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THE DECOMPOSITION OF TENSOR SPACES

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ABSTRACT. The problem of F-decomposition of tensor spaces is considered. F-decomposition of tensor spaces of the type $(1,1)$, $(1,2)$, $(1,3)$ is computed. Also F-decomposition for e-structures, when special conditions hold, is studied in this paper and explicit formulas for decomposition are given.

1. Introduction

Let E^p_q be a tensor space of tensors of the type (p,q) on E. (E is a real n-dimensional vector space.) If $A \in E^p_q$ we will write $A = (A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q})$, where $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ are components of the tensor $A \in E^p_q$ with respect to the basis of E^p_q.

D. Krupka [1], [2], [3] started to study traceless decomposition. Tensor $A \in E^p_q$ is called traceless if all traces are equal zero. D. Krupka proved next theorem, [3].

Theorem 1.1. *Let $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ be a tensor of the type (p,q). If $n+1 \geq p+q$, then there exists a unique decomposition of A of the form*

$$A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} + \sum_{t=1}^{min\{p,q\}} \sum_{\bigoplus} \delta^{i_{\rho_1} i_{\rho_2} \cdots i_{\rho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots j_{\sigma_t}} B^{\{\rho_1 \rho_2 \cdots \rho_t\} \cdots}_{\{\sigma_1 \sigma_2 \cdots \sigma_t\} \cdots}$$

where

$$\bigoplus = \begin{cases} \forall \rho_1, \rho_2 \cdots \rho_t \in \overline{1,p} & (\rho_1 < \rho_2 < \cdots < \rho_t) \\ \forall \sigma_1, \sigma_2 \cdots \sigma_t \in \overline{1,q} & (\sigma_i \textit{ are different to each other}) \end{cases}$$

and

$$\delta^{i_{\rho_1} i_{\rho_2} \cdots i_{\rho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots j_{\sigma_t}} = \delta^{i_{\rho_1}}_{j_{\sigma_1}} \delta^{i_{\rho_2}}_{j_{\sigma_2}} \cdots \delta^{i_{\rho_t}}_{j_{\sigma_t}}$$

such that tensors $B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}; B^{\{\rho_1 \rho_2 \cdots \rho_t\} \cdots}_{\{\sigma_1 \sigma_2 \cdots \sigma_t\} \cdots}$ are traceless.

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This theorem was generalized by J. Mikeš in [4], [5]. J. Mikeš introduced the concept F-traceless tensor in the following way.

Let F^i_j be an arbitrary affinor ($F^i_j \in E^1_1$ such that $F^\alpha_\alpha = 0$.) Tensor $A \in E^p_q$ is called F-traceless if the following condition holds.

$$\forall k = \overline{1,p}; \quad \forall r = \overline{1,q}; \quad A^{\cdots\, i_{k-1}\alpha i_{k+1}\,\cdots}_{\cdots\, j_{r-1}\beta j_{r+1}\cdots} F^\beta_\alpha = 0; \quad A^{\cdots\, i_{k-1}\alpha i_{k+1}\,\cdots}_{\cdots\, j_{r-1}\alpha j_{r+1}\cdots} = 0.$$

We also will work with e-structures. For these structures the condition $F^i_\alpha F^\alpha_j = e\delta^i_j$ holds, where $e = \pm\delta^i_j$.

J. Mikeš proved the following Theorem which generalizes Theorem 1.1 see [4], [5].

Theorem 1.2. *Let $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ be a tensor of the type (p,q). If $n > 2(p+q)$, then there exists a unique decomposition of A of the form*

$$A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} + \sum_{t=1}^{min\{p,q\}} \sum_{\bigoplus} \overset{*}{Q}{}^{i_{\rho_1} i_{\rho_2} \cdots i_{\rho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots j_{\sigma_t}} \overset{\diamond}{B}{}^{\cdots}_{\cdots}$$

where

$$\bigoplus = \begin{cases} \forall \rho_1, \rho_2 \cdots \rho_t \in \overline{1,p} & (\rho_1 < \rho_2 < \cdots < \rho_t) \\ \forall \sigma_1, \sigma_2 \cdots \sigma_t \in \overline{1,q} & (\sigma_i \textit{ are different to each other}) \\ \tau_1, \tau_2 \cdots \tau_t \in \{0,1\} & \end{cases}$$

$$* = \{\tau_1, \tau_2 \cdots \tau_t\}; \qquad \diamond = \begin{Bmatrix} \rho_1, & \rho_2 & \cdots & \rho_t \\ \sigma_1, & \sigma_2 & \cdots & \sigma_t \\ \tau_1, & \tau_2 & \cdots & \tau_t \end{Bmatrix}.$$

Tensors $B^{\cdots}_{\cdots}$ and $\overset{\diamond}{B}{}^{\cdots}_{\cdots}$ are F-traceless and

$$\overset{*}{Q}{}^{i_{\rho_1} i_{\rho_2} \cdots i_{\rho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots j_{\sigma_t}} = \overset{\{\tau_1\}}{F}{}^{i_{\rho_1}}_{j_{\sigma_1}} \overset{\{\tau_2\}}{F}{}^{i_{\rho_2}}_{j_{\sigma_2}} \cdots \overset{\{\tau_t\}}{F}{}^{i_{\rho_t}}_{j_{\sigma_t}}; \qquad \overset{\{0\}}{F}{}^i_j = \delta^i_j; \quad \overset{\{1\}}{F}{}^i_j = F^i_j.$$

2. F-decomposition of tensors of the type $(1,1)$ and $(1,2)$

The F-decomposition of the tensors of the type $(1,1)$ for e-structures is obvious and it is given by the next theorem.

Theorem 2.1. *Let A be a tensor of the type $(1,1)$ and let F be a traceless e-structure. If $n > 0$ then there exists unique F-decomposition of the tensor A in the form*

$$A^i_j = B^i_j + \delta^i_j C + D F^i_j \tag{1}$$

where tensors B^i_j, C, D are given in the form

$$C = \frac{1}{n} A^\alpha_\alpha; \qquad D = \frac{1}{en} A^\alpha_\beta F^\beta_\alpha \tag{2}$$

Proof. Suppose that tensor B is traceless. Contract the decomposition in the formula (1) by the Kronecker's delta and by the affinor F. We immediately get the formulas (2).

Now we find F-decomposition for tensors of the type $(1,2)$. We also extend an area of validity of the general theorem 1.2.

Theorem 2.2. *Let A be a tensor of the type $(1,2)$ and let F be a traceless e-structure. If $n>2$ then there exists unique F-decomposition of the tensor A in the form*

$$A^h_{ij} = B^h_{ij} + \delta^h_i C_j + \delta^h_j D_i + E_i F^h_j + G_j F^h_i, \tag{3}$$

where tensors B, C_j, D_j, E_j, G_j have following form

$$C_j = \tfrac{1}{n(3e+1-n^2)}\left((2e-n^2)A^\alpha_{\alpha j} + nA^\alpha_{j\alpha} + nA^\alpha_{\gamma\beta}F^\gamma_j F^\beta_\alpha - 2A^\alpha_{\beta\gamma}F^\gamma_j F^\beta_\alpha\right),$$
$$D_j = \tfrac{1}{n(3e+1-n^2)}\left(nA^\alpha_{\alpha j} + (2e-n^2)A^\alpha_{j\alpha} - 2A^\alpha_{\gamma\beta}F^\gamma_j F^\beta_\alpha + nA^\alpha_{\beta\gamma}F^\gamma_j F^\beta_\alpha\right),$$
$$E_j = \tfrac{1}{n(3e+1-n^2)}\left((2-n^2)A^\alpha_{j\beta}F^\beta_\alpha + nA^\alpha_{\beta j}F^\beta_\alpha + nA^\alpha_{\alpha\beta}F^\beta_j - 2A^\alpha_{\beta\alpha}F^\beta_j\right),$$
$$G_j = \tfrac{1}{n(3e+1-n^2)}\left(nA^\alpha_{j\beta}F^\beta_\alpha + (2-n^2)A^\alpha_{\beta j}F^\beta_\alpha - 2A^\alpha_{\alpha\beta}F^\beta_j + nA^\alpha_{\beta\alpha}F^\beta_j\right),$$
$$B^h_{ij} = A^h_{ij} - \delta^h_i C_j - \delta^h_j D_i - E_i F^h_j - G_j F^h_i.$$

Proof. Suppose that tensor B is F-traceless. Contract the decomposition in the form (3) by the Kronecker's delta and by the affinor F. We obtain a system of algebraic equations in unknowns

$$C_j,\ D_j,\ E_j,\ G_j,\ C_\alpha F^\alpha_j,\ D_\alpha F^\alpha_j,\ E_\alpha F^\alpha_j,\ G_\alpha F^\alpha_j:$$

$$\begin{array}{rcccccl}
nC_j & + & D_j & + & E_\alpha F^\alpha_j & = & A^\alpha_{\alpha j},\\
C_j & + & nD_j & + & G_\alpha F^\alpha_j & = & A^\alpha_{j\alpha},\\
enE_j & + & eG_j & + & C_\alpha F^\alpha_j & = & A^\alpha_{j\beta}F^\beta_\alpha,\\
eE_j & + & enG_j & + & D_\alpha F^\alpha_j & = & A^\alpha_{\beta j}F^\beta_\alpha,\\
eE_j & + & nC_\alpha F^\alpha_j & + & D_\alpha F^\alpha_j & = & A^\alpha_{\alpha\beta}F^\beta_j,\\
eG_j & + & C_\alpha F^\alpha_j & + & nD_\alpha F^\alpha_j & = & A^\alpha_{\beta\alpha}F^\beta_j,\\
eC_j & + & enE_\alpha F^\alpha_j & + & eG_\alpha F^\alpha_j & = & A^\alpha_{\gamma\beta}F^\gamma_j F^\beta_\alpha,\\
eD_j & + & eE_\alpha F^\alpha_j & + & enG_\alpha F^\alpha_j & = & A^\alpha_{\beta\gamma}F^\gamma_j F^\beta_\alpha.
\end{array}$$

For $n > 2$, this system has unique solution. After some computing we obtain components C_j, D_j, E_j, G_j from theorem 2.2. Then we compute B after substitution in

$$B^h_{ij} = A^h_{ij} - \delta^h_i C_j - \delta^h_j D_i - E_i F^h_j - G_j F^h_i.$$

By contracting tensor B we check that B is really F-traceless.

The decomposition is more simple in the case when tensor A^h_{ij} has a special structure. Consider following cases:

1) If equation $A^h_{ij} = A^h_{ji}$ is true then

$$C_j = D_j = \tfrac{1}{n(3e+1-n^2)} \left((2e + n - n^2) A^\alpha_{\alpha j} + (n-2) A^\alpha_{\gamma\beta} F^\gamma_j F^\beta_\alpha \right),$$

$$E_j = G_j = \tfrac{1}{n(3e+1-n^2)} \left((2 + n - n^2) A^\alpha_{j\beta} F^\beta_\alpha + (n-2) A^\alpha_{\alpha\beta} F^\beta_j \right).$$

2) If equation $A^h_{ij} = -A^h_{ji}$ holds then

$$C_j = -D_j = \tfrac{1}{n(3e+1-n^2)} \left((2e - n - n^2) A^\alpha_{\alpha j} + (n+2) A^\alpha_{\gamma\beta} F^\gamma_j F^\beta_\alpha \right),$$

$$E_j = -G_j = \tfrac{1}{n(3e+1-n^2)} \left((2 - n - n^2) A^\alpha_{j\beta} F^\beta_\alpha + (n+2) A^\alpha_{\alpha\beta} F^\beta_j \right).$$

3. F-decomposition of tensors of the type $(1,3)$

We want to compute the F-decomposition of tensors of the type $(1,3)$ in this section. But when we solve this problem we obtain 48 algebraic equations in 48 unknowns. The authors have been unable to solve this system of equations by the software accessible to them. We only were successful in extending the validity of the general theorem 1.2.

Theorem 3.1. *Let A be a tensor of the type $(1,3)$ and let F be traceless e-structure. If $n > 4$ then there exists unique F-decomposition of the tensor A in the form*

$$A^h_{ijk} = B^h_{ijk} + \delta^h_i C_{jk} + \delta^h_j D_{ik} + \delta^h_k E_{ij} + G_{jk} F^h_i + H_{ik} F^h_j + I_{ij} F^h_k, \tag{4}$$

where B^h_{ijk} is F-traceless tensor, $C_{jk}, D_{ik}, E_{ij}, G_{jk}, H_{ik}, I_{ij}$ are tensors of the type $(0,2)$.

Proof. Suppose that tensor B is F-traceless. We contract equation (4) by the Kronecker's delta and by the affinor F. We obtain a system of 48 algebraic equation in 48 unknowns. The determinant of the matrix of this system is expressed by this form

$$Det(T) = n^{12}\,(n-2)^{16}\,(n+2)^{16}\,(n-4)^2\,(n+4)^2. \tag{5}$$

This system has unique solution if and only if $Det(T) \neq 0$. The dimension of the tensor space with traceless e- structure must be even, it follows from the algebraic theory. Therefore there exists unique decomposition for $n > 4$.

4. F-decomposition of tensors of the type $(1,3)$ under special conditions

We compute F-decompositions tensors of the type$(1,3)$ for special cases in next theorems.

Theorem 4.1. *Let A be the tensor of the type $(1,3)$, $n=6$ and let F be traceless e-structure $(e=1)$. Then there exists unique F-decomposition of the tensor A in the form*

$$A^h_{ijk} = B^h_{ijk} + \delta^h_i C_{jk} + \delta^h_j D_{ik} + \delta^h_k E_{ij} + G_{jk} F^h_i + H_{ik} F^h_j + I_{ij} F^h_k,$$

where

$$\begin{aligned} C_{jk} = \ & \tfrac{1}{960} \Big[184 A^\alpha_{\alpha jk} - 36 \left(A^\alpha_{j\alpha k} + A^\alpha_{kj\alpha} + A^\alpha_{\gamma j\beta} F^\beta_\alpha F^\gamma_k + A^\alpha_{\gamma\beta k} F^\beta_\alpha F^\gamma_j \right) + \\ & +9 \left(A^\alpha_{jk\alpha} + A^\alpha_{k\alpha j} + A^\alpha_{\gamma\beta j} F^\beta_\alpha F^\gamma_k + A^\alpha_{j\gamma\beta} F^\beta_\alpha F^\gamma_k + A^\alpha_{k\beta\gamma} F^\beta_\alpha F^\gamma_j + \right. \\ & \left. + A^\alpha_{\gamma k\beta} F^\beta_\alpha F^\gamma_j + A^\alpha_{\beta\alpha\gamma} F^\gamma_j F^\beta_k + A^\alpha_{\gamma\beta\alpha} F^\gamma_j F^\beta_k \right) - 6 \left(A^\alpha_{\alpha kj} + A^\alpha_{\beta\gamma j} F^\beta_\alpha F^\gamma_k + \right. \\ & + A^\alpha_{j\beta\gamma} F^\beta_\alpha F^\gamma_k + A^\alpha_{\beta k\gamma} F^\beta_\alpha F^\gamma_j + A^\alpha_{k\gamma\beta} F^\beta_\alpha F^\gamma_j + A^\alpha_{\alpha\beta\gamma} F^\gamma_j F^\beta_k + A^\alpha_{\beta\gamma\alpha} F^\gamma_j F^\beta_k + \\ & \left. + A^\alpha_{\gamma\alpha\beta} F^\gamma_j F^\beta_k \right) + 14 \left(A^\alpha_{\beta j\gamma} F^\beta_\alpha F^\gamma_k + A^\alpha_{\beta\gamma k} F^\beta_\alpha F^\gamma_j \right) + 4 A^\alpha_{\alpha\gamma\beta} F^\beta_k F^\gamma_j \Big] , \end{aligned}$$

$$\begin{aligned} D_{jk} = \ & \tfrac{1}{960} \Big[184 A^\alpha_{j\alpha k} - 36 \left(A^\alpha_{\alpha jk} + A^\alpha_{jk\alpha} + A^\alpha_{j\gamma\beta} F^\beta_\alpha F^\gamma_k + A^\alpha_{\beta\gamma k} F^\beta_\alpha F^\gamma_j \right) + \\ & +9 \left(A^\alpha_{kj\alpha} + A^\alpha_{\alpha kj} + A^\alpha_{\beta\gamma j} F^\beta_\alpha F^\gamma_k + A^\alpha_{\gamma j\beta} F^\beta_\alpha F^\gamma_k + A^\alpha_{\beta k\ \gamma} F^\beta_\alpha F^\gamma_j + \right. \\ & \left. + A^\alpha_{k\gamma\beta} F^\beta_\alpha F^\gamma_j + A^\alpha_{\beta\gamma\alpha} F^\gamma_j F^\beta_k + A^\alpha_{\alpha\beta\gamma} F^\gamma_j F^\beta_k \right) - 6 \left(A^\alpha_{k\alpha j} + A^\alpha_{\beta\alpha\gamma} F^\gamma_j F^\beta_k + \right. \\ & + A^\alpha_{\alpha\gamma\beta} F^\gamma_j F^\beta_k + A^\alpha_{\gamma\beta\alpha} F^\gamma_j F^\beta_k + A^\alpha_{\beta j\gamma} F^\beta_\alpha F^\gamma_k + A^\alpha_{\gamma\beta j} F^\beta_\alpha F^\gamma_k + A^\alpha_{k\beta\gamma} F^\beta_\alpha F^\gamma_j + \\ & \left. + A^\alpha_{\gamma k\beta} F^\beta_\alpha F^\gamma_j \right) + 14 \left(A^\alpha_{j\beta\gamma} F^\beta_\alpha F^\gamma_k + A^\alpha_{\gamma\beta k} F^\beta_\alpha F^\gamma_j \right) + 4 A^\alpha_{\gamma\alpha\beta} F^\gamma_j F^\beta_k \Big] , \end{aligned}$$

$$\begin{aligned} E_{jk} = \ & \tfrac{1}{960} \Big[184 A^\alpha_{jk\alpha} - 36 \left(A^\alpha_{j\alpha k} + A^\alpha_{\alpha kj} + A^\alpha_{j\beta\gamma} F^\beta_\alpha F^\gamma_k + A^\alpha_{\beta k\gamma} F^\beta_\alpha F^\gamma_j \right) + \\ & +9 \left(A^\alpha_{\alpha jk} + A^\alpha_{k\alpha j} + A^\alpha_{\gamma\beta j} F^\beta_\alpha F^\gamma_k + A^\alpha_{\beta j\gamma} F^\beta_\alpha F^\gamma_k + A^\alpha_{\beta\gamma k} F^\beta_\alpha F^\gamma_j + \right. \\ & \left. + A^\alpha_{k\beta\gamma} F^\beta_\alpha F^\gamma_j + A^\alpha_{\alpha\gamma\beta} F^\gamma_j F^\beta_k + A^\alpha_{\beta\alpha\gamma} F^\gamma_j F^\beta_k \right) - 6 \left(A^\alpha_{kj\alpha} + A^\alpha_{\beta\gamma j} F^\beta_\alpha F^\gamma_k + \right. \\ & + A^\alpha_{\gamma j\beta} F^\beta_\alpha F^\gamma_k + A^\alpha_{k\gamma\beta} F^\beta_\alpha F^\gamma_j + A^\alpha_{\gamma\beta k} F^\beta_\alpha F^\gamma_j + A^\alpha_{\alpha\beta\gamma} F^\gamma_j F^\beta_k + A^\alpha_{\beta\gamma\alpha} F^\gamma_j F^\beta_k + \\ & \left. + A^\alpha_{\gamma\alpha\beta} F^\gamma_j F^\beta_k \right) + 14 \left(A^\alpha_{j\gamma\beta} F^\beta_\alpha F^\gamma_k + A^\alpha_{\gamma k\beta} F^\beta_\alpha F^\gamma_j \right) + 4 A^\alpha_{\gamma\beta\alpha} F^\gamma_j F^\beta_k \Big] , \end{aligned}$$

$$\begin{aligned} G_{jk} = \ & \tfrac{1}{960} \Big[184 A^\alpha_{\beta jk} F^\beta_\alpha - 36 \left(A^\alpha_{j\beta k} F^\beta_\alpha + A^\alpha_{kj\beta} F^\beta_\alpha + A^\alpha_{\beta j\alpha} F^\beta_k + A^\alpha_{\beta\alpha k} F^\beta_j \right) + \\ & +9 \left(A^\alpha_{jk\beta} F^\beta_\alpha + A^\alpha_{k\beta j} F^\beta_\alpha + A^\alpha_{\beta\alpha j} F^\beta_k + A^\alpha_{j\beta\alpha} F^\beta_k + A^\alpha_{k\alpha\beta} F^\beta_j + \right. \\ & \left. + A^\alpha_{\beta k\alpha} F^\beta_j + A^\alpha_{\gamma\delta\beta} F^\beta_\alpha F^\gamma_j F^\delta_k + A^\alpha_{\delta\beta\gamma} F^\beta_\alpha F^\gamma_j F^\delta_k \right) - 6 \left(A^\alpha_{\beta kj} F^\beta_\alpha + A^\alpha_{\alpha\beta j} F^\beta_k + \right. \\ & + A^\alpha_{j\alpha\beta} F^\beta_k + A^\alpha_{\alpha k\beta} F^\beta_j + A^\alpha_{k\beta\alpha} F^\beta_j + A^\alpha_{\gamma\beta\delta} F^\beta_\alpha F^\gamma_j F^\delta_k + A^\alpha_{\beta\delta\gamma} F^\beta_\alpha F^\gamma_j F^\delta_k + \\ & \left. + A^\alpha_{\delta\gamma\delta} F^\beta_\alpha F^\gamma_j F^\delta_k \right) + 14 \left(A^\alpha_{\alpha j\beta} F^\beta_k + A^\alpha_{\alpha\beta k} F^\beta_j \right) + 4 A^\alpha_{\beta\gamma\delta} F^\beta_\alpha F^\gamma_j F^\delta_k \Big] , \end{aligned}$$

$$\begin{aligned}
H_{jk} = \; & \tfrac{1}{960}\Big[184A^{\alpha}_{j\beta k}F^{\beta}_{\alpha} - 36\left(A^{\alpha}_{\beta jk}F^{\beta}_{\alpha} + A^{\alpha}_{j\beta\alpha}F^{\beta}_{k} + A^{\alpha}_{\alpha\beta k}F^{\beta}_{j} + A^{\alpha}_{jk\beta}F^{\beta}_{\alpha}\right) + \\
& +9\left(A^{\alpha}_{kj\beta}F^{\beta}_{\alpha} + A^{\alpha}_{\beta kj}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha\beta j}F^{\beta}_{k} + A^{\alpha}_{\beta j\alpha}F^{\beta}_{k} + A^{\alpha}_{\alpha k\beta}F^{\beta}_{j} + \right. \\
& \left. + A^{\alpha}_{k\beta\alpha}F^{\beta}_{j} + A^{\alpha}_{\delta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\beta\delta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\right) - 6\left(A^{\alpha}_{k\beta j}F^{\beta}_{\alpha} + A^{\alpha}_{\beta\alpha j}F^{\beta}_{k} + \right. \\
& +A^{\alpha}_{\alpha j\beta}F^{\beta}_{k} + A^{\alpha}_{\beta k\alpha}F^{\beta}_{j} + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j} + A^{\alpha}_{\beta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\gamma\delta\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + \\
& \left. + A^{\alpha}_{\delta\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\right) + 14\left(A^{\alpha}_{j\alpha\beta}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha k}F^{\beta}_{j}\right) + 4A^{\alpha}_{\gamma\beta\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big],
\end{aligned}$$

$$\begin{aligned}
I_{jk} = \; & \tfrac{1}{960}\Big[184A^{\alpha}_{jk\beta}F^{\beta}_{\alpha} - 36\left(A^{\alpha}_{j\beta k}F^{\beta}_{\alpha} + A^{\alpha}_{j\alpha\beta}F^{\beta}_{k} + A^{\alpha}_{\alpha k\beta}F^{\beta}_{j} + A^{\alpha}_{\beta kj}F^{\beta}_{\alpha}\right) + \\
& +9\left(A^{\alpha}_{\beta jk}F^{\beta}_{\alpha} + A^{\alpha}_{k\beta j}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha j\beta}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha j}F^{\beta}_{k} + A^{\alpha}_{\alpha\beta k}F^{\beta}_{j} + \right. \\
& \left. + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j} + A^{\alpha}_{\beta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\delta\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\right) - 6\left(A^{\alpha}_{kj\beta}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha\beta j}F^{\beta}_{k} + \right. \\
& +A^{\alpha}_{\beta j\alpha}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha k}F^{\beta}_{j} + A^{\alpha}_{k\beta\alpha}F^{\beta}_{j} + A^{\alpha}_{\gamma\beta\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\beta\delta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + \\
& \left. + A^{\alpha}_{\delta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\right) + 14\left(A^{\alpha}_{j\beta\alpha}F^{\beta}_{k} + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j}\right) + 4A^{\alpha}_{\gamma\delta\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big].
\end{aligned}$$

Proof. Suppose that tensor B is F-traceless. We substitute $n = 6$ a $e = 1$ in system from theorem 4.1. The solution is given by the above mentioned systems. Now we check F-tracelessness of the tensor B, i.e.

$$B^{\alpha}_{\alpha jk} = B^{\alpha}_{j\alpha k} = B^{\alpha}_{jk\alpha} = B^{\alpha}_{\beta jk}F^{\beta}_{\alpha} = B^{\alpha}_{j\beta k}F^{\beta}_{\alpha} = B^{\alpha}_{jk\beta}F^{\beta}_{\alpha} = 0.$$

This complete the proof. □ Next theorem gives F-decomposition of the tensor A of the type $(1,3)$ for $n = 8$ and $e = 1$.

Theorem 4.2. *Let A be tensor of the type $(1,3)$, $n = 8$ and let F be traceless e-structure. Then there exists unique F-decomposition of the tensor A in the form*

$$A^h_{ijk} = B^h_{ijk} + \delta^h_i C_{jk} + \delta^h_j D_{ik} + \delta^h_k E_{ij} + G_{jk}F^h_i + H_{ik}F^h_j + I_{ij}F^h_k,$$

where

$$\begin{aligned}
C_{jk} = \; & \tfrac{1}{1440}\Big[193A^{\alpha}_{\alpha jk} - 26\left(A^{\alpha}_{j\alpha k} + A^{\alpha}_{kj\alpha} + A^{\alpha}_{\gamma j\beta}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\gamma\beta k}F^{\beta}_{\alpha}F^{\gamma}_{j}\right) + \\
& +4\left(A^{\alpha}_{jk\alpha} + A^{\alpha}_{k\alpha j} + A^{\alpha}_{\gamma\beta j}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{j\gamma\beta}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{k\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j} + \right. \\
& \left. +A^{\alpha}_{\gamma k\beta}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\beta\alpha\gamma}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\gamma\beta\alpha}F^{\gamma}_{j}F^{\beta}_{k}\right) - 6\left(A^{\alpha}_{\alpha kj} + A^{\alpha}_{\beta\gamma j}F^{\beta}_{\alpha}F^{\gamma}_{k} + \right. \\
& +A^{\alpha}_{j\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta k\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{k\gamma\beta}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\alpha\beta\gamma}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\beta\gamma\alpha}F^{\gamma}_{j}F^{\beta}_{k} + \\
& \left. +A^{\alpha}_{\gamma\alpha\beta}F^{\gamma}_{j}F^{\beta}_{k}\right) + 14\left(A^{\alpha}_{\beta j\gamma}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta\gamma k}F^{\beta}_{\alpha}F^{\gamma}_{j}\right) + A^{\alpha}_{\alpha\gamma\beta}F^{\beta}_{k}F^{\gamma}_{j}\Big],
\end{aligned}$$

$$
\begin{aligned}
D_{jk} = \; & \tfrac{1}{1440}\Big[193A^{\alpha}_{j\alpha k} - 26\Big(A^{\alpha}_{\alpha jk} + A^{\alpha}_{jk\alpha} + A^{\alpha}_{j\gamma\beta}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta\gamma k}F^{\beta}_{\alpha}F^{\gamma}_{j}\Big) + \\
& +4\Big(A^{\alpha}_{kj\alpha} + A^{\alpha}_{\alpha kj} + A^{\alpha}_{\beta\gamma j}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\gamma j\beta}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta k\;\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j} + \\
& +A^{\alpha}_{k\gamma\beta}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\beta\gamma\alpha}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\alpha\beta\gamma}F^{\gamma}_{j}F^{\beta}_{k}\Big) - 6\Big(A^{\alpha}_{k\alpha j} + A^{\alpha}_{\beta\alpha\gamma}F^{\gamma}_{j}F^{\beta}_{k} + \\
& +A^{\alpha}_{\alpha\gamma\beta}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\gamma\beta\alpha}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\beta j\gamma}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\gamma\beta j}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{k\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j} + \\
& +A^{\alpha}_{\gamma k\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}\Big) + 14\Big(A^{\alpha}_{j\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\gamma\beta k}F^{\beta}_{\alpha}F^{\gamma}_{j}\Big) + A^{\alpha}_{\gamma\alpha\beta}F^{\gamma}_{j}F^{\beta}_{k}\Big],
\end{aligned}
$$

$$
\begin{aligned}
E_{jk} = \; & \tfrac{1}{1440}\Big[193A^{\alpha}_{jk\alpha} - 26\Big(A^{\alpha}_{j\alpha k} + A^{\alpha}_{\alpha kj} + A^{\alpha}_{j\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta k\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}\Big) + \\
& +4\Big(A^{\alpha}_{\alpha jk} + A^{\alpha}_{k\alpha j} + A^{\alpha}_{\gamma\beta j}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta j\gamma}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta\gamma k}F^{\beta}_{\alpha}F^{\gamma}_{j} + \\
& +A^{\alpha}_{k\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\alpha\gamma\beta}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha\gamma}F^{\gamma}_{j}F^{\beta}_{k}\Big) - 6\Big(A^{\alpha}_{kj\alpha} + A^{\alpha}_{\beta\gamma j}F^{\beta}_{\alpha}F^{\gamma}_{k} + \\
& +A^{\alpha}_{\gamma j\beta}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{k\gamma\beta}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\gamma\beta k}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\alpha\beta\gamma}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\beta\gamma\alpha}F^{\gamma}_{j}F^{\beta}_{k} + \\
& +A^{\alpha}_{\gamma\alpha\beta}F^{\gamma}_{j}F^{\beta}_{k}\Big) + 14\Big(A^{\alpha}_{j\gamma\beta}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\gamma k\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}\Big) + A^{\alpha}_{\gamma\beta\alpha}F^{\gamma}_{j}F^{\beta}_{k}\Big],
\end{aligned}
$$

$$
\begin{aligned}
G_{jk} = \; & \tfrac{1}{1440}\Big[193A^{\alpha}_{\beta jk}F^{\beta}_{\alpha} - 26\Big(A^{\alpha}_{j\beta k}F^{\beta}_{\alpha} + A^{\alpha}_{kj\beta}F^{\beta}_{\alpha} + A^{\alpha}_{\beta j\alpha}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha k}F^{\beta}_{j}\Big) + \\
& +4\Big(A^{\alpha}_{jk\beta}F^{\beta}_{\alpha} + A^{\alpha}_{k\beta j}F^{\beta}_{\alpha} + A^{\alpha}_{\beta\alpha j}F^{\beta}_{k} + A^{\alpha}_{j\beta\alpha}F^{\beta}_{k} + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j} + \\
& + A^{\alpha}_{\beta k\alpha}F^{\beta}_{j} + A^{\alpha}_{\gamma\delta\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\delta\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big) - 6\Big(A^{\alpha}_{\beta kj}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha\beta j}F^{\beta}_{k} + \\
& +A^{\alpha}_{j\alpha\beta}F^{\beta}_{k} + A^{\alpha}_{\alpha k\beta}F^{\beta}_{j} + A^{\alpha}_{k\beta\alpha}F^{\beta}_{j} + A^{\alpha}_{\gamma\beta\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\beta\delta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + \\
& +A^{\alpha}_{\delta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big) + 14\Big(A^{\alpha}_{\alpha j\beta}F^{\beta}_{k} + A^{\alpha}_{\alpha\beta k}F^{\beta}_{j}\Big) + A^{\alpha}_{\beta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big] \;,
\end{aligned}
$$

$$
\begin{aligned}
H_{jk} = \; & \tfrac{1}{1440}\Big[193A^{\alpha}_{j\beta k}F^{\beta}_{\alpha} - 26\Big(A^{\alpha}_{\beta jk}F^{\beta}_{\alpha} + A^{\alpha}_{j\beta\alpha}F^{\beta}_{k} + A^{\alpha}_{\alpha\beta k}F^{\beta}_{j} + A^{\alpha}_{jk\beta}F^{\beta}_{\alpha}\Big) + \\
& +4\Big(A^{\alpha}_{kj\beta}F^{\beta}_{\alpha} + A^{\alpha}_{\beta kj}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha\beta j}F^{\beta}_{k} + A^{\alpha}_{\beta j\alpha}F^{\beta}_{k} + A^{\alpha}_{\alpha k\beta}F^{\beta}_{j} + \\
& + A^{\alpha}_{k\beta\alpha}F^{\beta}_{j} + A^{\alpha}_{\delta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\beta\delta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big) - 6\Big(A^{\alpha}_{k\beta j}F^{\beta}_{\alpha} + A^{\alpha}_{\beta\alpha j}F^{\beta}_{k} + \\
& +A^{\alpha}_{\alpha j\beta}F^{\beta}_{k} + A^{\alpha}_{\beta k\alpha}F^{\beta}_{j} + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j} + A^{\alpha}_{\beta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\gamma\delta\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + \\
& + A^{\alpha}_{\delta\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big) + 14\Big(A^{\alpha}_{j\alpha\beta}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha k}F^{\beta}_{j}\Big) + A^{\alpha}_{\gamma\beta\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big] \;,
\end{aligned}
$$

$$
\begin{aligned}
I_{jk} = \; & \tfrac{1}{1440}\Big[193A^{\alpha}_{jk\beta}F^{\beta}_{\alpha} - 26\Big(A^{\alpha}_{j\beta k}F^{\beta}_{\alpha} + A^{\alpha}_{j\alpha\beta}F^{\beta}_{k} + A^{\alpha}_{\alpha k\beta}F^{\beta}_{j} + A^{\alpha}_{\beta kj}F^{\beta}_{\alpha}\Big) + \\
& +4\Big(A^{\alpha}_{\beta jk}F^{\beta}_{\alpha} + A^{\alpha}_{k\beta j}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha j\beta}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha j}F^{\beta}_{k} + A^{\alpha}_{\alpha\beta k}F^{\beta}_{j} + \\
& + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j} + A^{\alpha}_{\beta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\delta\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big) - 6\Big(A^{\alpha}_{kj\beta}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha\beta j}F^{\beta}_{k} + \\
& +A^{\alpha}_{\beta j\alpha}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha k}F^{\beta}_{j} + A^{\alpha}_{k\beta\alpha}F^{\beta}_{j} + A^{\alpha}_{\gamma\beta\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\beta\delta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + \\
& + A^{\alpha}_{\delta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big) + 14\Big(A^{\alpha}_{j\beta\alpha}F^{\beta}_{k} + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j}\Big) + A^{\alpha}_{\gamma\delta\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big] \;.
\end{aligned}
$$

Proof. Suppose that tensor B is F-traceless. Substitute $n = 8$ a $e = 1$ in system from theorem 4.1. The solution is given by above-mentioned systems. Then we check F-tracelessness of the tensor B and this complete the proof. $\square$

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