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SYMMETRY FIELDS OF GEODESIC VECTOR FIELD

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Abstract. The Lie algebra y of the symmetry fields of the geodesic vector field on the tangent bundle of a smooth manifold allows the natural grading $y = \oplus_{i=0}^{\infty} h_i$, where the 0-rank fields h_0 correspond to the connection symmetry fields. One of the difficult questions is the completeness of the fields in the sence of that of solutions. Partly the problem is solved by the Kobayashi theorem, from which follows that h_0 is the Lie algebra of complete fields provided such is the geodesic field. Under the same condition the author found a subspace in h_1 consisting of complete fields. These can be constructed as

$$Y = A_s^i v^s \frac{\partial}{\partial x_i} + (\nabla_k A_s^i v^s v^k - \omega_{ts}^i A_k^t v^s v^k) \frac{\partial}{\partial v_i},$$

where (x, v) are standard coordinates of the tangent bundle of the manifold, ω— a connection form, A— a covariant constant tensor field of type (1,1). For the flat connection on R^n the infinite space of complete fields is presented.

This report touches a domain of research concerning the infinite dimension spaces appearing in the differential geometry. The samples of such ones are the smooth manifold diffeomorfism group, space of vector and tensor fields, and that of linear connections on smooth manifolds.

At present there are many works in which one or another class of infinite groups and algebras were more or less investigated. First of all, it is Lie algebras of the vector fields and the correspondent diffeomorfism groups.

One of the difficulties in investigation of this case is the absence, in general, of the correspondence between the Lie algebra and the diffeomorfism group. For the vector fields are not complete, the algebra of the infinitesimal automorphisms is too large to be the corresponding Lie algebra. On the other hand the subset of complete fields is not even a linear space. Palais [1] provided the relevant examples.

But anyway, in view of the last results in fluid mechanics we hardly can overevaluate the significance of research in this domain.

One of the less investigated class of objects is a vector field automorphism group. For null-field on manifold it is the diffeomorphism group. Some types of

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differential structures on such groups were defined by Omori [2] in 1973. In 1984 N. K. Smolentsev [3] has shown that if the field X on compact smooth manifold M is non-divergent and does not equal zero at any point, the diffeomorphism group leaving invariant volume and the field can be endowed with Omori's structure and consists of the non-divergent vector fields commuting with X.

In 1995 for the case of analytic manifold M^{2n} M. A. Parinov [4] has shown that if A is a covector field, whose differential forms a symplectic structure, the automorphism group A is a finite dimension group and its dimension does not exceed $2n^2+n$. Note, that for the non-analytic case this statement is not valid.

The case of the Hamiltonian fields on the co-tangent bundle T^*M of C^∞-manifold M^2 was investigated by V. V. Kozlov, [5] in 1988-95. Studying the dynamic systems generated by the Hamiltonian field V_H he come to the "symmetry fields" $\{U|[U,V_H]=0\}$. It turned out that the form and existence of them depends on the topology of M^2. Decomposing the symmetry field near the equilibrium point the author indroduced the concept of homogeneous field of rank k according to the degrees of polynomial components of decomposition. All is reduced then to the investigation of the homogeneous fields.

Within the above research, the symmetrical tensor fields appear naturally. For them, in spite of the symmetry assumption, there are no general results (except maybe for 1 and 2 degrees). Moreover no interesting differential operators are singled out for research.

We will consider here the most general question on the geodesic vector field symmetries on the tangent bundle TM of C^∞-manifold M and present some result concerning the completness of the symmetry fields. We will consider the decomposition of symmetry field near the null section M of the tangent bundle TM. As shown in the work of the author [6] the terms of the decomposition have the form

$$Y = A^i_s v^s \frac{\partial}{\partial x_i} + (\nabla_k A^i_s v^s v^k - \omega^i_{ts} A^t_k v^s v^k)\frac{\partial}{\partial v_i} = (A, \nabla A - \omega(A)),$$

where (x,v) are standard coordinates of the tangent bundle of the manifold, ω— a connection form, A— a symmetrical tensor field of type $(1,k)$ on the manifold M satisfying the generalized Jacobi equation

$$\{\nabla^2 A + R(A)\} = 0,$$

where R is the curvature tensor and braces mean symmetrisation. The fields A will be named the generalized Jacobi fields and we have the graded Lie algebra $y = \oplus_{i=0} h_i$ of the above equation solutions.

We will say, that the Jacobi field A on M is *complete*, if the field Y_A is complete (i.e. trajectories of this field are everywhere defined). Partially the question about the completeness is solved by Kobayashi's theorem.

Theorem (Kobayashi [1]). *If the geodesic vector field S is complete, then all infinitesimal automorphisms of the connection are complete.*

The homogeneous fields of rank 0, which are the solutions of Jacobi's equation, are the infinitesimal automorphisms of connection on M. Thus, under the condition of completeness of the field S, the finite dimensional subalgebra h_0 of algebra y consists of the complete fields. In theorem 2 below one more finite dimensional subalgebra of algebra y is described, consisting of the complete fields, under the assumption, that the geodesic vector field is such.

Theorem 1. *Let A be tensor field, such that A_x is nondegenerate linear operator on T_xM at every point $x \in M$, and let the relation* (1)

$$\nabla A(A^{-1}, A^{-1}) = \nabla A^{-1}$$

be valid. The geodesic vector field S is complete, if and only if at least one field Y_A or $Y_{A^{-1}}$ is complete.

Proof. By virtue of the nondegeneracy of the operators A_x the fiber diffeomorphism $\widetilde{A} : TM \to TM$; $\widetilde{A}(x, v) = (x, Av)$ is correctly defined. In standard coordinates on TM its differential $d\widetilde{A}$ has the form

$$d_{(x,v)}\widetilde{A} = \begin{pmatrix} E & O \\ \frac{\partial A}{\partial x} v & A \end{pmatrix}.$$

One can write the condition, that the fields S and $Y_{A^{-1}}$ are $\widetilde{A}$-related:

$$(d\widetilde{A})S = (E, \frac{\partial A}{\partial x} - A(\omega)) = (E, \nabla A - \omega(A)),$$

$$(Y_{A^{-1}})(\widetilde{A}) = (A^{-1}A, (\nabla A^{-1})(A, A) - \omega(A^{-1}A, A)).$$

From here we see, that the fields are $\widetilde{A}$-related, if and only if the relation (1) is valid. It remains to note, that the fields Y_A and S are $\widetilde{A}^{-1}$-related, which completes the proof.

Let us denote the space of the covariant constant fields of type (1,1) on M by the sign r_1. The covariant constant field A is uniquely defined by the operator A_x at arbitrary point $x \in M$. The algebra r_1 is therefore isomorphic to certain subalgebra of the matrix algebra $Mat_n(R)$, containing E.

Theorem 2. *If S is a complete field, then the space of fields Y_A, where $A \in r_1$, is Lie algebra consisting of the complete fields.*

Proof. Fields generated by the covariant constant tensors commutate among themselves, and thus, mentioned linear space is the trivial Lie algebra. We will denote it also by the symbol r_1.

We will find in r_1 the basis consisting of the fields, which are nondegenerate at each point of M.

As already noted, r_1 is subalgebra in $Mat_n(R)$, with identity matrix corresponding to the geodesic vector field S. By Wedderburn's theorem, r_1 decomposes in direct sum of radical and the matrix algebras over R, C and H. In the radical we will take basis consisting of nilpotents, in the matrix algebras— elements in the form of εt_{ij}, where t_{ij} is the matrix with 1 on (i, j)th place, $\varepsilon = 1$ in the case of R, $\varepsilon = 1$, i in the case of C and $\varepsilon = 1$, i, j, k in the case of H. All these elements form the basis of algebra r_1 as the vector space. By virtue of validity of relations $X^k = 0$ or $X^k = X$ for the mentioned matrixes, all eigenvalues of the operator A_x are equal to 0 or lie on the unit circle for each tensor field A from the mentioned basis and every point $x \in M$. We will add $2E$ to each basis tensor field. The obtained set of fields $A_1, \dots, A_r$ is also basis of r_1, with the tensor fields generating them being nondegenerate at every point of manifold M. Thus, one can consider the fields A^{-1}. From the fact that the field A_i is covariant constant follows, that the field A_i^{-1} is the constant too, and hence the relation (1) is valid. By theorem 1 the fields A_i and A_i^{-1} are complete under the assumption of the field S completeness. Let us consider the field $V = \alpha_1 Y_{A_1} + \dots + \alpha_r Y_{A_r}$. Let us denote the local flow of the field V on M by the sign g_t^V. Then by virtue of the commutativity of the basis fields one can write

$$g_t^V = g_t^{\alpha_1 Y_{A_1}} \times \dots \times g_t^{\alpha_r Y_{A_r}}.$$

The right hand side is defined globally on TM, hence, the field V is complete. The theorem is proved.

So we proved that upon the completeness of the geodesic vector field the space of the complete Jacobi fields has the dimension not less than dimension of infinitesimal automorphisms space plus that of covariant constant tensor fields of type (1,1). The same is valid therefore for the diffeomorphism group.

For the plane case R^n, the above dimension is not less than $2n^2 + n$ (compare with the result of Parinov). But we can give the infinite set of the complete fields in this case. We describe them as the solutions of the differentail equation system $\dot{x} = A(x, v); \dot{v} = (\nabla A)(x, v)$.

a) Let a Jacobi field A form an analytic vector function $A(v)$. The field Y_A will have the form $A(v)\frac{\partial}{\partial x}$ and will generate the ordinary shift along M, obviously complete.

b) Let A satisfies the conditions

$$\begin{cases} A(x, v) = A_i(v) x^i; \\ \langle (\nabla A)(v), v \rangle = 0, \end{cases}$$

where brackets mean scalar product. The solutions of the equation $\dot{v} = (\nabla A)(v)$ are complete and bounded from above in this case. Thus the solutions of the linear system $\dot{x} = A(v)x$ are also complete. Intergal surface of this system is the cylinder over the trajectory of the linear system.

c) For all fields not as above let us consider the condition

$$\langle A(x, v), x \rangle + \langle (\nabla A)(x, v), v \rangle = 0.$$

Together with $\nabla^2 A = 0$ it gives the number of fields tangent to the spheres $x^2 + v^2 = c$ and therefore complete. Straightforward consideration shows that this set is not empty.

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