DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 303–308

THE DUISTERMAAT–HECKMAN INTEGRATION FORMULA ON GENERALIZED FLAG MANIFOLDS

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Abstract. Let M be a $2n$-dimensional symplectic manifold with symplectic 2-form ω. The partition function of a classical system whose dynamics is governed by the Hamiltonian function H can be written as an integral $\int_M e^{i\phi H}\frac{\omega^n}{n!}$ ($\phi \in \mathbb{R}$). The Duistermaat-Heckman integration formula states that if the Hamiltonian vector field has only isolated zeros and it is almost periodic, then the previous integral can be expressed as a certain sum over the critical points of the Hamiltonian. We will study this integration formula for a class of Kähler manifolds, the generalized flag manifolds. These are homogeneous spaces of the form $G/C(T)$, where G is a compact Lie group and T a maximal torus in G. Equivalently, they can be expressed as adjoint orbits of the adjoint representation of G in its Lie algebra. In our approach we use well known results of R. Bott and H. Samelson about the Morse theory of height functions on an adjoint orbit, to give a Lie algebraic expression for the Duistermaat-Heckman formula, suitable for calculations.

1. Introduction

Let M be a compact symplectic manifold of dimension $2n$ with symplectic 2-form ω, and let T be a torus acting on M in a Hamiltonian way. This means that there is a momemt map $H : M \to \mathfrak{t}^*$ (the Lie algebra of T), also called Hamiltonian, having the properties $\iota_X\omega = -dH_X$ and $\omega(X,Y) = 0$ for all $X, Y \in \mathfrak{t}$. Here $\iota_X\omega$ is the contraction of the vector field X with the 1-form ω defined by $\iota_X\omega(Y) = \omega(X,Y)$ for all $Y \in TM$. The moment map is defined by $\langle X, H(m)\rangle = H_X(m)$, for all $m \in M$, $X \in \mathfrak{t}$, and assume in addition that for each $X \in \mathfrak{t}$ the infinitesimal action of $(X^*f)(m) = \frac{d}{dt}f(\exp tX \cdot m)\big|_{t=0}$ of X on M is given by the Hamiltonian vector field $\tilde{X}$ of the function H_X.

1991 *Mathematics Subject Classification*. Primary 53C80, 53C30, 53C21, 58F05; Secondary 57R70, 55N91.

Key words and phrases. Generalized flag manifolds, Duistermaat-Heckman formula, Morse theory, Homogeneous spaces, Equivariant cohomology.

The author was supported by a post-doctoral fellowship from the Greek State Scholarships Foundation (IKY), Programm #1 (1997-98) Grant #3.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In the paper [Du-He] Duistermaat and Heckman showed that if a $2n$-form μ on M is equivariantly closed, i.e. $(d+\iota_X)\mu = 0$ and $\mathcal{L}_X\mu = 0$, then the value of the integral $\int_M \mu$ is given in terms of contributions from the points in M which are fixed points of the vector field X.

Alternatively, their theorem assets that the push-forward $H_*(dm)$ of the symplectic (or Liouville) measure dm on M is a piecewise polynomial measure on $\mathfrak{t}^*$. Furthermore, as shown in [At-Bo] the theorem of Duistermaat-Heckman can be viewed as a special case of more general results about equivariant cohomology (which uses the operator $d+\iota_X$ mentioned earlier, instead of d), but we will not discuss this here.

Now, when the vector field X has only isolated zeros or equivalently when H has only isolated critical points X_{cr}, then the above theorem leads to an exact formula for the oscillatory integral $\int_M \exp(H)\frac{\omega^n}{n!}$ given by

$$\int_M \exp(H)\frac{\omega^n}{n!} = \sum_{X_{cr}} \frac{\exp(H(X_{cr}))}{D(X_{cr})}$$

where $D(X_{cr})$ are certain integers attached to the infinitesimal action of T at X_{cr}.

Recently, Picken [Pi] discussed the above integration formula on certain homogeneous spaces called generalized flag manifolds. These are spaces of the form $G/C(T)$, where G is a compact Lie group and $C(T)$ is the centralizer of a torus in G. Alternatively, these spaces are orbits under the adjoint representation of G in its Lie algebra $\mathfrak{g}$. The main attempt in that paper was to give a precise expression of the left-hand side of the above integration formula and of course to compute the right-hand side. The first goal was achived by using the Bruhat coordinates for generalized flag manifolds, but the second one was achived only for special examples.

In this paper we use well known results due to Bott and Samelson [Bo1], [Bo2], [Bo-Sa] about Morse theory for height functions on generalized flag manifolds to give an explicit general expression of the right-hand side of the Duistermaat-Heckman integration formula. It is given in terms of the roots of the Lie algebra of $\mathfrak{g}$ and it is suitable for computations.

For our purposes the Duistermaat-Heckman integration formula will have the form

$$\int_M \exp(H)\frac{\omega^n}{n!} = \sum_{X_{cr}} \frac{e^{H(X_{cr})}}{\frac{\det(\mathrm{Hess}(X_{cr}))^{\frac{1}{2}}}{\det(\omega)}} \tag{1}$$

(see also [Pi] (5.8), (5.36)).

2. Generalized flag manifolds

We shall give a summary of the basic structure of the generalized flag manifolds or adjoint orbits. Let G be a semisimple and compact Lie group. A *generalized flag manifold* is a homogeneous space $M = G/K$ whose isotropy subgroup K is the centralizer of a one-parameter subgroup $T_P = \overline{\exp \mathbb{R}P}$ of G which generates a torus in G. Here P is an element in the Lie algebra $\mathfrak{g}$ of G. The homogeneous space M is also called *adjoint orbit* because in fact it can be identifed with the adjoint orbit $\mathrm{Ad}(G)P$ of P under the action of the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. Without loss of generality the point P can be taken to lie in a fixed maximal torus T of G, which can also contain T_P. Hence we have that $G/K = G/C(T_P) \cong \mathrm{Ad}(G)P$.

Let G/K be a generalized flag manifold with $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebras of G and K respectively. The Killing form $B(X,Y) = \mathrm{tr}\,\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ of $\mathfrak{g}$ gives rise to an orthogonal decomposition of $\mathfrak{g}$ as the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. The subspace $\mathfrak{m}$ is identified with the tangent space T_0M at the identity coset $0 = eK$.

Let $\mathfrak{h}$ be the Lie algebra of T and $\mathfrak{h}^{\mathbb{C}}$ its complexification. Then $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let R be the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ and

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in R} \mathfrak{g}^{(\alpha)}$$

its root space decomposition. For any root $\alpha \in R$ we can choose elements $E_\alpha \in \mathfrak{g}^{(\alpha)}$ which have the properties $B(E_\alpha, E_{-\alpha}) = -1$ and $[E_\alpha, E_{-\alpha}] = -H_\alpha$, where H_α is determined by the equation $B(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}^{\mathbb{C}}$. The corresponding real decomposition of $\mathfrak{g}$ may be written as

$$\mathfrak{g} = \mathfrak{t} + \sum V_\alpha$$

where $V_\alpha^{\mathbb{C}} = \mathbb{C}E_\alpha \oplus \mathbb{C}E_{-\alpha}$, and the sum is taken over one representative from each pair of roots $\pm\alpha$ (there is no natural choice for each pair).

Since $\mathfrak{k}^{\mathbb{C}}$ contains $\mathfrak{h}^{\mathbb{C}}$, there exists a subset R_K of R such that

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in R_K} \mathfrak{g}^{(\alpha)}\,.$$

The Lie algebra of the isotropy subgroup $K = C(T_P)$ is just $\mathrm{Ker}(\mathrm{ad}\,P)$, and the corresponding real decompotition is given by

$$\mathfrak{k} = \mathfrak{t} + \sum_{\alpha(P)=0} V_\alpha\,.$$

Consequently, we obtain that $\mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in R_M} \mathfrak{g}^{(\alpha)}$ and the root system R is decomposed as $R = R_K \cup R_M$. The set R_M is called the set of *complementary roots*. There is a natural decomposition $R_M = R_M^+ \cup R_M^-$, where the *positive* complementary roots is the set $R_M^+ = \{\alpha \in R_M : \ \alpha(P) > 0\}$. Hence we obtain that $\mathfrak{m} = \sum_{\alpha \in R_M^+} V_\alpha$, and the set $\{E_\alpha : \ \alpha \in R_M\}$ constitutes a basis for $\mathfrak{m}^{\mathbb{C}}$.

Next we recall the *isotropy representation* $\mathrm{Ad}^{G/K}$ of K on $T_0M = \mathfrak{m}$ which is the homomorphism defined by $k \mapsto (dL_k)_0$ (here L_g is the diffeomorhism of G/K given by $L_g(hK) = ghK$ for all $g \in G$). For a generalized flag manifold he isotropy representation is decomposed into a finite number of mutually inequivalent irreducible submodules, which can be described as follows. Let $\overline{\alpha}_1, \ldots, \overline{\alpha}_k$ be the distinct linear funcionals on $\mathfrak{t}_P$ obtained by restricting the positive complementary roots to $\mathfrak{t}_P$. Let $\mathfrak{m}_i = \sum_{\alpha|_{\mathfrak{t}_P}=\overline{\alpha}_i} V_\alpha$. Then the required decomposition is

$$\mathfrak{m} = \sum_{i=1}^{k} \mathfrak{m}_i \, .$$

Finally, by using the embedding $\mathrm{Ad}(G)P \subset \mathfrak{g}$, the tangent space to $\mathrm{Ad}(G)P$ at P is identified with $\mathrm{Im}(\mathrm{ad}\, P)$, hence a typical point of the tangent space at P will have the form $\mathrm{ad}(P)X$ for some $X \in \mathfrak{g}$.

A generalized flag manifold acquires a natural G-invariant complex structure determined by the linear transformation J on $\mathfrak{m}$ given by

$$J = \sum_{\alpha \in R_M^+} \frac{1}{2\pi\alpha(P)} \, \mathrm{ad}(P)|_{V_\alpha} \, .$$

Notice that this complex structure depend on the choice of positive complementary roots and the choice of the point P.

Finally we will discuss G-invariant metrics on a generalized flag manifold $M = G/K$. These metrics are determined by $\mathrm{ad}(\mathfrak{k})$-invariant inner products on $\mathrm{Im}(\mathrm{ad}\, P)$. By Schur's lemma any such inner product g_P can be expressed as

$$g_P = -c_1 \; B|_{\mathfrak{m}_1} - \cdots - c_k \; B|_{\mathfrak{m}_k} \, , \qquad (c_i > 0) \, .$$

A G-invariant metric of special importance is a Kähler metric g_P^K defined by $g_P^K(\mathrm{ad}(P)X, \mathrm{ad}(P)Y) = \omega(X, JY)$, where ω is the canonical symplectic (Kirillov) form determined by the skew symmetric billinear map $\Omega(\mathrm{ad}(P)X, \mathrm{ad}(P)Y) = -B(P, [X, Y])$.

3. Critical points of height functions on generalized flag manifolds

For the following we refer to [Bo-Sa] and [Gu]. Let $M = G/K$ be a generalized flag manifold viewed as an adjoint orbit $\mathrm{Ad}(G)P$ as above, and take a point $Q \in \mathfrak{g}$. We consider the *height function*

$$H^Q(X) = \langle X, Q \rangle = -B(X, Q) \, .$$

Proposition 1. *Let X be a point in $\mathrm{Ad}(G)P$. Then X is a critical point of H^Q if and only if $[Q, X] = 0$.*

Proof. The point X is a critical point of H^Q if and only if $\frac{d}{dt} H^Q(\mathrm{Ad}(\exp tU)X)\big|_{t=0} = 0$ for all $U \in \mathrm{Im}\,\mathrm{ad}\, X$, if and only if $\langle [U, X], Q \rangle = 0$ for all $U \in \mathrm{Im}\,\mathrm{ad}\, X$, i.e $[Q, X] = 0$.

Corollary 1. *The set of critical points of* H^Q *is* $\mathrm{Ad}(G)P \cap \mathrm{Ker}(\mathrm{ad}\, Q)$ *or equivalentlly* $\bigcup(\mathrm{Ad}(G)P \cap \mathfrak{h})$, *where the union is taken over all Cartan subalgebras* $\mathfrak{h}$ *that contain* Q.

Without loss of generality we may assume that P is a critical point of H^Q, so $[Q, P] = 0$. Hence there exists a maximal torus whose Lie algebra contains both P and Q. Without loss of generality we may assume that T is such a maximal torus. We furthermore assume that Q is a regular point of $\mathfrak{t}$ which means that $\mathrm{Ker}\,\mathrm{ad}(Q)$ is abelian, hence $\mathfrak{t}$ is the only Cartan subalgebra that contains Q. As a consequence of this the critical set is $\mathrm{Ad}(G)P \cap \mathfrak{t}$. Next we will compute the Hessian of H^Q.

Proposition 2. *The Hessian of the function* H^Q *at a critical point* X *is given by*

$$H(U, V) = \langle \mathrm{ad}(X)\,\mathrm{ad}(Q)U, V\rangle$$

for all $U, V \in \mathrm{Im}\,\mathrm{ad}(P)$.

Proof.

$$H(U, V) = \frac{d^2}{dsdt} H^Q(\mathrm{Ad}(\exp(sU + tV)X))|_0 = \langle \mathrm{ad}(X)\,\mathrm{ad}(Q)U, V\rangle$$

If we evaluate the Hessian at the basis $\{E_\alpha :\ \alpha \in R_M\}$, we obtain that (over $\mathbb{C}$) the set of eigenvalues of H^Q at a critical point X is $\{\alpha(Q)\alpha(X) :\ \alpha \in R_M\}$.

4. The Duistermaat-Heckman integration formula on generalized flag manifolds

We are now in the position to give an expression of the right-hand side of the formula (1). We recall that the symplectic form ω is determined by the skew symmetric billinear map $\Omega(\mathrm{ad}(P)X, \mathrm{ad}(P)Y) = -B(P, [X, Y])$. This can be evaluated by its restriction on each of the sets $V_\alpha,\ \alpha \in R_M^+$. By taking $X = E_\alpha$ and $Y = E_{-\alpha}$ we obtain that its complexification is given by $\Omega(\mathrm{ad}(P)E_\alpha, \mathrm{ad}(P)E_{-\alpha}) = \alpha(P)$, hence $\det \Omega = \prod_{\alpha \in R_M^+}(\alpha(P))^2$. From the above analysis we obtain the following:

Theorem 1. *The Duistermaat-Heckman integration formula (1) on a generalized flag manifold* $M = G/C(T_P) = \mathrm{Ad}(G)P$ *is given by*

$$\int_M \exp(H^Q)\frac{\omega^n}{n!} = \left(\prod_{\alpha \in R_M^+} \alpha(P)\right) \sum_{X_{cr}} \frac{e^{\langle X_{cr}, Q\rangle}}{\prod_{\alpha \in R_M} \alpha(Q)\alpha(X_{cr})} \tag{2}$$

where the sum is taken over the critical points X_{cr} *of* H^Q.

Example. Let $M = SU(3)/T = SU(3)/S(U(1)\times U(1)\times U(1)) = \mathrm{Ad}(SU(3))P$, where $P = \mathrm{diag}(-i,0,i)$ is an element in the Cartan subalgebra of the complexification of $\mathfrak{su}(n)$, consisting of the set of matrices $\{\mathrm{diag}(ix_1, ix_2, ix_3) : x_1+x_2+x_3 = 0\}$. We fix an ordering so that the set of positive roots are $R^+ = \{x_1-x_2, x_1-x_3, x_2-x_3\}$. We choose a regular point $Q = \mathrm{diag}(0,2i,-2i)$ (i.e. $(x_i-x_j)(Q)\neq 0$), and let $H^Q(X) = \langle X,Q\rangle = -\frac{1}{6}B(X,Q) = -\operatorname{tr} XY$.

Without loss of generality the point P can be chosen to be a critical point of H^Q, say $P = X_1$, so the other critical points are $X_2 = \mathrm{diag}(i,0,i), X_3 = \mathrm{diag}(0,i,-i), X_4 = \mathrm{diag}(0,-i,i), X_5 = \mathrm{diag}(i,-i,0), X_6 = \mathrm{diag}(-i,i,0)$. (According to [Ra] the number of critical points for $SU(n)/S(U(n_1)\times\cdots\times U(n_k))$ is $r = \frac{n!}{n_1!\cdots n_k!}$ and in our case $r = 6$).

An easy computation shows that $H^Q(X_1) = H^Q(X_5) = -2$, $H^Q(X_2) = H^Q(X_6) = 2$, $H^Q(X_3) = 4$, $H^Q(X_4) = -4$. Furthermore, $\prod_{\alpha\in R^+}\alpha(P) = 2i$ and $\prod_{\alpha\in R}\alpha(Q)\alpha(X_i) = 4^5$, hence the right-hand side of (2) reduces to

$$\frac{2i}{4^5}\left[2e^{-2}+2e^{2}+e^{4}+e^{-4}\right].$$

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