev_m(f)$ corresponding to a function f "grow up" from a single point of f, namely from the point $(m, f(m)$ of the graph of f, and do not evaluate the whole function f. Such a differential structure, when interpreted as a collection of physically measurable scalar fields, would allow us only for a pointwise contact with the objets of N^M and, consequently, it would make a very fragmentaric report about the space N^M. The aim of the generalization discussed here is to generate a differential structure by such real mappings defined on N^M which evaluate the whole functions or at least their "longer parts" which would be more representative than a single point of the graph. By using a measure which is given on M we can define the following differential structure on N^M

$$\mathcal{F}_{\mu 0} = gen(\{\Theta_{\mu\alpha\beta} : \alpha \in \mathcal{M}^c,\ \beta \in \mathcal{N}\})\,,$$

where $\mathcal{M}^c = \{\alpha \in \mathcal{M} : supp\, f\ \ is\ \ compact\}$ and the generators $\Theta_{\mu\alpha\beta}$ are defined as

$$\Theta_{\mu\alpha\beta}(f) = \int \alpha \cdot (\beta \circ f)\ d\mu \equiv \int_M \alpha(m) \cdot \beta(f(m))d\mu(m)\,.$$

One can easily interpret the above formulation in the case when a single measure μ is replaced by a family of measures $\mu \equiv \{\mu_t\}_{t\in T}$, namely

$$\mathcal{F}_{\mu 0} = gen(\{\Theta_{\mu_t\alpha\beta} : \alpha \in \mathcal{M}^c,\ \beta \in \mathcal{N},\ t \in T\})\,.$$

Proposition. *The differential structure $\mathcal{F}_{\mu 0}$ is proper.*

Proof. Let $(K, \mathcal{K})$ be a differential space and let us consider a smooth mapping $h : (K \times M, \mathcal{K} \sqcap \mathcal{M}) \longrightarrow (N, \mathcal{N})$. Then, for any $k \in K$, we obtain

$$\Theta_{\mu_t\alpha\beta} \circ \hat{h}(k) = \int_M \alpha(m) \cdot \beta(\hat{h}(k)(m))d\mu_t(m) = \int_M \alpha(m) \cdot \beta(h(k,\ m))d\mu_t(m)\,,$$

which can be written in the form (without k)

$$\Theta_{\mu_t\alpha\beta} \circ \hat{h} = \int_M \alpha(m) \cdot \beta(h(\cdot,\ m))d\mu_t(m)\,,$$

which means that $\Theta_{\mu_t\alpha\beta} \circ \hat{h} \in \mathcal{K}$.

Analogously we define the remaining differential structures $\mathcal{F}_{\mu n}$, for $\mu \equiv \{\mu_t\}_{t\in T}$ and $n = 1, 2, \ldots$,

$$\mathcal{F}_{\mu n} = gen(\{\Theta_{\mu_t\alpha\beta},\ \Gamma^X_{\mu_t\alpha\beta} : \alpha \in \mathcal{M}^c,\ \beta \in \mathcal{N},\ t \in T,\ X \in \mathbf{X}^n(M)\})\,,$$

where the generators $\Gamma^X_{\mu_t\alpha\beta}$ are defined by the formula

$$\Gamma^X_{\mu_t\alpha\beta}(f) = \int_M \alpha(m) \cdot \Gamma^X_\beta(j^n_m(f))d\mu(m) \equiv \int_M \alpha(m) \cdot X(m)(\beta \circ f)d\mu(m)\,,$$

where $\mathbf{X}^n(M)$ denotes the module of all smooth vector fields of n-th order [3].

The above integral formulas generalize the locally defined differential structures induced on N_M from the differential space $J^n(M,\ N)$, $\mathcal{F}(J^n(M,\ N))$, for $n = 0, 1, 2, \ldots$ (see [3]). In order to obtain the local formulation from the above integral formulas one needs to choose the particular family of measures. It is quite evident that the suitable measures are the Dirac measures, i.e. the we should take the family $\{\mu_m\}_{m\in M}$ of measures whose every element μ_m is a measure concentrated to a single point m. From the physical point of view, if $(M, \mathcal{M})$ was a phase space then the family $\{\mu_m\}_{m\in M}$ of the Dirac measures would have an interpretation as the family of the so called *pure states*. By applying the measures $\{\mu_m\}_{m\in M}$ to the above integral formulas we obtain the differential structure on N^M induced from the space of jets.

For a differential space $(M, \mathcal{M})$ the concept of a smooth vector field is understood algebraically, i.e. it is a mapping from $\mathcal{M}$ into $\mathcal{M}$ which fulfils the Leibniz derivation condition. A natural question is what should be a differential structure defined on $\mathcal{M}$ with respect to which the "algebraically smooth" vector fields become the smooth mappings in the category of differential spaces. One can prove that the suitable differential structure on $\mathcal{M}$ is the structure $\mathcal{F}_{\mu\,\infty}$, which in the particular case of Dirac measures is the structure induced from the differential space of the infinite jets.

Of course, the most natural situation is when $(M, \mathcal{M})$ is a topological locally compact group. Then there is an invariant measure μ defined (modulo a positive coefficient) on M, which determines the unique differential structure on N^M.

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