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SOME REMARKS ON FLOER HOMOLOGY INVARIANTS

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Abstract. This article is an introductory description of work-in-progress. Here is presented a short review of the Floer homology, especially of its topological aspects. In analogy with Morse homology we look at the three theories: symplectic, instantonic and Seiberg-Witten cases. Finally, I emphasize some features and describe a few directions for investigation.

1. Topological Quantum Field Theories

It makes sense to consider that a good framework for the Floer homology is the context of TQFTs [1,2,31]. From the mathematical point of view TQFTs can be viewed as a scheme for different topological theories. Physicists think that a good mathematical insight on the whole picture will create a theory, that would be important also for theoretical physics. But here, so far, we shall look at TQFTs just as a formal scheme.

Formally, a *field theory* takes a manifold M^{d+1}, a collection V of some functions (sections, forms, just functions, etc.) ϕ on M, an action-functional $S(\phi) = \int_{m \in M} L(\phi(m), \partial\phi(m)) : V \to \mathbf{R}$ and a collection of observables $W : V \to \mathbf{C}$ and tries to calculate the partition function $Z(M,k) = \int_{\phi \in V} D\phi e^{2\pi i k S(\phi)}$ and the correlation functions of observables $\ll W_1 \cdots W_N \gg = (\int_{\phi\in V} D\phi e^{2\pi i k S(\phi)} W_1(\phi) \cdots W_N(\phi))/Z(M,k)$. Usually, the *geometric quantization* considers a manifold Σ^d, associates to it the classical phase space $\mathcal{M}(\Sigma)$ of solutions of Euler-Lagrange equations $\delta S_\phi = 0$ on $\Sigma \times (-\infty, +\infty)$, constructs an operator $\overline{\partial}$, a vector space $Q(\mathcal{M}(\Sigma)) := H^0(\mathcal{M}(\Sigma), \overline{\partial})$ and for different M^{d+1} with $\partial M = \Sigma$ defines $Z(M) \in Q(\mathcal{M}(\Sigma))$. Such a quantum field theory is called *topological* if it does not depend on different geometric structures considered on M.

A structure on a manifold M is said *topological* if (i) it has one local model and (ii) its global perturbations are trivial. Examples: orientations, vector bundles, $G-$structures, $G-$bundles, cohomology classes, symplectic structures. In contrast,

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examples of *geometric* structures are metrics, connections, almost-complex structures. The above-mentioned scheme would be:

1. Take a manifold M and a topological structure on it. The invariant will be associated to M and to the considered topological structure.
2. Introduce some additional geometric structure on M.
3. Using the geometric structure define some differential equations on M.
4. Count in some sense the number of solutions. This will be the value of the invariant.
5. Prove that it is independent of the choice of the additional structure, usually by showing that under infinitesimal variation of this structure the variation of the invariant is null. Therefore one should be able to define the invariant only for generic geometric structures.

This scheme is usually called *perturbative.* A *non-perturbative* one would be to deduce the invariant, for example, from a set of axioms, like the *Atiyah axioms of TQFTs.* There one associates to an oriented compact manifold Σ^d, a complex vector space $\mathcal{H}(\Sigma)$, and to an oriented compact manifold M^{d+1} with $\partial M = \Sigma$, an element $Z(M) \in \mathcal{H}(\Sigma)$ such that:

1) The association is functorial in the sense that diffeomorphisms go to isomorphisms and composition to composition.
2) Involutivity: $\mathcal{H}(-\Sigma) = \mathcal{H}(\Sigma)^*$
3) $\mathcal{H}(\emptyset) = \mathbf{C}$ and $Z(\Sigma \times I) = id_{\mathcal{H}(\Sigma)}$
4) Multiplicativity: $\mathcal{H}(\Sigma_1 \sqcup \Sigma_2) = \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2)$
5) Composition: $M = M_1 \cup M_2, \partial M_1 = -\Sigma_1 \cup \Sigma, \partial M_2 = -\Sigma \cup \Sigma_2 \Rightarrow$
$\Rightarrow Z(M) = Z(M_2) \circ Z(M_1) : \mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$
6) If $\mathcal{H}(\Sigma)$ has a hermitian structure, we require $Z(-M) = \overline{Z(M)}$

As a consequence, for $\partial M = \emptyset, M = M_1 \cup_\Sigma M_2$, the number $Z(M) = (Z(M_1), Z(M_2))$ is given by the natural map $(\cdot,\cdot) : \mathcal{H}(\Sigma) \otimes \mathcal{H}(\Sigma)^* \to \mathbf{C}$. This is called *gluing formula.* Sometimes it is better to consider other coefficients than $\mathbf{C}$, for example $\mathbf{Z}$, some grading of $\mathcal{H}(\Sigma)$ etc. A few examples of such theories:

1) *Hodge theory.* The topological structure is M itself, the geometrical one is a Riemannian metric and the equations are just the Laplace equations for $q-$forms. The invariants $b_q(M)$, the Betti numbers, are just the dimensions of the spaces of solutions and these numbers are metric independent.
2) *Donaldson theory.* The topological structure on M is a $SU(2)-$bundle and a collection of homology classes. The geometrical structure is a conformal class of metrics. We consider the anti-self-dual equations $*F_\omega = -F_\omega$ on the quotient $\mathcal{A}(\mathcal{M})/\mathcal{G}$ of the space of connections over the space of gauge transformations. The invariant is the integral over the Uhlenbeck completion of the moduli space of the wedge product of the images via a canonical morphism of the considered homology classes, which turns out to be a polynomial function of them.
3) *Seiberg-Witten theory.* Here the topological structure is a $Spic^{\mathbf{C}}$-structure on M, while the geometrical one is a metric on M. The Seiberg-Witten equations are written for a connection on the determinant bundle and a section of the positive

spinor bundle. The invariant is the corresponding power of the Chern class c_1 of the base point bundle, integrated over the moduli space. In the case when the dimension of the moduli space is zero, we just count the points with signs.

2. Morse homology

As it is well-known, the Morse theory expresses the homology of a finite-dimensional manifold M in terms of critical points of smooth functions on it. The functions are Morse or Bott-Morse and the idea is to calculate the relative homology $H_*(M_b, M_a)$, where $M_a = \{f < a\} \subset M_b = \{f < b\}$ and to express the results in the form of Morse polynomials/series $M_f(t) = P(t) + (1+t)R(t)$, where $P(t)$ is the Poincaré characteristic polynomial. Among the other results of Floer, he has also given an homology interpretation of Morse theory [4,13,27], starting from the ideas introduced by Witten in 1982. We shall follow the TQFTs scheme:

1. Consider M a finite-dimensional oriented connected closed manifold.
2. Take a Morse function $f : M \to \mathbf{R}$ and a Riemannian metric g.
3. Write the differential equations of the negative gradient lines of f:

$$(1) \left\{ \begin{array}{l} \frac{dl}{dt} = -grad\, f(l(t)) \\ l : \mathbf{R} \to M \end{array} \right.$$

For $t \to \pm\infty, l(t)$ tends to some critical points of f. So, for $x_\pm \in crit(f)$ take $\tilde{\mathcal{M}}(x_-, x_+) = \{l(t) \mid l(\cdot)\ solution\ of\ (1),\ l(t) \to \pm x\ for\ t \to \pm\infty\}$ and the relative moduli spaces $\mathcal{M}(x_-,x_+) = \tilde{\mathcal{M}}(x_-, x_+)/\mathbf{R}$, where the action of $\mathbf{R}$ on $\tilde{\mathcal{M}}(x_-, x_+)$ is $(r \cdot l)(t) = l(t+r)$.

4. First, we have to prove three kind of results:

a) Transversality results. Let us call $\mathcal{U}(\alpha) = \{\ m \in M \mid \phi_t(m) \to \alpha\ for\ t \to -\infty\}$ the unstable and $\mathcal{S}(\alpha) = \{m \in M \mid \phi_t(m) \to \alpha\, for\, t \to +\infty\}$ the stable manifolds, where $\phi_t(m)$ is the negative gradient flow of f. We have $\mathcal{M}(x_-,x_+) = (\mathcal{U}_{x_-} \cap \cap \mathcal{S}_{x_+})/\mathbf{R}$. From the Morse lemma we have the homeomorphisms $\mathcal{U}(x_-) \approx \mathbf{R}^{\mu(x_-)}$, $\mathcal{S}(x_+) \approx \mathbf{R}^{dimM-\mu(x_+)}$, where μ denotes the Morse index. $\mathcal{M}(x_-,x_+) \approx S^{\mu(x_-)-1} \cap \cap S^{dimM-\mu(x_+)-1}$ and for generic f the intersection is transverse. Using also the compactness of M we conclude all in the following

Lemma 1. *For generic f, $\mathcal{M}(x_-,x_+)$ is a smooth manifold of dimention $\mu(x_-)- -\mu(x_+) - 1$. If $\mu(x_-) = \mu(x_+) + 1$, then $\mathcal{M}(x_-,x_+)$ is finite.*

b) Compactness results. We shall not insist here on Sobolev spaces estimates, since our goal is the topological picture. Therefore we shall just mention that for $\mu(x_-) - \mu(x_+) \geq 2$ the relative moduli spaces can be compactified to manifolds with corners, constructing a gluing map, which adds exactly the broken gradient lines. In particular,

Lemma 2. *For $\mu(x_-) - \mu(x_+) = 2$, $\mathcal{M}(x_-,x_+)$ can be compactified to a manifold with boundary by adding the broken gradient lines $\bigcup_{\mu(q)=\mu(x_-)-1} \mathcal{M}(x_-,q) \times \times \mathcal{M}(q,x_+)$ as boundary points*

c) Orientation results. Starting from a choice of orientation on stable manifolds, for critical points α, we orient the unstable ones by $[\mathcal{S}_\alpha][\mathcal{U}_\alpha] = [M]$. Then we consider the normal bundles,so $[\mathcal{S}_\alpha][\mathcal{N}_{\mathcal{S}_\alpha}] = [M]$ and $[\mathcal{U}_\alpha][\mathcal{N}_{\mathcal{U}_\alpha}] = [M]$, and define the orientation of $\mathcal{M}(\alpha,\beta)$ by $[\mathcal{M}(\alpha,\beta)][\mathcal{N}_{\mathcal{S}_\beta} \oplus \mathcal{N}_{\mathcal{U}_\alpha}][\nabla f] = [M]$. We can also orient the boundary and corners.

Now we can construct the Witten complex and define the Morse homology and cohomology. Let $C_i = C^i$ be the $\mathbf{Z}$-module generated by the critical points of index i,

$$\partial : C_i \to C_{i-1}, \quad \partial[\alpha] = \textstyle\sum_{\mu(\beta_j)=i-1} \#\mathcal{M}(\alpha,\beta_i)[\beta_i],$$
$$\partial^* : C^i \to C^{i+1}, \quad \partial^*[\alpha] = \textstyle\sum_{\mu(\gamma_k)=i+1} \#\mathcal{M}(\gamma_k,\alpha)[\gamma_k].$$

In both cases # denotes the number of points with orientation. Using the orientation and compactness results, $\partial\partial[\alpha] = \sum_{\beta,\gamma} \#\mathcal{M}(\alpha,\beta)\#\mathcal{M}(\beta,\gamma) = = \sum_\gamma \#(\cup_\beta \mathcal{M}(\alpha,\beta) \times \mathcal{M}(\beta,\gamma)) = \sum_\gamma \#\partial\mathcal{M}(\alpha,\gamma) = 0$, as we have here the boundary of an oriented 1-dimensional manifold. Without orientation results, we could define the homology with coefficients only in $\mathbf{Z}_2$. Verifying the Eilenberg-Steenrod axioms, we obtain:

Theorem. *$H_i(M,\mathbf{Z}) \cong H_i(C_*,\partial)$ and $H^i(M,\mathbf{Z}) \cong H^i(C^*,\partial^*)$.*

Corollary 1 (Morse inequalities). *$\exists k_i \geq 0$ such that $\sum_i (m_i - b_i)t^i = (1+t)\sum_i k_i t^i$.*

Proof. $k_i := m_i - rank\, ker(\partial : C_i \to C_{i-1}) \geq 0$.

Corollary 2 (Poincaré duality). *$H^*(M,\mathbf{Z}) \cong H_{n-*}(M,\mathbf{Z})$.*

Proof. Consider the function $-f$.

5. The above theorem shows that the Morse homology is independent of the choice of metric and Morse function, but we wish to conclude this as in the case of Floer homology, where we do not have an a priori homology [4]. Consider the time-dependent gradient lines, given by the field $-grad_{g_t} f_t + \frac{\partial}{\partial t}$ on $M \times \mathbf{R}$, where the family $F = (f_t, g_t)$ of functions and metrics is such that $f_t = f_1$, $g_t = g_1$ for $t \geq 1$, $f_t = f_{-1}$, $g_t = g_{-1}$ for $t \leq -1$ and the dependence of t is smooth. Define the unstable $\overline{\mathcal{U}}_\alpha = \{\ (x,t) \in M \times \mathbf{R} \mid \phi_s(x,t) \to (\alpha, -\infty)\ for\ s \to -\infty\}$ and the stable manifolds $\overline{\mathcal{S}}_\alpha$. For generic (f_t, g_t), $\mathcal{M}_F(\alpha,\beta) = \overline{\mathcal{U}}_\alpha \cap \overline{\mathcal{S}}_\beta/\mathbf{R}$ is an oriented manifold of dimension $\mu(\alpha) - \mu(\beta)$, which allows us to define the chain map $F_{-11*} : C_*^{-1} \to C_*^1$, $F_{-11*}[\alpha] := \sum_\beta \#\mathcal{M}_F(\alpha,\beta)[\beta]$. Then we can show that the composition of two such chain homomorphisms also arises from a time-dependent flow and we can construct an homotopy operator, which shows that the composition $F_{-11} \circ F_{1-1}$ induces the same map as the identity, which arises from the constant family.

Note, that for the intersection $\overline{\mathcal{U}}_\alpha \cap \overline{\mathcal{S}}_\beta$ to be transverse we have to assume some more generic properties of (f,g) than those necessary to define the homology [4]. Naturally, for *non-compact* manifolds we may ask: can one classify pairs sufficiently generic to define an homology, but not sufficiently generic to define the main one; or at least can one obtain other homologies than the main one?

Fukaya [20], considering a family of Morse functions such that $f_1 + \cdots + f_n = 0$, gave an interpretation of cup-products, and more general Massey products. Cup-products have been also obtained by Austin and Braam [4], by considering higher-dimensional moduli spaces. They have developed a Bott-Morse homology theory, which is related to equivariant cohomology [3] and equivariant Morse theory.

3. Symplectic Floer homology

The three types of Floer homology will be shortly described in the next three sections. Its origins lie in the attempts to solve the famous in symplectic topology **Arnold conjecture.** *A symplectomorphism $f : X \to X$ of a finite-dimensional symplectic manifold X has at least as many fixed points as many critical points should have a function on it.*

On $X \times \overline{X}$ the diagonal and the graph of f are Lagrangian submanifolds. Floer suggested to consider a more general situation, namely two arbitrary Lagrangian submanifolds Λ_1 and Λ_2 of a compact symplectic manifold (M^{2n}, ω). On $\Omega(\Lambda_1, \Lambda_2) = \{l : [0,1] \to M \mid l(0) \in \Lambda_1,\, l(1) \in \Lambda_2\}$ we have the *symplectic action*:

(2) $S(l) = \int\limits_{D^2} \phi^* \omega,$

where $\phi : D^2 \to M$; $\partial D^2 = \partial_0 D^2 \cup \partial_1 D^2 \cup \partial_2 D^2$, say the parameter vary for $\partial_2 D^2$ from 0 to $\frac{2\pi}{3}$, for $\partial_0 D^2$ from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$ and for $\partial_1 D^2$ from $\frac{4\pi}{3}$ to 2π; $\phi|_{\partial_0 D^2} = l$; $\phi|_{\partial_1 D^2} \in \Lambda_1$; $\phi|_{\partial_2 D^2} \in \Lambda_2$; $\phi(1) = p \in \Lambda_1 \cap \Lambda_2$. This action is defined on each connected component of $\Omega(\Lambda_1, \Lambda_2)$, and there is only one component if $\pi_1(M) = 0$. Moreover, if $\pi_2(M) = \pi_1(\Lambda_1) = \pi_1(\Lambda_2) = 0$, S is single-valued, otherwise it can be multi-valued. A simple calculation shows that the critical points of S are just the constant paths, i.e. the points of $\Lambda_1 \cap \Lambda_2$. (We assume that the intersection of Λ_1 and Λ_2 is generic). Now we have in principle the first three steps in the calculation of homology, but we still miss an useful interpretation of gradient lines, which would allow us to study the topology of the relative moduli spaces. Floer solved this problem by introducing an other geometrical structure, an almost-complex structure J, compatible with ω. As a consequence, $grad\, S(l) = -J \frac{dl}{ds}$, or:

Proposition 1. *The gradient lines of S are the mappings $u : [0,1] \times \mathbf{R} \to M$ which are J-holomorphic (M. Gromov):*

$$\frac{\partial u}{\partial t} = J \frac{\partial u}{\partial s}$$

$\forall (s,t) \in [0,1] \times \mathbf{R}$, *with boundary conditions $u(0 \times \mathbf{R}) \subset \Lambda_1$, $u(1 \times \mathbf{R}) \subset \Lambda_2$.*

Let us denote $\mathcal{M}(\alpha, \beta)$ the relative moduli space. We can apply the Conley index theory to define a function $\mu : crit(S) \to \mathbf{Z}$. This theory interprets the notion of index not in terms of directions in $\omega(\Lambda_1, \Lambda_2)$, but of directions in M. Therefore we do not have the problem of infinite index. Summarizing step four,

Proposition 2. *For generic Λ_1, Λ_2 and J, $\mathcal{M}(\alpha, \beta)$ is an oriented smooth finite-dimensional manifold of dimension $\mu(\alpha) - \mu(\beta) - 1$. For $\mu(\alpha) - \mu(\beta) = 1$, $\mathcal{M}(\alpha, \beta)$ is compact. For $\mu(\alpha) - \mu(\beta) = 2$ it can be compactified by adding the boundary points $\cup_{\mu(q) = \mu(\alpha) - 1} \mathcal{M}(\alpha, q) \times \mathcal{M}(q, \beta)$.*

Finally, step five.

Proposition 3. *The above construction is independent of the choice of J and when Λ_2 is deformed to Λ_2' by Lagrangian submanifolds,* $HF_*(\Lambda_1, \Lambda_2) \cong \cong HF_*(\Lambda_1, \Lambda_2')$.

The symplectic Floer homology $HF_*(\Lambda_1, \Lambda_2)$ leads to results of the form $\#\Lambda_1 \cap \Lambda_2 \geq \sum_{i=0}^{2n} dim HF_*(\Lambda_1, \Lambda_2)$. Stability under deformation leads to a solution of Arnold conjecture for symplectomorphisms in the identity component, because for $\pi_2(M) = 0$ the symplectic Floer homology coincides with the homology of M [14,15].

Remark to Section 3. Recently, the Arnold conjecture was proved in the general case independently by Lu and Tian [33] and by Fykaya and Ono [34]. The homotopy assumption on the manifold is removed by a careful study of the homotopy of critical manifolds (in equivariant setting).

4. Instantonic Floer homology

Let Y^3 be an oriented connected closed 3-manifold. Recall that in this case, equivalence under homeomorphisms implies equivalence under diffeomorphisms; the (generalized) Poincaré conjecture says that homotopy-equivalence implies homeo-equivalence. Nevertheless there are classical examples of homology 3-spheres which have different homotopy type. Recall that every $SU(2)$-bundle over Y^3 is trivial. The group of gauge transformations $\mathcal{G}$ acts on the space $\mathcal{A}$ of connections on this bundle. This action is not free, but there are only 3 possibilities for the stabilizers $\mathcal{G}^a = \{g \in \mathcal{G} \mid d^*a = a\}$, namely $\{\pm 1\}$, $U(1)$ or $SU(2)$. Those in the first class are called irreducible and they form an open dense subset $\mathcal{A}^*$ of $\mathcal{A}$. Every connection in the second class is gauge equivalent to a connection of the form $\begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}$. All the connections in the third class are gauge-equivalent to the trivial one; their general form is $\begin{pmatrix} df & 0 \\ 0 & -df \end{pmatrix}$. For homology 3-spheres the second class is empty. On $\mathcal{A}/\mathcal{G}$ we have the *Chern-Simons functional*

$$(3)\quad CS(a) = \frac{1}{8\pi^2}\int_Y tr(a \wedge da + \tfrac{2}{3} a \wedge a \wedge a) = \frac{1}{8\pi^2}\int_W tr(F_{\tilde{a}} \wedge F_{\tilde{a}}),$$

where $\tilde{a}$ is the pull-back of a, $F_{\tilde{a}} = d\tilde{a} + \tilde{a} \wedge \tilde{a}$ on a 4-manifold W with $\partial W = Y$. Indeed, $CS(g^*a) = CS(a)$ for $g \in \mathcal{G}_0$ the connected component of the identity, and in general $CS(g^*a) - CS(a) = deg\, g = \frac{1}{24\pi^2}\int_Y tr(gdg \wedge gdg \wedge gdg) \in \mathbf{Z}$, called the Wess-Zumino functional. A metric on Y induces an L^2-metric on $\mathcal{A}^*/\mathcal{G}_0$ and we presume all the spaces to be Sobolev complete. Simple calculations show that $crit(CS) = flat\ \ connections$ and with respect to the introduced metric $grad_{[a]}CS = -\frac{1}{4\pi^2} *_Y F_a$. Let $\mathcal{C}(a_-, a_+) = \{a(t) : \mathbf{R} \to \mathcal{A} \mid \frac{da(t)}{dt} = \frac{1}{4\pi^2} *_Y F_a,\ a(t) \to b_\pm\ as\, t \to \pm\infty,\ b_\pm\ gauge-equivalent\ to\ a_\pm\}$. Floer found a nice link between CS-functional and ASD-connections:

Lemma. *Let $\tilde{a}$ be a connection 1-form on the trivial bundle over $Y^3 \times \mathbf{R}$, with vanishing dt-term, i.e. in temporal gauge. Then $\tilde{a}$ is anti-self-dual, that is $F_{\tilde{a}} + + *_{Y\times\mathbf{R}} F_{\tilde{a}} = 0$, if ond only if for $a_t = a|_{M\times\{t\}}$ we have $\frac{da_t}{dt} = *_Y F_a$.*

Some technical, but simple perturbations of the Chern-Simons functional have crucial rôle. Let $CS' = CS + h_\gamma$ where $h_\gamma : \mathcal{A}/\mathcal{G} \to \mathbf{R}$, $h_\gamma(a) = \int_{D^2} h(\hat{\gamma}_x(a))$, $(\gamma, h) \in \Pi = \bigcup_{m\in\mathbf{N}} \Gamma_m \times \mathcal{F}_m$, $\Gamma_m = \{\gamma : \bigvee_{i=1}^m S_i^1 \times D^2 \to M \mid \gamma_x : \bigvee_{i=1}^m S_i^1 \times \{x\} \to \to M,\ \gamma_i : S_i^1 \times D^2 \to M$ *are smooth embeddings for* $\forall\, x \in D^2,\ \ \forall\, i \in \overline{1,n}\ \}$, $\hat{\gamma}_x : \mathcal{A}/\mathcal{G} \to \underbrace{SU(2) \times \cdots \times SU(2)}_{m\ times}$ denote the family of induced holonomy maps for $\forall x \in D^2$, and $\mathcal{F}_m = \{h : \underbrace{SU(2) \times \cdots \times SU(2)}_{m\ times} \to \mathbf{R} \mid$ *h invariant under the adjoint action of* $SU(2)\}$.

Floer transversality theorem. *Let Y^3 be an homology 3-sphere. Then under the above perturbation of CS-functional all the critical points are non-degenerate and except for the trivial connection all are irreducible. Moreover, there is a function $\mu : crit(CS') \to \mathbf{Z}$ such that there are only finitely many critical points of a fixed index, and such that $\mathcal{M}(a_-, a_+)$ is an oriented manifold of dimension $\mu(a_-) - \mu(a_+) - dim\,\mathcal{G}^{a_+}$. μ is defined only mod 8. (Compare with the Taubes gluing theorem* [28].*)*

Now the construction of the homology is already standard. The chain complex (C_*, ∂) is generated only be the irreducible critical points of CS', i.e. by all except the trivial connection $\mathbf{0}$, for which we also take μ to vanish. We still have $\partial\partial = 0$, since the gradient lines which broke at the trivial connection form manifolds of dimension $dim\,\mathcal{M}(a_-, 0) + dim\,\mathcal{M}(0, a_+) = \mu(a_-) - \mu(a_+) - dim\,\mathcal{G}^0 = 2 - 3 = -1$, i.e. empty sets. See [16, 18,19,23,29].

An other perturbation $CS'' = CS - f_\varepsilon$ due to Fukaya [18] withdraw the assumption of homology 3-sphere. Consider the *Wilson loop functional* $\mathcal{W}_{\rho,\alpha}(\omega) = = Tr[\rho(g_\omega)]$, where $\rho : G \to V$ is a representation of a Lie group on a vector space, $\alpha \in \Omega(Y, x_0)$ a loop, $\omega \in \mathcal{A}$ a connection on the principal bundle $P(M, G)$, $\hat{\alpha}_\omega : \pi^{-1}(x_0) \to \pi^{-1}(x_0)$ is the parallel translation along α and $\alpha_\omega(p) = pg_\omega$. α_ω is independent of the choice of $p \in \pi^{-1}(x_0)$, ω in its gauge class and α in its homology class in $H_1(Y, \mathbf{Z})$. Let $H_1(Y, \mathbf{Z})$ be free (we will consider here only this case), $\{\gamma_1, \dots, \gamma_{b_1}\}$ a basis for it, $G = SU(2)$ and ρ its defining representation. Extend $\gamma_i : S^1 \to M$ to an embedding $\gamma_i^0 : S^1 \times D^2 \to Y$, define the loop $\gamma_{i,x}(t) := \gamma_i^0(t, x)$, $t \in S^1$ and choose a function $u : D^2 \to \mathbf{R}$ with compact support, such that $\int_{D^2} u(x)dx = 1$. Then define $\mathcal{W}_i(x, \omega) := \mathcal{W}_{\rho,\gamma_{i,x}}(\omega)$, $1 \le i \le b_1$, and for $\varepsilon > 0$, $f_\varepsilon : \mathcal{A}/\mathcal{G} \to \mathbf{R}$, $f_\varepsilon = \varepsilon\Sigma_{i=1}^{b_1} \int_{x\in D^2} \mathcal{W}_i(x, \omega)u(x)dx$. Fukaya showed that for sufficiently small ε, CS'' has finitely many non-degenerate critical points with well-defined index $mod\,8$. Its gradient field is $*_Y F_a - grad_a f_\varepsilon$, so for a homology 3-sphere the Fukaya homology coincides with the Floer one. The $mod\,8$ ambiguity have been replaced by Fintushel and Stern [12] by a certain $\mathbf{Z}$-grading of the instantonic homology.

A conjecture of Atiyah and Floer states that symplectic and instantonic Floer homology coincide. Let $Y_1 \cup_{\Sigma_g} Y_2$ be a Heegaard splitting of Y along a Riemann surface Σ_g of genus g. The flat $SU(2)$-connections on Σ_g which extend to Y_1, respectively Y_2 form two Lagrangian subspaces L_1^{3g-3} and L_2^{3g-3} in the space L_Σ^{6g-6} of $SU(2)$-connections on Σ_g (this has nothing to do with particular $SU(2)$), which carries a natural symplectic structure (with singularities). Therefore on one side we have the symplectic homology $FH_*(L_1, L_2)$, on the other - the instantonic homology $FH_*(Y^3)$, both constructed on the same set of critical points, the flat connections on Y^3. Dostoglou and Salamon [11] considered the ASD equations on $U \times \Sigma$, where $U \subset \mathbf{R}^2$ is an open subset, and in the limit $\Sigma \to 0$ they obtained the J-holomorphic maps from U to L_Σ. Yoshida [32], on the other side, have considered the representations of $\pi_1(\Sigma_g)$ in $SU(2)$. We have not said about the independence of the choice of metric and perturbation, since actually the proof in the Morse homology case already discussed, have been adapted from the instantonic one [16].

5. Seiberg-Witten-Floer homology

As in the previous section, I shall only point out the specific features of the Seiberg-Witten-Floer homology with respect to the general Floer homology scheme. For details and complete presentation see [10,24,25,30].

Let X^4 be an oriented connected compact 4-manifold. For A a $U(1)$-connection and ψ a section of the positive spinor bundle $S^+ \otimes L$ we have the Seiberg-Witten equations:

$$(4) \quad \left\{ \begin{array}{l} \mathcal{D}_A \psi = 0 \\ F_A^+ = \psi \cdot \hat{\psi} \end{array} \right. ,$$

where $\mathcal{D} = \gamma \circ \nabla_A$ is the Dirac operator (elliptic, self-adjoint, first-order), γ is the Clifford multiplication map, F_A^+ is the self-adjoint part of the curvature 2-form, $\psi \cdot \hat{\psi}$ is given in local coordinates by $(\psi \cdot \hat{\psi})_{ij} = \frac{1}{4}\langle e_i \cdot e_j \cdot \psi, \psi\rangle e^i \wedge e^j$, $\langle , \rangle$ is the inner product of sections of $S^+ \otimes L$, $\{e_i\}$ form a local orthonormal basis for TX, $\{e^i\}$ is the dual basis in T^*X and e_i acts on ψ by Clifford multiplication. The first idea is to make the equations (4) play the same rôle as the ASD equations for the instantonic homology, that is to interpolate their solutions between some critical points. Therefore, we shall consider $X^4 = Y^3 \times \mathbf{R}$ with last variable t and we shall separate the variables.

Lemma. *For a pair (A, ψ) in temporal gauge, the equations (4) induce the following equations on Y:*

$$(5) \quad \left\{ \begin{array}{l} \frac{d\psi}{dt} = -\partial_A \psi \\ \frac{d}{dt}A = - *_Y F_A + \sigma(\psi, \psi) \end{array} \right. ,$$

where the 1-form $\sigma(\psi, \psi)$ is given in local coordinates by $\frac{1}{2}\langle e_i \cdot \psi, \psi\rangle e^i$, $\mathcal{D}_A = \partial_t + \partial_{A(t)}$

Clearly, our critical points $\mathcal{M}_C$ (modulo gauge transformations) will be the solutions of:

$$(6) \quad \left\{ \begin{array}{l} \partial_A \psi = 0 \\ *_Y F_A = \sigma(\psi, \psi) \end{array} \right. .$$

For $b^1(Y) > 0$, (5) can be perturbed by a term in the second equation to assure that $\mathcal{M}_C$ is a finite set of points with orientations. A relative Morse index $\mu(\alpha) - \mu(\beta)$ can be defined, which is integer-valued and does not have a *mod* 8 ambiguity as in the instantonic case [24]. This fact is due mainly to the absence of reducible solutions ($\psi \equiv 0$). The $b^1(Y) = 0$ case is treated in [30], where a different perturbation in the same equation is applied.

But as one can expect, some supplementary perturbations were necessary to prove independence of the choice of Riemannian metric and perturbation. Moreover,some new phenomena came into evidence.Namely,for $b^1(Y)$=1 the metric and the perturbation can not be both chosen arbitrarily. For two given metrics g_0 and g_1, it is possible to find perturbations such that the invariant does not change along a path of metrics g_t between g_0 and g_1. However, a perturbation good simultaneously for all (or an open dense subset of) the metrics can not be find. Also, for $b^1(Y) = 0$, the SWF-groups can change as one passes through the codimension 1 space of metrics where the ordinary Dirac operator has non-trivial kernel.

6. Some directions for investigation

6.1. Uniqueness of Floer homology. The main obstruction to "uniqueness" is the absence of a natural Mayer-Vietoris sequence and a natural excision. I will mention the Floer's idea [7,17] to make a ±1-Dehn surgery by a knot to obtain two manifolds Y' and Y'' from a 3-manifold Y^3. The sequence is

$$\dots \to H_*(Y'') \to H_*(Y) \to H_*(Y') \to \dots$$

Cohen, Jones and Segal [9] attempted to develop a homotopy theory for "Floer functions", which would induce the homology. It is possible to do some analysis on infinite-dimensional manifolds (which is highly non-trivial), but the second task is still not reached. Remarks in sections 2 and 5 suggest that we have to consider somehow the pairs (f, g) of functions/perturbations and metrics rather than only functions.

6.2. Relation between Donaldson and Seiberg-Witten invariants. Donaldson and Seiberg-Witten invariants can be interpreted in the spirit of the Atiyah axioms of $TQFT$s, i.e. for a closed 4-manifold $X = X_1 \cup_Y X_2$ which is cut along a closed 3-manifold Y, the invariants $\psi(X_1)$ and $\psi(X_2)$ have values in $FH_*(Y)$, respectively $FH^*(Y)$;and we should have a gluing formula $\psi(X) = (\psi(X_1), \psi(X_2))$. Note [30], that for the Seiberg-Witten case, it is possible to analyze nicely the relative moduli spaces $\mathcal{M}(\partial X_1, a)$. Recall also that already Floer showed that a cobordism X from Y_1 to Y_2 induces a graded homomorphism $X_j : FH_j(Y_1) \to FH_{j+b(X)}(Y_2)$, $j \in \mathbf{Z}_8$, $b(X) = 3(b_1(X) - b_2(X))$. The calculations of Donaldson and Seiberg-Witten invariants are complex-geometric at origins, and it is reasonable to look for a topological method of calculation.

6.3. Geometrical interpretation of Floer homology. Let $\gamma \in \Omega(Y, x)$ be a loop at $x \in Y^3$. A connection ω on the (trivial) $SU(2)$-bundle over Y induces an horizontal lift $h_u : \Omega(Y, x) \to SU(2)$, $u \in \pi^{-1}(x)$. Since ω is flat, h_u depends only of the homotopy class of γ, so $h_u : \pi_1(Y, x) \to SU(2)$. To make this map independent of the choice of $u \in \pi^{-1}(x)$, observe that $h_{ug} = g^{-1}h_u g$. Therefore, a flat connection ω determinates an element of the quotient $Hom(\pi_1(Y), SU(2))/SU(2)$, where we have the adjoint action of $SU(2)$. Conversely, let $f \in Hom(\pi_1(Y), SU(2))$, $\tilde{Y} \to Y$ the universal cover ($\tilde{Y}$ is a $\pi_1(Y)$-principal bundle over Y). Let $Q = \tilde{Y} \times_f SU(2)$. This is a trivial $SU(2)$-bundle over

Y, which admits a natural flat connection. f and $g^{-1}fg$, $g \in SU(2)$ determine gauge-equivalent flat connections. Therefore,

(7) $\mathcal{A}^{flat}(Y)/\mathcal{G} \cong Hom(\pi_1(Y), SU(2))/SU(2)$.

This is a geometric interpretation of the critical points in the definition of instantonic Floer homology. The Casson invariant (half the Euler-Poincaré characteristic of the homology) have been introduced just with this geometric interpretation in mind [29]. We could consider above any flat bundle (Q, ω), no matter of Y^3 and $SU(2)$. For example, for an oriented compact surface Σ of genus g and an $U(1)$-bundle over Σ, we have $\mathcal{A}^{flat}(\Sigma)/\mathcal{G} \cong Hom(\pi_1(\Sigma), U(1)) = H^1(\Sigma, U(1)) = = U(1)^{2g}$.

In contrast, the SW equations on a 3-manifold are difficult to interpret geometrically, except for the cases when Y admits a metric of positive scalar curvature (SWF groups are trivial), or when $Y = \Sigma \times S^1$, Σ a surface (the SWF groups are the ordinary homology groups of the symplectic products of Σ). The proof of the last statement suggests that one should regard the equivariant theory. See the papers [5,6,25] for a development of this theory.

There exist generalizations of SW equations on a 4-manifold for the group $SU(2)$ instead of $U(1)$. That fact suggested some relations between the classical Donaldson and Seiberg-Witten invariants [26]. Therefore we think that a generalization for 3-manifolds should give new insights to the problem of geometrical interpretation of SWF_*. A kind of formula (7) should be established.

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