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GEOMETRICAL ASPECTS IN THE DYNAMICS OF AN AUTOMOBIL WITH TWO TRAILERS

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ABSTRACT. We study some properties related to the dynamics, control, stability and integrability of an automobil with two trailers.

1. Introduction

Let us consider a simple kinematic model of an automobil with two trailers. Its dynamics is described by the following chained form system:

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & u_1\,, \\ \dot{x}_2 & = & u_2\,, \\ \dot{x}_3 & = & x_2 u_1\,, \\ \dot{x}_4 & = & x_3 u_1\,, \\ \dot{x}_5 & = & x_4 u_1\,. \end{array}\right. \tag{1.1}$$

The goal of our paper is to prove that we can express this chained form system as a left invariant control system on a matrix Lie group and then using this realisation to study some of its geometrical and dynamical properties.

2. Matrix realisation of the system (1.1)

Let G_5 be the Lie group of all 5×5 matrices of the form:

$$X = \begin{bmatrix} 1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 1 & x_1 & x_1^2/2 & x_1^3/6 \\ 0 & 0 & 1 & x_1 & x_1^2/2 \\ 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

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where $x_i \in \mathbb{R}$, for each $i = 1, 2, 3, 4, 5$. A basis for the nilpotent Lie algebra $\mathfrak{g}_5$ of this Lie group is given by $\{A_1, A_2, A_3, A_4, A_5\} = \{A_1, A_2, [A_2, A_1], [[A_2, A_1], A_1], [[[A_2, A_1], A_1], A_1]\}$, where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A long, but straightforward computation shows us that:

Theorem 2.1 *The chained form system* (1.1) *can be put in the equivalent form:*

$$\dot{X} = X(u_1 A_1 + u_2 A_2) \tag{2.1}$$

or in other words, the system (1.1) *can be viewed as a left invariant control system on* G_5.

Theorem 2.2 *The system* (1.1) *is controllable and it is a triple Lie bracket one.*

3. An optimal control problem

Let J be the cost function given as usually by

$$J(u_1, u_2) = \frac{1}{2} \int_0^{t_f} (c_1 u_1^2 + c_2 u_2^2) dt\,, \quad c_1 > 0, c_2 > 0\,. \tag{3.1}$$

Then we can prove:

Theorem 3.1 *The controls which minimize J and steer the system (2.1) from $X = X_0$ at $t = 0$ to $X = X_f$ at $t = t_f$ are given by:*

$$u_1 = \frac{1}{c_1} P_1\,, \quad u_2 = \frac{1}{c_2} P_2\,, \tag{3.2}$$

where P_i's are the solutions of:

$$\begin{cases} \dot{P}_1 &= \frac{1}{c_2} P_2 P_3\,, \\ \dot{P}_2 &= -\frac{1}{c_1} P_1 P_3\,, \\ \dot{P}_3 &= -\frac{1}{c_1} P_1 P_4\,, \\ \dot{P}_4 &= -\frac{1}{c_1} P_1 P_5\,, \\ \dot{P}_5 &= 0\,. \end{cases} \tag{3.3}$$

Proof. Simply apply Krishnaprasad's theorem [?] to this special case. Then the extremal controls are given by (3.2), where P_i's are solutions of the reduced Hamilton's equations on $\mathfrak{g}_5^* \cong \mathbb{R}^5$, i.e.

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \\ \dot{P}_4 \\ \dot{P}_5 \end{bmatrix} = \begin{bmatrix} 0 & P_3 & P_4 & P_5 & 0 \\ -P_3 & 0 & 0 & 0 & 0 \\ -P_4 & 0 & 0 & 0 & 0 \\ -P_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \nabla H\,,$$

where

$$H(P_1, P_2, P_3, P_4, P_5) = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 , \tag{3.4}$$

or equivalently

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \\ \dot{P}_4 \\ \dot{P}_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{c_2}P_2P_3 , \\ -\frac{1}{c_1}P_1P_3 , \\ -\frac{1}{c_1}P_1P_4 , \\ -\frac{1}{c_1}P_1P_5 , \\ 0 \end{bmatrix} \cdot \nabla H ,$$

as required. □

Remark 3.1 It is not hard to see that the function C given by

$$C(P_1, P_2, P_3, P_4, P_5) = -P_3P_5 + \frac{1}{2}P_4^2$$

is a Casimir of our configuration $(\mathfrak{g}_5^*)_-$, i.e.

$$(\nabla C)^t \cdot \begin{bmatrix} 0 & P_3 & P_4 & P_5 & 0 \\ -P_3 & 0 & 0 & 0 & 0 \\ -P_4 & 0 & 0 & 0 & 0 \\ -P_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \nabla f = 0 ,$$

for each $f \in C^\infty(\mathbb{R}^5, \mathbb{R})$.

4. The Hamilton-Poisson structure of the system (3.3) and some of its properties

It is not hard to see that the system (3.3) can be put in the equivalent form:

$$\begin{cases} \dot{P}_1 &= \frac{1}{c_2}P_2P_3 , \\ \dot{P}_2 &= -\frac{1}{c_1}P_1P_3 , \\ \dot{P}_3 &= -\frac{1}{c_1}P_1P_4 , \\ \dot{P}_4 &= -\frac{k}{c_1}P_1 , \end{cases} \tag{4.1}$$

where

$$P_5 = k .$$

In all that follows we shall assume that

$$c_1 = c_2 = k = 1 . \tag{4.2}$$

Then we can prove:

Theorem 4.1 *The equations (4.1) - (4.2) may be explicitely integrated by elliptic functions.*

Proof. It is known that

$$P_1^2 + P_2^2 = l$$

and

$$\frac{1}{2}P_4^2 - P_3 = m$$

are constants of motion. Then an easy computation shows us that

$$P_2 = \frac{1}{6}P_4^3 - cP_4 + d\,,$$

where $c, d \in \mathbb{R}$. Therefore

$$\begin{aligned} P_1 &= \pm\sqrt{l - P_2^2} \\ &= \pm\sqrt{l - (\frac{1}{6}P_4^3 - cP_4 + d)}\,. \end{aligned}$$

So,

$$\dot{P}_4 = \mp\sqrt{l - d - \frac{1}{6}P_4^3 + cP_4}\,,$$

or equivalently

$$t = \mp\int_{P_4(0)}^{P_4} \frac{dP_4}{\sqrt{l - d - \frac{1}{6}P_4^3 + cP_4}}$$

which shows that P_4, and hence P_3, P_1 and P_2 are elliptic functions of time as required. □

Theorem 4.2 *The system (4.1) – (4.2) has a Hamilton-Poisson realisation with the phase space $\mathbb{R}^4$, with the Poisson bracket given by the matrix*

$$\Pi = \begin{bmatrix} 0 & P_3 & P_4 & 1 \\ -P_3 & 0 & 0 & 0 \\ -P_4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{4.3}$$

and the Hamiltonian H given by:

$$H(P_1, P_2, P_3, P_4) = \frac{1}{2}(P_1^2 + P_2^2)\,.$$

Proof. Indeed, one can easily check that

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \\ \dot{P}_4 \end{bmatrix} = \begin{bmatrix} 0 & P_3 & P_4 & 1 \\ -P_3 & 0 & 0 & 0 \\ -P_4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} P_2P_3 \\ -P_1P_3 \\ -P_1P_4 \\ -P_1 \end{bmatrix},$$

as required. □

Remark 4.1 It is not hard to see that the function C given by:

$$C(P_1, P_2, P_3, P_4) = -P_3 + \frac{1}{2}P_4^2$$

is a Casimir of our configuration $(\mathbb{R}^4, \Pi)$.

Let us observe now that the equilibrium states of our system (4.1) – (4.2) are

$$(0, 0, \alpha, \beta), \quad (0, \alpha, 0, \beta) \tag{4.4}$$

where $\alpha, \beta \in \mathbb{R}$.

Theorem 4.3 *The equilibrium states (4.4) are all spectrally stable.*

Proof. The proof can be obtained immediately using the matrix of the linearized system at the equilibrium of interest and so we shall omit any other details. □

Remark 4.2 It is an open problem to decide if the equilibrium states (4.4) are nonlinear stable or they are unstable.

Let us finish this section with some remarks concerning the nonlinear stability of the equilibrium state $(0,0,\alpha,0)$, $\alpha\in\mathbb{R}$, of our system (4.1) – (4.2) via a linear control. For beginning let us consider the controlled system:

$$\left\{\begin{array}{lcl} \dot{P}_1 & = & P_2P_3+P_4\,, \\ \dot{P}_2 & = & -P_1P_3\,, \\ \dot{P}_3 & = & -P_1P_4\,, \\ \dot{P}_4 & = & -P_1\,. \end{array}\right. \tag{4.5}$$

Theorem 4.4 *The controlled system (4.5) is an Hamilton-Poisson system with the phase space $\mathbb{R}^4$, the Poisson bracket given by the matrix (4.3) and the Hamiltonian given by*

$$H(P_1,P_2,P_3,P_4)=\frac{1}{2}(P_1^2+P_2^2+P_4^2)\,.$$

Proof. Indeed, one easily check that

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \\ \dot{P}_4 \end{bmatrix} = \begin{bmatrix} 0 & P_3 & P_4 & 1 \\ -P_3 & 0 & 0 & 0 \\ -P_4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ 0 \\ P_4 \end{bmatrix} = \begin{bmatrix} P_2P_3+P_4 \\ -P_1P_3 \\ -P_1P_4 \\ -P_1 \end{bmatrix},$$

as required. □

Now, we shall prove via the energy-Casimir method, [?], [?], that the equilibrium state $(0,0,\alpha,0)$ of our system (4.5) is nonlinear stable. More exactly we have:

Theorem 4.5 *The equilibrium state $(0,0,\alpha,0)$, $\alpha\in\mathbb{R}$, of the system (4.5) is nonlinear stable.*

Proof. Consider the energy-Casimir function:

$$H_\varphi(P_1,P_2,P_3,P_4)=\frac{1}{2}(P_1^2+P_2^2+P_4^2)+\varphi(-P_3+\frac{1}{2}P_4^2)\,,$$

where $\varphi:\mathbb{R}\to\mathbb{R}$ is an arbitrary smooth real valued function defined on $\mathbb{R}$. Let φ', φ'' denote its first and second derivative. Now, the first variation of H_φ is given by:

$$\delta H_\varphi = P_1\delta P_1+P_2\delta P_2+P_4\delta P_4+\varphi'(\cdot)(-\delta P_3+P_4\delta P_4)\,.$$

This equals zero at the equilibrium of interest if and only if

$$\varphi'(-\alpha)=0\,. \tag{4.6}$$

The second variation of H_φ at the equilibrium of interest is given via (4.6) by:

$$\delta^2 H_\varphi(0,0,\alpha,0)=(\delta P_1)^2+(\delta P_2)^2+(\delta P_4)^2+\varphi''(-\alpha)(\delta P_3)^2\,,$$

which is positive definite if we choose φ such that:

$$\varphi''(-\alpha) > 0\,.$$

So we can conclude that the equilibrium state $(0,0,\alpha,0)$, $\alpha \in \mathbb{R}$, is nonlinear stable as required. □

5. Numerical integration of the system (3.3)–(4.2) via the Lie-Trotter integrator

In this section we shall discuss the numerical integration of the system (3.3) – (4.2) and we shall point out some of its geometrical properties from the Poisson geometry point of view.

To begin with, let us observe that the Hamiltonian vector field X_H splits as follows:

$$X_H = X_{H_1} + X_{H_2}\,,$$

where

$$H_1 = \frac{1}{2}P_1^2, \quad H_2 = \frac{1}{2}P_2^2\,.$$

The integral curves of X_{H_1} and X_{H_2} are given by:

$$P(t) = \exp(tX_{H_1})\cdot P(0) = \phi_1(t, P(0))$$

and respectively

$$P(t) = \exp(tX_{H_2})\cdot P(0) = \phi_2(t, P(0))\,.$$

Now, following [?], [?], [?], [?], [?], we obtain an explicit integrator for the system (3.3) – (4.2) namely:

$$P^{k+1} = \phi_1(t, \phi_2(t, P^k))\,,$$

or explicitely:

$$\left\{\begin{array}{lcl} P_1^{k+1} & = & P_1^k + tP_3(0)P_2^k\,, \\ P_2^{k+1} & = & P_2^k - tP_1^2(0)P_3^k + t^2\frac{P_1(0)}{2}P_4^k - t^3\frac{P_1^3(0)}{6}P_5^k\,, \\ P_3^{k+1} & = & P_3^k - tP_1(0)P_4^k + t^2\frac{P_1^2(0)}{2}P_5^k\,, \\ P_4^{k+1} & = & P_4^k - tP_1(0)P_5^k\,, \\ P_5^{k+1} & = & P_5^k\,. \end{array}\right. \tag{5.1}$$

Some of its properties are sketched in the following theorem:

Theorem 5.1 (i) *The numerical integrator (5.1) preserves the Poisson structure given by the matrix*

$$\Pi = \begin{bmatrix} 0 & P_3 & P_4 & P_5 & 0 \\ -P_3 & 0 & 0 & 0 & 0 \\ -P_4 & 0 & 0 & 0 & 0 \\ -P_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\,,$$

which is in fact the minus-Lie-Poisson structure of $\mathfrak{g}_5^*$.

(ii) *The numerical integrator (5.1) preserves the Casimir C of our configuration* $(\mathbb{R}^5, \Pi)$, *given by:*

$$C(P_1, P_2, P_3, P_4, P_5) = P_2 P_4 - \frac{1}{2} P_3^2 \,.$$

(iii) *Its restriction to each coadjoint orbit*

$$C(P_1, P_2, P_3, P_4, P_5) = constant$$

gives rise to an symplectic integrator with respect to the Kostant-Kirilov-Souriau symplectic form.

(iv) *The numerical integrator (5.1) does not preserve the Hamiltonian H given by*

$$H(P_1, P_2, P_3, P_4, P_5) = \frac{1}{2}(P_1^2 + P_2^2) \,.$$

Proof. The items (i) – (iii) hold because ϕ_1 and ϕ_2 are flows of some Hamiltonian vector fields, hence they are Poisson maps.

Item (iv) is essentially due to the fact that

$$\{H_1, H_2\} \neq 0 \,. \qquad \square$$

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